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# CLIQUE IRREDUCIBILITY OF SOME ITERATIVE CLASSES OF GRAPHS

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#### Abstract

In this paper, two notions, the clique irreducibility and clique vertex irreducibility are discussed. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique vertex irreducible if every clique in Ghas a vertex which does not lie in any other clique of G. It is proved that L(G) is clique irreducible if and only if every triangle in G has a vertex of degree two. The conditions for the iterations of line graph, the Gallai graphs, the anti-Gallai graphs and its iterations to be clique irreducible and clique vertex irreducible are also obtained.

**Keywords:** line graphs, Gallai graphs, anti-Gallai graphs, clique irreducible graphs, clique vertex irreducible graphs.

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#### 1. INTRODUCTION

We consider only finite, simple graphs G = (V, E) with |V| = n and |E| = m.

A clique of a graph G is a maximal complete subgraph of G. A graph G is clique irreducible if every clique in G of size at least two, has an edge which does not lie in any other clique of G and it is clique reducible if it

is not clique irreducible [7]. A graph G is clique vertex irreducible if every clique in G has a vertex which does not lie in any other clique of G and it is clique vertex reducible if it is not clique vertex irreducible.

The line graph of a graph G, denoted by L(G), is a graph whose vertex set corresponds to the edge set of G and any two vertices in L(G) are adjacent if the corresponding edges in G are incident. The iterations of L(G)are recursively defined by  $L^1(G) = L(G)$  and  $L^{n+1}(G) = L(L^n(G))$ , for  $n \ge 1$  [5].

The Gallai graph of a graph G, denoted by  $\Gamma(G)$ , is a graph whose vertex set corresponds to the edge set of G and any two vertices in  $\Gamma(G)$  are adjacent if the corresponding edges in G are incident on a common vertex and they do not lie in a common triangle [4]. The anti-Gallai graph of a graph G, denoted by  $\Delta(G)$ , is a graph whose vertex set corresponds to the edge set of G and any two vertices in  $\Delta(G)$  are adjacent if the corresponding edges lie in a triangle in G [4]. Both the Gallai graph and the anti-Gallai graph are spanning subgraphs of the line graph and their union is the line graph. Though L(G) has a forbidden subgraph characterization, both these do not have the vertex hereditary property and hence cannot be characterized using forbidden subgraphs [4].

In [1], it is proved that there exist infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai and anti-Gallai graphs. The existence of a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be *H*-free for any finite graph *H* is proved. The relationship between the chromatic number, the radius and the diameter of a graph and its Gallai and anti-Gallai graphs are also obtained. In [4], it has been proved that  $\Gamma(G)$  is isomorphic to *G* only for cycles of length greater than three. Also, computing the clique number and the chromatic number of  $\Gamma(G)$  are NP-complete problems.

A graph G is clique-Helly if any family of mutually intersecting cliques has non-empty intersection [6]. It is hereditary clique-Helly if all the induced subgraphs of G are clique-Helly [6]. It is also proved in [6] that a graph G is hereditary clique-Helly, if it does not contain any Hajós' graph as an induced subgraph.

The complement of a graph G is denoted by  $G^c$  and the graph induced by a set of vertices  $v_1, v_2, \ldots, v_n$  is denoted by  $\langle v_1, v_2, \ldots, v_n \rangle$ . A complete graph, a path and a cycle on n vertices are denoted by  $K_n$ ,  $P_n$  and  $C_n$ respectively. The complete bipartite graph is denoted by  $K_{m,n}$ , where mand n are the number of vertices in each of the partition. A vertex of degree one is called a pendant vertex and an edge incident to a pendant vertex is called a pendant edge. A diamond is the graph  $K_4 - \{e\}$ , where e is any edge of  $K_4$ .



Hajós' graphs

In this paper, the graphs G for which L(G) and  $L^2(G)$  are clique vertex irreducible are characterized and it is deduced that  $L^n(G)$  for  $n \ge 3$  is clique vertex irreducible if and only if G is  $K_3, K_{1,3}$  or  $P_k$  where  $k \le n + 3$ . After characterizing the graphs G such that L(G),  $L^2(G)$ ,  $L^3(G)$  and  $L^4(G)$  are clique irreducible, we prove that  $L^n(G)$ ,  $n \ge 5$ , is clique irreducible if and only if it is non-empty and  $L^4(G)$  is clique irreducible. The Gallai graphs which are clique irreducible and clique vertex irreducible are characterized. A forbidden subgraph characterization for clique vertex irreducibility of  $\Gamma(G)$  is obtained. Also, the forbidden subgraphs for the anti-Gallai graphs and all its iterations to be clique irreducible and clique vertex irreducible are ducible are obtained.

All graph theoretic terminology and notations not mentioned here are from [2].

## 2. The Iterations of the Line Graph

**Theorem 1.** Let G be a graph. The line graph L(G) is clique vertex irreducible if and only if G satisfies the following conditions

- (1) Every triangle in G has at least two vertices of degree two,
- (2) Every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two lying in a triangle.

**Proof.** Let G be a graph which satisfies the conditions (1) and (2). The cliques of L(G) are induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three, the edges in G which are incident on a vertex of degree two and which do not lie in a triangle and by the edges in G which lie in a triangle. By (2), the cliques in L(G)

induced by the vertices corresponding to the edges in G which are incident on a vertex, have a vertex which does not lie in any other clique of L(G). By (1), the cliques in L(G) induced by the vertices which correspond to the edges in G which lie in a triangle, have a vertex which does not lie in any other clique of L(G). Therefore, G is clique vertex irreducible.

Conversely, assume that L(G) is a clique vertex irreducible graph. Let  $\langle u_1, u_2, u_3 \rangle$  be a triangle in G. Let  $e_1, e_2, e_3$  be the vertices in L(G) which correspond to the edges  $u_1u_2, u_2u_3, u_3u_1$  in G.  $T = \langle e_1, e_2, e_3 \rangle$  is a clique in L(G). If  $d(u_i) > 2$  for two  $u_i$ s,  $u_1$  and  $u_2$ , then there exist  $v_1$  and  $v_2$  (not necessarily different, but different from  $u_3$ ) such that  $u_i$  is adjacent to  $v_i$  for i = 1, 2. But then, the vertices  $e_1$  and  $e_3$  will be present in the clique induced by the edges incident on the vertex  $u_1$  and the vertices  $e_2$  and  $e_3$  will be present in the clique induced by the edges incident on the vertex  $u_2$ . Therefore, every vertex in T belongs to another clique in L(G) which is a contradiction to the assumption that L(G) is clique vertex irreducible. Hence every triangle in G has at least two vertices of degree two.

Now, let  $u \in V(G)$  and  $N(u) = \{u_1, u_2, \ldots, u_p\}$ , where  $p \ge 2$  and if p = 2 then  $u_1$  is not adjacent to  $u_2$ . Let  $e_i$  be the vertex in L(G) corresponding to the edge  $uu_i$  in G for  $i = 1, 2, \ldots, p$ . Let C be the clique  $\langle e_1, e_2, \ldots, e_p \rangle$  in L(G). If u has no pendant vertex attached to it then every  $u_i$  has a neighbor  $v_i \ne u$  for  $i = 1, 2, \ldots, p$ . The  $v_i$ s are not necessarily pairwise different. Moreover, some  $v_i$  can be equal to some  $u_j$  with  $j \ne i$ , except in the case p = 2. Therefore, for each i, every  $e_i$  in L(G) will be present in another clique, either induced by the edges incident on the vertex  $u_i$  in G or by the edges in a triangle containing u and  $u_i$  in G. But this is a contradiction to the assumption that L(G) is clique vertex irreducible. Hence, every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

**Theorem 2.** Let G be a connected graph. The second iterated line graph  $L^2(G)$  is clique vertex irreducible if and only if G is one of the following graphs.

(i) 
$$K_2$$
 (ii)  $K_3$  (iii)  $P_3$  (iv)  $P_4$  (v)  $P_5$  (vi)  $K_{1,3}$  (vii)

**Proof.** By Theorem 1,  $L^2(G)$  is clique vertex irreducible if and only if

- (1) Every triangle in L(G) has at least two vertices of degree two,
- (2) Every vertex of degree greater than one in L(G) has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

By (2), every non-pendant edge in G must have a pendant edge attached to it on one end vertex and the degree of that end vertex must be two.

Case 1. L(G) has a triangle.

A triangle in L(G) corresponds to a triangle or a  $K_{1,3}$  (need not be induced) in G. Let it correspond to a triangle in G. If any of the vertices of this triangle has a neighbor outside the triangle, then two vertices in the corresponding triangle in L(G) have neighbors outside the triangle, which is a contradiction. Therefore, since G is connected, in this case G must be  $K_3$ .

If the triangle in L(G) corresponds to a  $K_{1,3}$  in G, then two of the edges of this  $K_{1,3}$  cannot have any other edge incident on any of its end vertices. Therefore, G cannot have a vertex of degree greater than three. Moreover, two vertices of  $K_{1,3}$  in G must be pendant vertices. Again, by (2) and since G is connected, we conclude that G is either  $K_{1,3}$  or the graph (vii).

Case 2. L(G) has no triangle.

Since L(G) has no triangle, G cannot have a  $K_3$  or a vertex of degree greater than or equal to 3. Therefore, since G is connected, G must be a path or a cycle of length greater than three. Again, by (2), G cannot be a path of length greater than five or a cycle. Therefore G is  $K_2$ ,  $P_3$ ,  $P_4$  or  $P_5$ .

**Corollary 3.** Let G be a connected graph. The n<sup>th</sup> iterated line graph  $L^n(G)$  is clique vertex irreducible if and only if G is  $K_3, K_{1,3}$  or  $P_k$  where  $n+1 \leq k \leq n+3$ , for  $n \geq 3$ .

**Theorem 4.** The line graph L(G) is clique irreducible if and only if every triangle in G has a vertex of degree two.

**Proof.** Let G be a graph such that every triangle in G has a vertex of degree two. Let C be a clique in L(G).

Case 1. The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three.

An edge of C can be present in another clique of L(G) if and only if the corresponding pair of edges in G lies in a triangle. Thus, if every edge of C lies in another clique of L(G), then G has an induced  $K_p$ , where p is at least four. But, this contradicts the assumption that every triangle in G has a vertex of degree two.

Case 2. The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree two and which do not lie in a triangle.

In this case, C is  $K_2$  which always has an edge of its own.

Case 3. The clique C is induced by the vertices corresponding to the edges which lie in a triangle T in G.

Since T has a vertex v of degree two, the vertices corresponding to the edges which are incident on v induce an edge in C which does not lie in any other clique of L(G). Therefore, G is clique irreducible.

Conversely, assume that G is a clique irreducible graph. Let  $\langle u_1, u_2, u_3 \rangle$  be a triangle in G. Let  $e_1, e_2, e_3$  be the vertices in L(G) which correspond to the edges  $u_1u_2, u_2u_3, u_3u_1$  of G.  $T = \langle e_1, e_2, e_3 \rangle$  is a clique in L(G). If  $d(u_i) > 2$  for each *i*, there exist  $v_1, v_2, v_3$  such that  $u_i$  is adjacent to  $v_i$  for i = 1, 2, 3 ( $v_1, v_2$  and  $v_3$  are not necessarily different, but they are different from  $u_1, u_2$  and  $u_3$ ). Then the edges  $e_1e_2, e_2e_3$  and  $e_3e_1$  of L(G) will be present in the cliques induced by edges which are incident on the vertices  $u_1, u_2$  and  $u_3$  respectively. Therefore, every edge in T is in another clique of L(G), which is a contradiction.

**Theorem 5.** The second iterated line graph  $L^2(G)$  is clique irreducible if and only if G satisfies the following conditions

- (1) Every triangle in G has at least two vertices of degree two,
- (2) Every vertex of degree three has at least one pendant vertex attached to it,
- (3) G has no vertex of degree greater than or equal to four.

**Proof.** Let G be a graph such that  $L^2(G)$  is clique irreducible. By Theorem 4, every triangle in L(G) has a vertex of degree two. Then, we have the following cases.

Case 1. The triangle in L(G) corresponds to a triangle in G.

Let  $\langle u_1, u_2, u_3 \rangle$  be a triangle in G. Let  $e_1, e_2, e_3$  be the vertices in L(G) which correspond to the edges  $u_1u_2, u_2u_3, u_3u_1$  of G. At least one of the vertices of the triangle  $\langle e_1, e_2, e_3 \rangle$  in L(G) must be of degree two. Let  $e_1$  be a vertex of degree two in L(G). Since  $e_2$  and  $e_3$  belong to  $N(e_1)$  in  $L(G), e_1$  has no other neighbors in L(G). Therefore, the corresponding end vertices,  $u_1$  and  $u_2$  in G have no other neighbors. Hence (1) holds.

Case 2. The triangle in L(G) corresponds to a  $K_{1,3}$  (need not be induced) in G.

Let  $e_1, e_2, e_3$  be the vertices in L(G) corresponding to the edges  $uu_1$ ,  $uu_2, uu_3$  in G. At least one of the vertices of the triangle  $\langle e_1, e_2, e_3 \rangle$  in L(G)must be of degree two. Let  $e_1$  be a vertex of degree two in L(G). Vertices  $e_2$  and  $e_3$  belong to  $N(e_1)$  in L(G) and hence  $e_1$  has no other neighbors in L(G). Therefore, the corresponding end vertices, u and  $u_1$  in G have no other neighbors. Since u has no other neighbors (3) holds and since  $u_1$  has no other neighbors (2) holds.

Conversely, assume that G is a graph which satisfies all the three conditions. A triangle in L(G) corresponds to a triangle or a  $K_{1,3}$  (need not be induced) in G. A triangle in L(G) which corresponds to a triangle in G has at least one vertex of degree two by (1). Again, a triangle in L(G) which corresponds to a  $K_{1,3}$  in G has at least one vertex of degree two by (2) and (3). Therefore, every triangle in L(G) has at least one vertex of degree two and by Theorem 4,  $L^2(G)$  is clique irreducible.

**Theorem 6.** Let G be a connected graph. If  $G \neq K_3$  then,  $L^3(G)$  is clique irreducible if and only if G satisfies the following conditions

- (1) G is triangle free,
- (2) G has no vertex of degree greater than or equal to four,
- (3) At least two of the vertices of every  $K_{1,3}$  in G are pendant vertices,
- (4) If uv is an edge in G, then either u or v has degree less than or equal to two.

**Proof.** Let  $L^{3}(G)$  be clique irreducible. By Theorem 5, L(G) satisfies:

- (1') Every triangle in L(G) has at least two vertices of degree 2,
- (2') Every vertex of degree three in L(G) has at least one pendant vertex attached to it,
- (3') L(G) has no vertex of degree greater than or equal to 4.

A triangle in L(G) corresponds to a triangle or a  $K_{1,3}$  (need not be induced) in G. Every triangle in L(G) has at least two vertices of degree two implies that every triangle in G has its three vertices of degree two. i.e., G is a triangle, because G is connected. Since  $G \neq K_3$ , G must be triangle free. Also, every  $K_{1,3}$  in G has at least two pendant vertices and the degree of a vertex cannot exceed three. Therefore (1), (2) and (3) hold. Again (3') implies that no edge in G can have more than three edges incident on its end vertices. Therefore, (4) holds.

Conversely, assume that the given conditions hold. Since G is triangle free, a triangle in L(G) corresponds to a  $K_{1,3}$  (need not be induced) in G. Therefore, by (2) and (3) every triangle in L(G) has at least two vertices of degree two.

Let e be a vertex of degree three in L(G) and let uv be the corresponding edge in G. Since e is of degree three in L(G), the number of edges incident on u in G together with the number of edges incident on v in Gis three. If u (or v) has three more edges incident on it then u (or v) will be of degree at least four which is a contradiction to the condition (2). Therefore, u has two neighbors and v has one neighbor (or vice versa) in G. Let  $u_1$  and  $u_2$  be the neighbors of u, and let  $v_1$  be the neighbor of vin G. Then  $\langle u, v, u_1, u_2 \rangle = K_{1,3}$  in G and hence at least two of  $v, u_1$  and  $u_2$  must be pendant vertices. Since v is not a pendant vertex,  $u_1$  and  $u_2$ must be pendant vertices. Therefore, e has two pendant vertices attached to it in L(G) corresponding to the edges  $uu_1$  and  $uu_2$  in G. Hence (2') is satisfied.

Again, (2), (3) and (4) together imply (3'). Since the conditions (1'), (2') and (3') are satisfied, by Theorem 5,  $L^3(G)$  is clique irreducible.

**Theorem 7.** Let G be a connected graph. The fourth iterated line graph  $L^4(G)$  is clique irreducible if and only if G is  $K_3, K_{1,3}, P_n$  with  $n \ge 5$  or  $C_n$  with  $n \ge 4$ .

**Proof.** Let  $L^4(G)$  be clique irreducible. Then by Theorem 6, if  $L(G) \neq K_3$  then L(G) must be triangle free. If  $L(G) = K_3$  then G is either  $K_3$  or  $K_{1,3}$ . If L(G) is triangle free then G is triangle free and cannot have vertices of degree greater than or equal to three. Therefore, G is either a path or a cycle of length greater than three.

Conversely, if G is  $K_3, K_{1,3}, P_n$  or  $C_n$  then  $L^4(G)$  is either a triangle, a path or a cycle and all of them are clique irreducible.

**Corollary 8.** For  $n \ge 5$ ,  $L^n(G)$  is clique irreducible if and only if it is non-empty and  $L^4(G)$  is clique irreducible.

## 3. The Gallai Graphs

**Theorem 9.** The Gallai graph  $\Gamma(G)$  is clique vertex irreducible if and only if for every  $v \in V(G)$ , every maximal independent set I in N(v) with  $|I| \ge 2$ contains a vertex u such that  $N(u) - \{v\} = N(v) - I$ .

**Proof.** Let G be a graph such that its Gallai graph  $\Gamma(G)$  is clique vertex irreducible. A clique C in  $\Gamma(G)$  of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex  $v \in V(G)$  whose other end vertices form a maximal independent set I of size at least two in N(v). Let  $I = \{v_1, v_2, \ldots, v_p\}$ , where  $p \ge 2$ , be a maximal independent set in N(v). Let  $e_i$  be the vertex in  $\Gamma(G)$  corresponding to the edge  $vv_i$  in G for  $i = 1, 2, \ldots, p$ . Let C be the clique  $\langle e_1, e_2, \ldots, e_p \rangle$  in  $\Gamma(G)$ . Let  $e_i$  be the vertex in C which does not belong to any other clique in G. Therefore,  $e_i$  has no neighbors in  $\Gamma(G)$  other than those in C. Hence,  $N(v_i) - \{v\} = N(v) - I$ .

Conversely, assume that for every  $v \in V(G)$ , every maximal independent set  $I = \{v_1, v_2, \ldots, v_p\}$  in N(v) contains a vertex u such that  $N(u) - \{v\} = N(v) - I$ . If C is a clique of size one, it contains a vertex of its own. Otherwise, let C be defined as above. By our assumption, there exists a vertex  $u = v_i$  such that  $N(u) - \{v\} = N(v) - I$ . Therefore,  $e_i$  has no neighbors outside C. Hence C has a vertex  $e_i$  of its own.

**Theorem 10.** If  $\Gamma(G)$  is clique vertex reducible, then G contains one of the graphs in Figure 1 as an induced subgraph.

**Proof.** Let G be a graph such that  $\Gamma(G)$  is clique vertex reducible and let C be a clique in  $\Gamma(G)$  such that each vertex of C belongs to some other clique in  $\Gamma(G)$ . Consider the order relation  $\preceq$  among the vertices of C where  $e \preceq e'$  if  $N[e] \preceq N[e']$ . If  $\preceq$  is a total ordering, then every vertex adjacent to the minimum vertex e is also adjacent to all the vertices in C. Therefore, by maximality of C, e cannot have neighbors outside C. This is a contradiction to the assumption that e belongs to some other clique of  $\Gamma(G)$ . So, there exist two vertices  $e_1$  and  $e_2$  in C which are not comparable. That is, there exist vertices  $f_1$  and  $f_2$  of  $\Gamma(G)$  such that  $e_i$  is adjacent to  $f_j$  if and only if i = j. Let  $vv_1$  and  $vv_2$  be the edges corresponding to  $e_1$  and  $e_2$ , respectively. Then  $v_1$  and  $v_2$  are non-adjacent. Let  $u_1$  and  $u_2$  be the end points of  $f_1$  and  $f_2$ , respectively, which are both different from v,  $v_1$  and  $v_2$ .



Figure 1

Case 1. Both  $f_1$  and  $f_2$  correspond to the edges incident to v. In this case,  $u_1$  and  $u_2$  are adjacent to v,  $u_i$  is adjacent to  $v_j$  if and only if  $i \neq j$  and  $u_1$  and  $u_2$  can be either adjacent or not. Therefore  $\langle v, v_1, v_2, u_1, u_2 \rangle$  is the graph (i) or (ii) in Figure 1.

Case 2. None of  $f_1$  and  $f_2$  correspond to the edges incident to v. In this case,  $u_1$  and  $u_2$  are adjacent to  $v_1$  and  $v_2$ , respectively, and not to v. If  $u_1 = u_2$  then G contains an induced  $C_4$ . If  $u_1 \neq u_2$  and G does not contain an induced  $C_4$ , then  $\langle v, v_1, v_2, u_1, u_2 \rangle$  is either  $P_5$  or  $C_5$ .

Case 3. Exactly one of  $f_1$  and  $f_2$  correspond to the edges incident to v, say  $f_1$ .

In this case,  $u_1$  is adjacent to both v and  $v_2$  and is not adjacent to  $v_1$ . The vertex  $u_2$  is adjacent to  $v_2$  and is not adjacent to v. If  $u_2$  is adjacent to  $v_1$  then G contains an induced  $C_4$ . Otherwise,  $\langle v, v_1, v_2, u_1, u_2 \rangle$  is the graph (vi) or (vii) in Figure 1.

**Theorem 11.** The Gallai graph  $\Gamma(G)$  is clique irreducible if and only if for every  $v \in V(G)$ ,  $\langle N(v) \rangle^c$  is clique irreducible.

**Proof.** A clique C in  $\Gamma(G)$  of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex  $v \in V(G)$  whose other end vertices form a maximal independent set I of size

at least two in N(v). Therefore, C has an edge which does not belong to any other clique of  $\Gamma(G)$  if and only if I has a pair of vertices both of which together does not belong to any other maximal independent set in N(v). But, this happens if and only if every clique of size at least two in  $\langle N(v) \rangle^c$  has an edge which does not belong to any other clique in  $\langle N(v) \rangle^c$ , since a maximal independent set in a graph corresponds to a clique in its complement.

**Theorem 12.** The second iterated Gallai graph  $\Gamma^2(G)$  is clique irreducible if and only if for every  $uv \in E(G)$ , either  $\langle N(u) - N(v) \rangle$  and  $\langle N(v) - N(u) \rangle$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible.

**Proof.** By Theorem 11,  $\Gamma^2(G)$  is clique irreducible if and only if for every  $e \in V(\Gamma(G)), \langle N(e) \rangle^c$  is clique irreducible.

Let  $e = uv \in E(G)$ ,  $N(u) - N(v) = \{u_1, u_2, \dots, u_p\}$  and  $N(v) - N(u) = \{v_1, v_2, \dots, v_l\}$ . Also let  $e_i = uu_i$  for  $i = 1, 2, \dots, p$  and  $f_j = vv_j$  for  $j = 1, 2, \dots, l$ .  $N_{\Gamma(G)}(e) = \{e_1, e_2, \dots, e_p, f_1, f_2, \dots, f_l\}$ .  $\langle N(e) \rangle^c$  is clique irreducible if and only if every maximal independent set I in  $\langle N(e) \rangle$  has a pair of vertices of its own.  $e_i$  is not adjacent to  $e_j$  if and only if  $u_i$  is adjacent to  $u_j$ . Similarly,  $f_i$  is not adjacent to  $f_j$  if and only if  $v_i$  is adjacent to  $v_j$ . So,  $I = \{e_{i_1}, e_{i_2}, \dots, e_{i_k}, f_{j_1}, f_{j_2}, \dots, f_{j_l}\}$  if and only if  $\{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$  is a clique in  $\langle N(u) - N(v) \rangle$  and  $\{v_{j_1}, v_{j_2}, \dots, v_{j_l}\}$  is a clique in N(v) - N(u). Therefore, every maximal independent set I in  $N_{\Gamma(G)}(e)$  has a pair of vertices of its own if and only if either both  $\langle N(u) - N(v) \rangle$  and  $\langle N(v) - N(u) \rangle$  are clique vertex irreducible or one among them is a clique and the other is clique irreducible.

**Theorem** [6]. If G is hereditary clique-Helly, then it is clique irreducible.

**Theorem 13.** If  $\Gamma(G)$  is clique reducible then G contains one of the graphs in Figure 2 as an induced subgraph.

**Proof.** Let  $\Gamma(G)$  be a clique reducible graph. By Theorem [6],  $\Gamma(G)$  contains at least one of the Hajós' graph as an induced subgraph. The Hajós' graphs is an induced subgraph of  $\Gamma(G)$  if and only if G contains one of the graphs in Figure 2 as an induced subgraph. Hence the theorem.



Note. The converse is not necessarily true.

Figure 2

Let G be the graph in Figure 3.  $V(G) = \{v, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5, w_7, w_7, w_8\}$ . Let  $\langle v, v_1, v_2, v_3, u_1, u_2, u_3 \rangle$  be the graph (i) in Figure 2 and let  $w_i$ s for  $i = 1, 2, \ldots, 8$  induce a complete graph. Also, let  $w_1$  be adjacent to  $\{v_1, v_2, v_3\}$ ,  $w_2$  be adjacent to  $\{v_1, v_2, u_3\}$ ,  $w_3$  be adjacent to  $\{v_1, u_2, v_3\}$ ,  $w_4$  be adjacent to  $\{v_1, u_2, u_3\}$ ,  $w_5$  be adjacent to  $\{u_1, v_2, v_3\}$ ,  $w_6$  be adjacent to  $\{u_1, v_2, u_3\}$ ,  $w_7$  be adjacent to  $\{u_1, u_2, v_3\}$ ,  $w_8$  be adjacent to  $\{u_1, u_2, u_3\}$ , and v adjacent to  $w_i$  for  $i = 1, 2, \ldots, 8$ .



Figure 3

In  $\Gamma(G)$  the vertices corresponding to the edges with one end vertex v induces  $K_6$  minus a perfect matching in which the vertices of each of the eight triangles are adjacent to another vertex each. The remaining vertices induce the graph  $H = 4K_{1,8}$ . Therefore,  $\Gamma(G)$  is clique irreducible.

## 4. The Iterations of the Anti-Gallai Graphs

**Theorem 14.** The anti-Gallai graph  $\Delta(G)$  is clique vertex irreducible if and only if G neither contains  $K_4$  nor one of the Hajós' graphs as an induced subgraph.

**Proof.** Let G be a graph which does neither contain  $K_4$  nor one of the Hajós' graphs as an induced subgraph. The cliques of  $\Delta(G)$  are induced by the vertices corresponding to the edges of G incident on a vertex of degree at least 3 whose other end vertices induce a complete graph and by the vertices corresponding to the edges which lie in a triangle. In the first case G contains an induced  $K_4$ , which is a contradiction. Therefore, the cliques of  $\Delta(G)$  are induced by the edges which lie in a triangle. Let  $\langle u_1, u_2, u_3 \rangle$  be a triangle in G. Let  $e_1, e_2, e_3$  be the vertices in  $\Delta(G)$  corresponding to the edges  $u_1u_2, u_2u_3, u_3u_1$  in G. Then  $\langle e_1, e_2, e_3 \rangle$  is a clique in  $\Delta(G)$ . If a vertex  $e_i$  for i = 1, 2, 3 lies in another clique of  $\Delta(G)$ , then the edge corresponding to  $e_i$  in G has a neighbor  $v_i$  for i = 1, 2, 3.  $v_i \neq v_j$  if  $i \neq j$  and  $v_1, v_2, v_3$  are not adjacent to  $u_3, u_1, u_2$ , respectively, since otherwise G contains a  $K_4$ , which is a contradiction. Then,  $\langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle$  is one of the Hajós' graphs, a contradiction. Hence, G is clique vertex irreducible.

Conversely, assume that G is clique vertex irreducible. If G contains  $K_4$  or one of the Hajós' graphs as an induced subgraph, then there exists a clique in  $\Delta(G)$ , corresponding to a triangle in G, which shares each of its vertices with some other clique of  $\Delta(G)$ .

**Lemma 1.** If G is  $K_4$ -free then  $\Gamma(G)$  is diamond free.

**Proof.** Let G be a graph which does not contain  $K_4$  as an induced subgraph. Therefore, a triangle in  $\Delta(G)$  can only be induced by a triangle in G. If two vertices of the triangle in  $\Delta(G)$  have a common neighbor, then it forces G to have a  $K_4$ , a contradiction. Therefore,  $\Delta(G)$  is diamond free. **Theorem 15.** The second iterated anti-Gallai graph  $\Delta^2(G)$  is clique vertex irreducible if and only if G does not contain  $K_4$  as an induced subgraph.

**Proof.** By Theorem 14,  $\Delta^2(G)$  is clique vertex irreducible if and only if  $\Delta(G)$  does neither contain  $K_4$  nor one of the Hajós' graphs as an induced subgraph.

Let G be a graph which does not contain  $K_4$  as an induced subgraph. Therefore, G does not contain  $K_5$  as an induced subgraph and hence  $\Delta(G)$  does not contain  $K_4$  as an induced subgraph. Again, by Lemma 1,  $\Delta(G)$  cannot have diamond as an induced subgraph and hence it does not contain any of the Hajós' graph as an induced subgraph. Hence,  $\Delta^2(G)$  is clique vertex irreducible.

Conversely, assume that  $\Delta^2(G)$  is clique vertex irreducible. If G contains  $K_4$  as an induced subgraph then in  $\Delta(G)$  the vertices corresponding to the edges of this  $K_4$  induce  $K_6$  minus a perfect matching which is the fourth Hajós' graph, a contradiction. Therefore, G does not contain  $K_4$  as an induced subgraph.

**Theorem 16.** The  $n^{th}$  iterated anti-Gallai graph  $\Delta^n(G)$  is clique vertex irreducible if and only if G does not contain  $K_{n+2}$  as an induced subgraph.

**Proof.** By Theorem 15,  $\Delta^n(G)$  is clique vertex irreducible if and only if  $\Delta^{n-2}(G)$  does not contain  $K_4$  as an induced subgraph.  $\Delta^{n-2}(G)$  does not contain  $K_4$  as an induced subgraph if and only if  $\Delta^{n-3}(G)$  does not contain  $K_5$  as an induced subgraph. Proceeding like this, we get that  $\Delta(G)$  does not contain  $K_{n+1}$  as an induced subgraph if and only if G does not contain  $K_{n+2}$  as an induced subgraph. Therefore,  $\Delta^n(G)$  is clique vertex irreducible if and only if G does not contain  $K_{n+2}$  as an induced subgraph.

**Theorem** [3]. If a graph G has no induced diamond, then every edge of G belongs to exactly one clique.

**Theorem 17.** The anti-Gallai graph  $\Delta(G)$  is clique irreducible if and only if G does not contain  $K_4$  as an induced subgraph.

**Proof.** Let G be a graph which does not contain  $K_4$  as an induced subgraph. By Lemma 1 and Theorem [3],  $\Delta(G)$  is clique irreducible.

Conversely, if G contains a  $K_4 = \langle u_1, u_2, u_3, u_4 \rangle$ , then it follows that the clique in  $\Delta(G)$ , corresponding to the triangle  $\langle u_1, u_2, u_3 \rangle$  in G, shares each

of its edges with some other clique. Therefore, if  $\Delta(G)$  is clique irreducible, then G cannot have  $K_4$  as an induced subgraph.

**Theorem 18.** The  $n^{th}$  iterated anti-Galli graph  $\Delta^n(G)$  is clique irreducible if and only if G does not contain an induced  $K_{n+3}$ .

**Proof.** By Theorem 17,  $\Delta^n(G)$  is clique irreducible if and only if  $\Delta^{n-1}(G)$  does not contain an induced  $K_4$ .  $\Delta^{n-1}(G)$  does not contain an induced  $K_4$  if and only if  $\Delta^{n-2}(G)$  does not contain an induced  $K_5$ . Proceeding like this, we get,  $\Delta(G)$  does not contain an induced  $K_{n+2}$  if and only if G does not contain an induced  $K_{n+2}$  if and only if G does not contain an induced  $K_{n+3}$ . Therefore,  $\Delta^n(G)$  is clique irreducible if and only if G does not contain an induced  $K_{n+3}$ .

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