Discussiones Mathematicae
Graph Theory 28 (2008) 323–333

ON ACYCLIC COLORINGS OF DIRECT PRODUCTS

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Abstract

A coloring of a graph G is an acyclic coloring if the union of any two color classes induces a forest. It is proved that the acyclic chromatic number of direct product of two trees T_1 and T_2 equals $\min\{\Delta(T_1)+1, \Delta(T_2)+1\}$. We also prove that the acyclic chromatic number of direct product of two complete graphs K_m and K_n is $mn - m - 2$, where $m \geq n \geq 4$. Several bounds for the acyclic chromatic number of direct products are given and in connection to this some questions are raised.

Keywords: coloring, acyclic coloring, distance-two coloring, direct product.

2000 Mathematics Subject Classification: 05C15.

1. Introduction

The *direct product* $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$. Vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \times H$ if $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. The direct product is one of the four standard graph products and has been studied from various points of view (see [8]).

Work partly supported by the Ministry of Science of Slovenia under grants P1–0297 and L1–9338.

One of the most notorious open problems for direct products of graphs is the Hedetniemi's conjecture, which claims that

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}\,.$$

The conjecture was partially confirmed by El-Zahar and Sauer in [7], where they proved that the product of two four chromatic graphs is four chromatic. Many other results on the conjecture are collected in a survey by Zhu [13]. The fractional chromatic number of direct products was considered in [12].

Acyclic colorings were introduced by Grünbaum [5]. He conjectured that planar graphs are acyclically 5-colorable. The conjecture is a generalization of Stein's theorem about $(1, 2, 2)$-partition of planar graphs, and was later confirmed by Borodin [4]. Many other classes of graphs were studied later: acyclic colorings of graphs on surfaces, locally planar graphs and random graphs were considered [1, 2, 11]. The acyclic chromatic number of Cartesian products of trees and cycles was established in [9] and [10].

An *n-coloring* of a graph G is a function $f : V(G) \to \{1, \dots, n\}$. We say that f is a *coloring* if it is an n-coloring for some n. A coloring of a graph is a *proper coloring* if any two neighboring vertices receive distinct colors, and it is an *acyclic coloring* if it is proper and union of any two color classes induces a forest. The *acyclic chromatic number* of a graph G, denoted as $\chi_a(G)$, is the least n such that the graph G is acyclically n-colorable.

A proper coloring of a graph is a *distance-two coloring* if any two vertices at distance two receive distinct colors. We denote by $\chi_2(G)$ the least n such that the graph G admits a distance-two coloring with n colors.

A *G-layer* in $G \times H$ is a set of vertices $G^y = \{(x, y)|x \in V(G)\}$, where $y \in V(H)$, and an *H-layer* is $G^x = \{(x, y)|y \in V(H)\}$, where $x \in V(G)$. The *neighborhood* of a vertex $v \in V(G)$ is a set of vertices $N(v) = \{u \in V(G)|uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. By $\Delta(G)$ we denote the largest degree over all vertices in G.

2. Bounds for Acyclic Chromatic Number of Direct Products

Lemma 2.1. *For any graphs G and H, $\chi_a(G \times H) \geq \min\{\Delta(G) + 1, \Delta(H) + 1\}$.*

Proof. Let $x \in V(G)$ and $y \in V(H)$ be vertices with $\deg_G(x) = \Delta(G)$ and $\deg_H(y) = \Delta(H)$. Let $\Delta(G) = \alpha$ and $\Delta(H) = \beta$ and let $x_1, \dots, x_\alpha$ be neighbors of x in G and $y_1, \ldots, y_\beta$ neighbors of y in H. The lemma is clear if $\alpha = 1$ and $\beta = 1$, since in this case the direct product will have at least one edge and therefore $\chi_a(G) \geq 2$ (and it is trivial if $\alpha = 0$ or $\beta = 0$), so assume that $\alpha, \beta \geq 2$. Consider the subgraph of $G \times H$ induced by $N[x] \times N[y]$. It follows from the definition of the direct product that for every $i \leq \alpha$ and $j \leq \beta$, the vertex (x_i, y) is adjacent to (x, y_j). Therefore the complete bipartite graph $K_{\alpha,\beta}$ is a subgraph of $N[x] \times N[y]$.

Suppose that f is an acyclic coloring of $G \times H$. If vertices (x_i, y) receive pairwise distinct colors, then clearly f is a coloring with at least $\Delta(G) + 1$ colors, since every vertex (x, y_j) is colored by a color different from the colors of (x_i, y). Otherwise there exist $a, b \leq \alpha$, such that $f(x_a, y) = f(x_b, y)$. If also $f(x, y_c) = f(x, y_d)$ for some $c, d \leq \beta$, then $(x_a, y)(x, y_c)(x_b, y)(x, y_d)$ is a bichromatic 4-cycle, a contradiction. Therefore, in this case, the vertices (x, y_j) receive pairwise distinct colors and f is a coloring with at least $\Delta(H) + 1$ colors. ■

It follows from Lemma 2.1 that an upper bound for $\chi_a(G \times H)$ in terms of $\chi_a(G)$ and $\chi_a(H)$ is not possible. Clearly, such a bound would fail for products of trees, since for any tree T, $\chi_a(T) = 2$ and $\Delta(T)$ can be arbitrary large.

Alon, McDiarmid and Reed proved that for every graph G, $\chi_a(G) \leq \lceil 50\Delta(G)^{4/3} \rceil$ (see [1]). Since $\Delta(G \times H) = \Delta(G)\Delta(H)$, it follows from this result, that $\chi_a(G \times H) \leq \lceil 50\Delta(G)^{4/3}\Delta(H)^{4/3} \rceil$. We think that an improvement of this bound is possible, therefore we suggest the following problem.

Problem 2.2. Find a sharp upper bound for $\chi_a(G \times H)$ in terms of $\Delta(G)$ and $\Delta(H)$.

Lemma 2.3. *For any tree T and any graph G, $\chi_a(G \times T) \leq \chi_2(G)$.*

Proof. Let T be a tree and G an arbitrary graph. Suppose that f is a distance-two coloring of G and let

$$f' : V(G \times T) \to \{1, \ldots, n\}$$

be the coloring induced by f, that is

$$f'(x, y) = f(x)$$

for all $x \in V(G)$ and $y \in V(T)$. We claim that f' is an acyclic coloring of $G \times T$. Assume on the contrary that $C = (x_1, y_1)(x_2, y_2) \ldots (x_{2n}, y_{2n})(x_1, y_1)$

is a bichromatic cycle in $G \times T$. Then $f'(x_{2k-1}, y_{2k-1}) = i$ and $f'(x_{2k}, y_{2k}) = j$, $i \neq j$, and therefore $f(x_{2k-1}) = i$ and $f(x_{2k}) = j$ (for $k = 1, \ldots, n/2$). Since f is a distance-two coloring, we find that $x_m = x_{m+2}$ for $m = 1, \ldots, n$. It follows that C is of the form $(x_1, y_1)(x_2, y_2) \ldots (x_1, y_{2n-1})(x_2, y_{2n})(x_1, y_1)$. If $y_k \neq y_l$ for $k \neq l$, then $y_1 y_2 \ldots y_{2n} y_1$ is a cycle in T. Otherwise there exist k and l, $k \neq l$, such that $y_k = y_l$ and therefore $y_k y_{k+1} \ldots y_l$ is a cycle in T, a contradiction in either case. We conclude that f' is an acyclic coloring of $G \times T$. ■

In the following theorem we show that bounds from Lemma 2.1 and Lemma 2.3 are sharp for products of trees.

Theorem 2.4. *Let T_1 and T_2 be two trees. Then $\chi_a(T_1 \times T_2) = \min\{\Delta(T_1) +1, \Delta(T_2) + 1\}$.*

Proof. By Lemma 2.1 it suffices to prove that $\chi_a(T_1 \times T_2) \leq \min\{\Delta(T_1) +1, \Delta(T_2) + 1\}$. We claim that $\chi_2(T) = \Delta(T) + 1$ for any tree T. To prove this, run a BFS algorithm on T and color the children of each vertex by pairwise distinct colors that are different also from the color of their father and forefather. Clearly, the obtained coloring f is a distance-two coloring and it needs exactly $\Delta(T) + 1$ colors. It follows from Lemma 2.3, that $\chi_a(T_1 \times T_2) \leq \min\{\Delta(T_1) + 1, \Delta(T_2) + 1\}$. ■

Observation 2.5. *The direct product of an n-cycle and a complete graph on two vertices is either a $2n$-cycle, if n is odd or a disjoint union of two n-cycles, if n is even. Conversely, the projection of a cycle in $G \times K_2$ to $V(G)$ is a cycle in G.*

Consider the direct product $G \times K_2$ and suppose that $f : V(G) \to \{1, \ldots, n\}$ is an acyclic coloring of G. Let $V(K_2) = \{u, v\}$ and let $f' : V(G \times K_2) \to \{1, \ldots, n\}$ be the coloring induced by f, more precisely

$$f'(g, u) = f'(g, v) = f(g)$$

for every $g \in V(G)$. We claim that f' is an acyclic coloring of $G \times K_2$. Indeed, if $(x_1, u)(x_2, v) \ldots (x_n, v)(x_1, u)$ is a bichromatic cycle in $G \times K_2$, then $f'(x_{2k-1}, u) = i$ and $f'(x_{2k}, v) = j$ (for $k = 1, \ldots, n/2$) and herefrom $f(x_{2k-1}) = i$ and $f(x_{2k}) = j$. Since $x_1 x_2 \ldots x_n x_1$ is a cycle in G we conclude that the coloring f is not acyclic. We have proved the following lemma.

Lemma 2.6. *For any graph $G, \chi_a(G \times K_2) \leq \chi_a(G)$.*

We follow with a lower bound for $\chi_a(G \times K_2)$.

Theorem 2.7. *For any graph $G, \chi_a(G \times K_2) \geq \sqrt{\frac{\chi_a(G)+1/8}{2}} + \frac{1}{4}$.*

Proof. Let $G \times K_2$ be a direct product and f an acyclic n-coloring of $G \times K_2$. Let $V(K_2) = \{u, v\}$ and

$$F_{i,j} = \{g \in V(G) \,|\, f(g, u) = i,\ f(g, v) = j\}.$$

For every cycle with vertices in $F_{i,j}$ we have an even cycle in $G \times K_2$ (see Observation 2.5), such that all even vertices of the cycle are colored by i and odd vertices by j. Since f is an acyclic coloring of $G \times K_2$ we infer that the graph induced by $F_{i,j}$ is a forest (for $1 \leq i, j \leq n$). Moreover, it follows directly from the definition of the direct product that $F_{i,i}$ is an independent set (for $1 \leq i \leq n$). Consider the forests $F_{i,j}$ and $F_{k,\ell}$ and the edges

$$E_{i,j,k,\ell} = \{uv \,|\, u \in F_{i,j},\ v \in F_{k,\ell}\}$$

with one end vertex in $F_{i,j}$ and the other in $F_{k,\ell}$. We claim that the subgraph H of G with the vertex set $F_{i,j} \cup F_{k,\ell}$ and the edge set $E_{i,j,k,\ell}$ is a forest. Indeed, a cycle in H induces an even cycle in $G \times K_2$ (see Observation 2.5) with even vertices colored by i (resp. j) and odd vertices by ℓ (resp. k).

Let f' be a coloring of G such that each forest $F_{i,j}, i \neq j$ is colored by two colors and each independent set $F_{i,i}$ by one color, so that $f'(u) \neq f'(v)$ if u and v are vertices of two distinct sets $F_{i,j}$ $(1 \leq i, j \leq n)$. Clearly, f' is a coloring with $2n^2 - n$ colors. Since the graphs induced by $F_{i,j}$ are forests and there is no cycle with edges in $E_{i,j,k,\ell}$ $(1 \leq i, j, k, \ell \leq n)$ we infer that the coloring f' is acyclic. We have proved that $\chi_a(G \times K_2) = n$ implies $\chi_a(G) \leq 2n^2 - n$ which implies the desired lower bound. ∎

Since $G \times K_2$ and $K_2 \times H$ are subgraphs of $G \times H$ we have the following corollary.

Corollary 2.8. *For any nontrivial graphs G and H, $\chi_a(G \times H) \geq \max\{\sqrt{\frac{\chi_a(G)+1/8}{2}} + \frac{1}{4}, \sqrt{\frac{\chi_a(H)+1/8}{2}} + \frac{1}{4}\}$.*

In Figure 1 we present a graph G for which $\chi_a(G \times K_2) < \chi_a(G)$. We draw two copies of G which correspond to the two layers of $G \times K_2$, the edges of $G \times K_2$ are not drawn for clearity reasons. It is easy to see that the given coloring is an acyclic 3-coloring of $G \times K_2$. Moreover, the graph G is not acyclically 3-colorable. We also mention that the lower bound from Theorem 2.7 is not sharp for acyclically 2 and 3-colorable products. In fact $\chi_a(G \times K_2) = 2$ implies that $G \times K_2$ does not have any cycles, which in turn implies that G has no cycles (see Observation 2.5) and hence $\chi_a(G) = 2$. It turns out that if $\chi_a(G \times K_2) = 3$, then $\chi_a(G) \leq 5$, we do not give the proof of this.

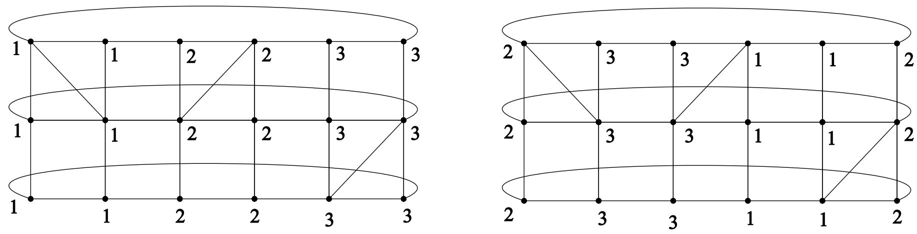

Figure 1. A graph G, with $\chi_a(G) = 4$ and $\chi_a(G \times K_2) = 3$.

Question 2.9. Is $\chi_a(G \times K_2) \geq c\,\chi_a(G)$ for some constant $c > 0$?

Question 2.10. Is $\chi_a(G \times H) \geq \min\{\chi_a(G), \chi_a(H)\}$ for every graphs G and H?

3. Direct Products of Complete Graphs

In this section we give the acyclic chromatic number for products of complete graphs.

Theorem 3.1. *The acyclic chromatic number of direct product of complete graphs is*

$$\chi_a(K_m \times K_n) = \begin{cases} n & \textit{if } m = 2, \\ 5 & \textit{if } m = n = 3, \\ 6 & \textit{if } m = 4 \textit{ and } n = 3, \\ 2m - 1 & \textit{if } m > 4 \textit{ and } n = 3, \\ mn - m - 2 & \textit{if } m \geq n \geq 4. \end{cases}$$

Proof. Let $V(K_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(K_n) = \{y_1, y_2, \ldots, y_n\}$.

We first prove that $\chi_a(K_2 \times K_n) = n$. By Lemma 2.6 we have to show that $\chi_a(K_2 \times K_n) \geq n$. Suppose that there is an acyclic coloring f of $K_2 \times K_n$ with less than n colors. Then at least two vertices in a K_n-layer receive the same color. Without loss of generality let $f(x_1, y_1) = f(x_1, y_2) = 1$. Since the vertices $(x_2, y_i), 3 \leq i \leq n$ are adjacent to (x_1, y_1) and (x_1, y_2), we find that they must be colored by pairwise distinct colors, so let $f(x_2, y_i) = i - 1$ for $3 \leq i \leq n$. Since the vertex (x_2, y_2) can not receive the color 1, we may assume that $f(x_2, y_2) = 2$. This forces $f(x_1, y_i) = f(x_2, y_i)$ for $4 \leq i \leq n$. Therefore $f(x_2, y_1) = 2$ and $f(x_1, y_3) = 1$, to provide a proper coloring. The obtained coloring is not acyclic since the first three vertices of both K_n-layers form a bichromatic 6-cycle, a contradiction.

Note that $K_3 \times K_3 = C_3 \Box C_3$. It is known that $\chi_a(C_3 \Box C_3) = 5$ (see [9]).

Next, consider the case when $m = 4$ and $n = 3$. In Table 1 an acyclic coloring of $K_4 \times K_3$ is depicted, thus $\chi_a(K_4 \times K_3) \leq 6$. Suppose that 5 colors suffice. Then one color appears four times or two colors appear three times. The same color can be used only within one layer (otherwise the coloring is not proper). Hence in the case when one color appears four times (in a K_4-layer), all vertices of the other two K_4-layers must receive pairwise distinct colors, hence we need 9 colors in this case. The remaining case is when two colors appear three times. Two K_3-layers, each colored by one color, imply a bichromatic 6-cycle. If there are two K_4-layers with three vertices of the same color we obtain either a bichromatic 4-cycle or a bichromatic 6-cycle. But we obtain a bichromatic 4-cycle also in the last case with a monochromatic K_3-layer and three vertices of the same color in a K_4-layer. Hence we need at least 6 colors to color $K_4 \times K_3$ acyclically and the formula holds in this case.

Table 1. An optimal acyclic coloring of $K_4 \times K_3$.

1	1	2	2
3	4	4	3
5	6	5	6

Next, we show that $\chi_a(K_m \times K_n) = mn - m - 2$ for $m \geq n \geq 4$. For this case an optimal acyclic coloring of $K_m \times K_n$ is depicted in Table 2, where rows correspond to K_m-layers and columns correspond to K_n-layers and vertices denoted by $*$ are colored by pairwise distinct colors, where each of them is colored by a color $> m+2$. We need $mn-m-2$ colors for this coloring.

To prove the optimality of the coloring given in 2 we show that we need at least $mn-m-2$ colors to color $K_m \times K_n$ acyclically.

Table 2. An optimal acyclic coloring of $K_m \times K_n$.

1	3	$*$	$*$	$*$	$*$
1	4	5	6	$\cdots$	$m+2$
2	4	$*$	$*$	$*$	$*$
2	3	5	6	$\cdots$	$m+2$
$*$	$*$	$*$	$*$	$*$	$*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$*$	$*$	$*$	$*$	$*$	$*$

Case 1. If there exists a K_n-layer with at least four vertices of the same color, then all vertices of all other K_n-layers must be colored by pairwise distinct colors (as soon as we color any two vertices of $\bigcup_{i=2}^m K_n^{x_i}$ by the same color, we obtain a bichromatic 4-cycle). Thus we need at least $(m-1)n$ colors in this case.

Case 2. Suppose that a color appears three times in a K_n-layer. Without loss of generality assume that three vertices of $K_n^{x_1}$ are colored by 1.

Case 2.1. If additional two vertices of $K_n^{x_1}$ are colored by the same color ($\neq 1$), then we need at least $m(n-1)$ colors to color the vertices of $\bigcup_{i=2}^m K_n^{x_i}$, since they must be colored by pairwise distinct colors.

Case 2.2. Otherwise the vertices in $K_n^{x_1}$ are colored by colors $1, 2, \ldots, n-2$, where only the color 1 appears three times.

Case 2.2.1. Suppose that none of the colors $1, 2, \ldots, n-2$ appears in $\bigcup_{i=2}^m K_n^{x_i}$. Since three vertices of $K_n^{x_1}$ are colored by 1, we find that no color appears in different K_n-layers of $\bigcup_{i=2}^m K_n^{x_i}$ (otherwise we obtain a bichromatic 4-cycle). Moreover, in every layer $K_n^{x_i}, i \geq 2$, there is at most one color which appears twice and no color appears three times (otherwise we obtain a bichromatic 4 or 6-cycle). Therefore we need at least $(m-1)(n-1)+n-2 = mn-m-1$ colors to color $K_m \times K_n$ acyclically.

Case 2.2.2. Suppose that one of the colors $1, 2, \ldots, n-2$ appears in $K_n^{x_i}$ for some $i \geq 2$. Then assume, without loss of generality, that a vertex of $K_n^{x_2}$ is colored by 2 and therefore vertices of $\bigcup_{i=3}^{m} K_n^{x_i}$ are colored by pairwise distinct colors. Moreover, a color used in $\bigcup_{i=3}^{m} K_n^{x_i}$ can not repeat in $K_n^{x_2}$ (otherwise a bichromatic 4-cycle would exist) and at most one color of $K_n^{x_2}$ appears twice in $K_n^{x_2}$. Hence we need $(m-2)n + n - 1 = mn - n - 1$ colors in this case.

Case 3. Suppose that no color appears three times in a K_n-layer.

Case 3.1. Suppose that at least three colors appear twice in a K_n-layer, without loss of generality assume $K_n^{x_1}$. Then vertices of $\bigcup_{i=2}^{m} K_n^{x_i}$ must be colored by pairwise distinct colors, hence we need at least $(m-1)n$ colors.

Case 3.2. If two colors appear twice in a K_n-layer then there are at most three K_n-layers with two colors repeated twice. If there are three K_n-layers with two colors repeated twice, then we may without loss of generality assume that we have the coloring depicted in Table 3. Observe that as soon as we color any two other vertices with the same color, we obtain a bichromatic 4-cycle. In this case $mn - 6$ colors are needed.

Table 3. The coloring from the Case 3.2.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$*$	$*$	$*$	$*$	$\ldots$
2	3	6	$*$	$\ldots$
2	4	5	$*$	$\ldots$
1	4	6	$*$	$\ldots$
1	3	5	$*$	$\ldots$

Case 3.2.1. Now suppose that two colors appear twice in at most two K_n-layers (and at least in one K_n-layer).

Assume that vertices of $K_n^{x_1}$ are colored by colors $1, 2, \ldots, n-2$, where only the colors 1 and 2 appear twice. Again, there is no color that repeats in two different $K_n^{x_i}$-layers for $i \geq 2$. If none of the colors $1, 2, \ldots, n-2$ appears in $\bigcup_{i=2}^{m} K_n^{x_i}$ then we need at least $2(n-2) + (m-2)(n-1) = mn - m - 2$ colors. If one of the colors $1, 2, \ldots, n-2$ appears in $K_n^{x_i}$ for $i \geq 2$, then we find by analogous arguments as in Case 2.2.2 that $mn - n - 2$ colors are needed.

Case 3.2.2. Finally, assume that in every K_n-layer there is at most one color which appears twice. Since the direct product is commutative and $m \geq n$ we can also assume that in every K_m-layer there is at most one color which appears twice. Clearly, if there are less than three K_n-layers or less than three K_m-layers, such that two vertices in this layer receive the same color, then the coloring uses at least $mn - m - 2$ colors. So assume, that at least three K_m and three K_n-layers have two vertices of the same color. Then one of the situations below will occur; note that these are (up to an isomorphism) the only possible cases.

Case a

$\vdots$	$\vdots$	$\vdots$	
$*$	$*$	$*$	$\dots$
3	3	$*$	$\dots$
2	2	$*$	$\dots$
1	1	$*$	$\dots$

Case b

$\vdots$	$\vdots$	$\vdots$	
$*$	$*$	$*$	$\dots$
3	$*$	3	$\dots$
2	2	$*$	$\dots$
1	1	$*$	$\dots$

Case c

$\vdots$	$\vdots$	$\vdots$	
$*$	$*$	$*$	$\dots$
$*$	3	3	$\dots$
2	$*$	2	$\dots$
1	1	$*$	$\dots$

Case d

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$*$	$*$	$*$	$*$	$\dots$
3	$*$	$*$	3	$\dots$
2	$*$	2	$*$	$\dots$
1	1	$*$	$*$	$\dots$

In *Case a* only the first two K_n-layers admit two vertices of the same color. If the third K_n-layer in *Case b* have two vertices of the same color, then in no other K_n-layer a color appears twice. In *Case c*, the vertices where a color repeats in a K_n-layer are precisely those marked with $*$. But in this case only the first three K_n and the first three K_m-layers have two vertices of the same color, hence $mn - 6 \geq mn - m - 2$ colors are needed. In *Case d*, if the first K_n-layer has two vertices of the same color, then none of the other K_n-layers has. Otherwise second, third and fourth K_n-layer and first three K_m-layers have two vertices of the same color and none of the others. In this case again $mn - 6$ colors are needed.

Table 4. An optimal acyclic coloring of $K_m \times K_3$.

1	2	3	$\dots$	m
1	2	3	$\dots$	m
1	$*$	$*$	$\dots$	$*$

Finally, we show that $\chi_a(K_m \times K_3) = 2m - 1$ for $m > 4$. In Table 4 an acyclic coloring of $K_m \times K_3$ is given. Vertices denoted by $*$ are colored by pairwise distinct colors, where each of them is colored by a color greater than m. Since this coloring uses $2m-1$ colors, $\chi_a(K_m \times K_3) \leq 2m-1$ for $m > 4$. The proof that $\chi_a(K_m \times K_3) \geq 2m-1$ basically follows the case analysis of the previous case and is left to the reader. ■

We mention that $\chi_a(G \times H) \leq mn - m - 2$ for any graphs G and H of order m and n, $m \geq n \geq 4$ and that the coloring given in Table 2 is an acyclic coloring of $G \times H$.

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Received 13 December 2007
Revised 18 February 2008
Accepted 20 February 2008