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MONOCHROMATIC PATHS AND QUASI-MONOCHROMATIC CYCLES IN EDGE-COLOURED BIPARTITE TOURNAMENTS

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Abstract

We call the digraph D an m-coloured digraph if the arcs of D are coloured with m colours. A directed path (or a directed cycle) is called *monochromatic* if all of its arcs are coloured alike. A directed cycle is called *quasi-monochromatic* if with at most one exception all of its arcs are coloured alike.

A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

(i) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and

(ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

In this paper it is proved that if D is an m-coloured bipartite tournament such that: every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic, and D has no induced particular 6-element bipartite tournament $\widetilde{T}_6$, then D has a kernel by monochromatic paths.

Keywords: kernel, kernel by monochromatic paths, bipartite tournament.

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1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph, and let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of D, respectively. An arc $(u_1, u_2) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D (resp. symmetrical part of D) which is denoted by $\mathrm{Asym}(D)$ (resp. $\mathrm{Sym}(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. If S is a nonempty subset of $V(D)$ then the subdigraph $D[S]$ induced by S is the digraph having vertex set S, and whose arcs are those arcs of D joining vertices of S.

A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D, that is an arc from z to some vertex in N. A digraph D is called *kernel-prefect digraph* when every induced subdigraph of D has a kernel. Sufficient conditions for the existence of kernels in a digraph have been investigated by several authors, Von Neumann and Morgenstern [14], Richardson [11], Duchet and Meyniel [3] and Galeana-Sánchez and Neumann-Lara [4]. The concept of kernel is very useful in applications. Clearly, the concept of kernel by monochromatic paths generalizes those of kernel.

A digraph D is called *a bipartite tournament* if its set of vertices can be partitioned into two sets V_1 and V_2 such that: (i) every arc of D has an endpoint in V_1 and the other endpoint in V_2, and (ii) for all $x_1 \in V_1$ and for all $x_2 \in V_2$, we have $|\{(x_1, x_2), (x_2, x_1)\} \cap A(D)| = 1$. We will write $D = (V_1, V_2)$ to indicate the partition.

If $\mathcal{C} = (z_0, z_1, \ldots, z_n, z_0)$ is a directed cycle and if $z_i, z_j \in V(\mathcal{C})$ with $i \leq j$ we denote by $(z_i, \mathcal{C}, z_j)$ the $z_i z_j$-directed path contained in $\mathcal{C}$, and $\ell(z_i, \mathcal{C}, z_j)$ will denote its length; similarly $\ell(\mathcal{C})$ will denote the length of $\mathcal{C}$.

If D is an m-coloured digraph, then the closure of D, denoted by $\mathcal{C}(D)$ is the m-coloured multidigraph defined as follows: $V(\mathcal{C}(D)) = V(D)$, $A(\mathcal{C}(D)) = A(D) \cup \{(u, v)$ with colour $i \mid$ there exists an uv-monochromatic directed path coloured i contained in $D\}$.

Notice that for any digraph D, $\mathcal{C}(\mathcal{C}(D)) \cong \mathcal{C}(D)$ and D has a kernel by monochromatic paths if and only if $\mathcal{C}(D)$ has a kernel.

In [13] Sands *et al.* have proved that any 2-coloured digraph has a kernel by monochromatic paths; in particular they proved that any 2-coloured tournament has a kernel by monochromatic paths. They also raised the following problem: Let T be a 3-coloured tournament such that every directed cycle of length 3 is quasi-monochromatic; must $\mathcal{C}(T)$ have a kernel? (This question remains open.) In [12] Shen Minggang proved that if in the problem we ask that every transitive tournament of order 3 be quasi-monochromatic, the answer will be yes; and the result is best possible for m-coloured tournaments with $m \geq 5$. In 2004 [9] presented a 4-coloured tournament T such that every directed cycle of order 3 is quasi-monochromatic; but T has no kernel by monochromatic paths. The known sufficient conditions for the existence of kernel by monochromatic paths in m-coloured ($m \geq 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of certain subdigraphs. In [5] it was proved that if T is an m-coloured tournament such that every directed cycle of length at most 4 is quasi-monochromatic then $\mathcal{C}(T)$ is kernel-perfect. A generalization of this result was obtained by Hahn, Ille and Woodrow in [10]; they proved that if T is an m-coloured tournament such that every directed cycle of length k is quasi-monochromatic and T has no polychromatic directed cycles of length ℓ, $\ell < k$, for some $k \geq 4$, then T has a kernel by monochromatic paths. (A directed cycle is polychromatic if it uses at least three different colours in its arcs). Results similar to those in [12] and [5] were proved for the digraph obtained from a tournament by the deletion of a single arc, in [7] and [6], respectively. Kernels by monochromatic paths in bipartite tournaments were studied in [8]; where it is proved that if T is a bipartite tournament such that every directed cycle of length 4 is monochromatic, then T has a kernel by monochromatic paths.

We prove that if T is a bipartite tournament such that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic and T has no induced subtournament isomorphic to $\widetilde{T}_6$, then T has a kernel by monochromatic paths.

$\widetilde{T}_6$ is the bipartite tournament defined as follows:

$V(\widetilde{T}_6) = \{u, v, w, x, y, z\}$,

$A(\widetilde{T}_6) = \{(u,w), (v,w), (w,x), (w,z), (x,y), (y,u), (y,v), (z,y)\}$ with $\{(u,w), (w,x), (y,u), (z,y)\}$ coloured 1 and $\{(v,w), (w,z), (x,y), (y,v)\}$ coloured 2. (See Figure 1).

We will need the following result.

Theorem 1.1 Duchet [2]**.** *If D is a digraph such that every directed cycle has at least one symmetrical arc, then D is a kernel-perfect digraph.*

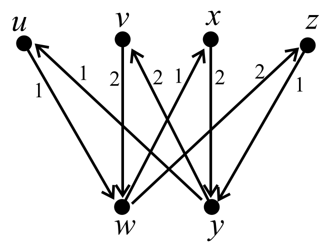

Figure 1

2. The Main Result

The following lemmas will be useful in the proof of the main result.

Lemma 2.1. *Let $D = (V_1, V_2)$ be a bipartite tournament and $C = (u_0, u_1, \ldots, u_n)$ a directed walk in D. For $\{i, j\} \subseteq \{0, 1, \ldots, n\}$, $(u_i, u_j) \in A(D)$ or $(u_j, u_i) \in A(D)$ if and only if $j - i \equiv 1 (\mathrm{mod}\, 2)$.*

Lemma 2.2. *For a bipartite tournament $D = (V_1, V_2)$, every closed directed walk of length at most 6 in D is a directed cycle of D.*

Lemma 2.3. *Let D be an m-coloured bipartite tournament such that every directed cycle of length 4 is quasi-monochromatic and every directed cycle of length 6 is monochromatic. If for $u, v \in V(D)$ there exists a uv-monochromatic directed path and there is no vu-monochromatic directed path (in D), then at least one of the following conditions hold:*

(i) $(u, v) \in A(D)$,

(ii) *there exists (in D) a uv-directed path of length 2,*

(iii) *there exists a uv-monochromatic directed path of length 4.*

Proof. Let D, u, v be as in the hypothesis. If there exists a uv-directed path of odd length, then it follows from Lemma 2.1 that $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Since there is no vu-monochromatic directed path in D, then $(u, v) \in A(D)$ and Lemma 2.3 holds. So, we will assume that every uv-directed path has even length. We proceed by induction on the length of a uv-monochromatic directed path.

Clearly Lemma 2.3 holds when there exists a uv-monochromatic directed path of length at most 4. Suppose that $T = (u = u_0, u_1, u_2, u_3, u_4, u_5,$

$u_6 = v)$ is a uv-monochromatic directed path of length 6. It follows from Lemma 2.1 that $(u, u_5) \in A(D)$ or $(u_5, u) \in A(D)$ and also $(u_1, v) \in A(D)$ or $(v, u_1) \in A(D)$. If $(u, u_5) \in A(D)$ or $(u_1, v) \in A(D)$ then we obtain a uv-directed path of length two, and we are done. So, we will assume that $(u_5, u) \in A(D)$ and $(v, u_1) \in A(D)$. Thus $(u = u_0, u_1, u_2, u_3, u_4, u_5, u_0 = u)$ is a directed cycle of length 6 which is monochromatic and has the same colour as T. Also $(u_1, u_2, u_3, u_4, u_5, u_6 = v, u_1)$ is a directed cycle coloured as T. Hence $(v = u_6, u_1, u_2, u_3, u_4, u_5, u_0 = u)$ is a vu-monochromatic directed path, a contradiction. Suppose that Lemma 2.3 holds when there exists a uv-monochromatic directed path of even length ℓ with $6 \leq \ell \leq 2n$. Now assume that there exists a uv-monochromatic directed path say $T = (u = u_0, u_1, \ldots, u_{2(n+1)} = v)$ with $\ell(T) = 2(n+1)$; we may assume w.l.o.g. that T is coloured 1.

From Lemma 2.1 we have that for each $i \in \{0, 1, \ldots, 2(n+1) - 5\}$, $(u_{i+5}, u_i) \in A(D)$ or $(u_i, u_{i+5}) \in A(D)$. We will analyze two possible cases:

Case a. For each $i \in \{0, 1, \ldots, 2(n+1) - 5\}$, $(u_{i+5}, u_i) \in A(D)$.
In this case $C_6 = (u_i,\ u_{i+1},\ u_{i+2},\ u_{i+3},\ u_{i+4},\ u_{i+5},\ u_i)$ is a directed cycle with $\ell(C_6) = 6$; so it is monochromatic and coloured 1 (as (u_i, u_{i+1}) is coloured 1). Let $k \in \{1, 2, 3, 4, 5\}$ such that $k \equiv 2(n+1) \pmod 5$, then $(v = u_{2(n+1)}, u_{2(n+1)-5}, u_{2(n+1)-10}, \ldots,\ u_k) \cup (u_k, T, u_5) \cup (u_5, u_0)$ is a vu-monochromatic directed path in D, a contradiction.

Case b. For some $i \in \{0, 1, \ldots, 2(n+1) - 5\}$, $(u_i, u_{i+5}) \in A(D)$.
Notice that from Lemma 2.1, there exists an arc between u_1 and $u_{2(n+1)}$ and also there exists an arc between u_0 and u_{2n+1}. If $(u_1, u_{2(n+1)}) \in A(D)$ or $(u_0, u_{2n+1}) \in A(D)$, then we obtain a uv-directed path of length two, and we are done. So, we will assume that $(u_{2(n+1)}, u_1) \in A(D)$ and $(u_{2n+1}, u_0) \in A(D)$. Observe that: If for some $i \in \{1, \ldots, 2(n+1) - 5\}$, $(u_{2(n+1)}, u_i) \in A(D)$ and the arcs $(u_{2(n+1)}, u_i)$ and (u_{2n+1}, u_0) are coloured 1, then $(v = u_{2(n+1)}, u_i) \cup (u_i, T, u_{2n+1}) \cup (u_{2n+1}, u_0)$ is a vu-directed path coloured 1, contradicting the hypothesis. Hence we have:

(a) If for some $i \in \{1, 2, \ldots, 2(n+1) - 5\}$ we have $(u_{2(n+1)}, u_i) \in A(D)$, then $(u_{2(n+1)}, u_i)$ is not coloured 1 or (u_{2n+1}, u_0) is not coloured 1.

Case b.1. (u_{2n+1}, u_0) is not coloured 1.
Recall that for some $i \in \{0, 1, \ldots, 2(n+1) - 5\}$, $(u_i, u_{i+5}) \in A(D)$. Let $\{i_0, j_0\} \subseteq \{0, 1, \ldots, 2(n+1)\}$ be such that $j_0 - i_0 = \max\{j - i \,|\, \{i, j\} \subseteq$

$\{0, 1, \ldots, 2(n+1)\}$ and $(u_i, u_j) \in A(D)\}$; clearly $j_0 - i_0 \geq 5$. Now we will analyze several possibilities:

Case b.1.1. $i_0 \geq 2$ and $j_0 \leq 2n$.
Since $(u_{i_0}, u_{j_0}) \in A(D)$, it follows from Lemma 2.1 that $j_0 - i_0 \equiv 1 (\mathrm{mod}\, 2)$ so $(j_0+2)-(i_0-2) \equiv 1(\mathrm{mod}\, 2)$ and $((u_{i_0-2}, u_{j_0+2}) \in A(D)$ or $(u_{j_0+2}, u_{i_0-2}) \in A(D))$; the selection of $\{i_0, j_0\}$ implies $(u_{j_0+2},\ u_{i_0-2}) \in A(D)$. Thus $(u_{i_0-2},\ u_{i_0-1},\ u_{i_0},\ u_{j_0},\ u_{j_0+1},\ u_{j_0+2},\ u_{i_0-2})$ is a directed cycle of length 6 and hence monochromatically coloured 1. Now $(u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, u_{2(n+1)} = v)$ is a uv-monochromatic directed path whose length is less than $\ell(T)$. Then the assertion of Lemma 2.3 follows from the inductive hypothesis.

Case b.1.2. $i_0 = 0$.
When $j_0 \leq 2n-3$; it follows from Lemma 2.1 and the choice of $\{i_0, j_0\}$ that $(u_{j_0+4}, u_{i_0} = u_0) \in A(D)$. Thus $(u_0 = u_{i_0}, u_{j_0}, u_{j_0+1}, u_{j_0+2}, u_{j_0+3}, u_{j_0+4}, u_0)$ is a monochromatic directed cycle (it has length 6), coloured 1. Hence $(u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, v)$ is a uv-monochromatic directed path whose length is less than $\ell(T)$ and the assertion follows from the inductive hypothesis. When $j_0 \geq 2n-1$, we have $j_0 = 2n-1$ (recall $(u_{2n+1}, u_0) \in A(D)$, $j_0 - i_0 \equiv 1(\mathrm{mod}\, 2)$, $i_0 = 0$). So, $(u_0 = u_{i_0}, u_{j_0} = u_{2n-1}, u_{2n}, u_{2n+1}, u_0)$ is a directed cycle of length 4 which by hypothesis is quasi-monochromatic. Since (u_{2n+1}, u_0) is not coloured 1, then (u_{i_0}, u_{j_0}) is coloured 1, and $(u = u_{i_0}, u_{j_0} = u_{2n-1}, u_{2n}, u_{2n+1}, u_{2n+2} = v)$ is a uv-monochromatic directed path coloured 1 of length 4.

Case b.1.3. $i_0 = 1$.
When $j_0 \leq 2n-2$, we have $(u_{j_0+4}, u_{i_0} = u_1) \in A(D)$ (Lemma 2.1 and the choice of $\{i_0, j_0\}$). Thus $(u_1 = u_{i_0}, u_{j_0}, u_{j_0+1}, u_{j_0+2}, u_{j_0+3}, u_{j_0+4}, u_{i_0})$ is a directed cycle of length 6 (monochromatic and coloured 1). Hence $(u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, v)$ is a uv-monochromatic directed path whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

When $j_0 \geq 2n$, we have $j_0 = 2n$ (as $j_0 - i_0 \equiv 1(\mathrm{mod}\, 2)$, $i_0 = 1$ and $(u_{2n+2}, u_1) \in A(D)$). Hence $(u_1 = u_{i_0}, u_{j_0} = u_{2n}, u_{2n+1}, u_0, u_1)$ is a directed cycle of length 4, from the hypothesis it is quasi-monochromatic and (u_{2n+1}, u_0) is not coloured 1, so (u_{i_0}, u_{j_0}) is coloured 1. Therefore $(u_0, u_1 = u_{i_0}, u_{j_0} = u_{2n}, u_{2n+1}, u_{2n+2} = v)$ is a uv-monochromatic directed path, coloured 1, of length 4.

Case b.1.4. $j_0 = 2n + 1$.
When $i_0 \geq 4$, we have $(u_{2n+1} = u_{j_0}, u_{i_0-4}) \in A(D)$ ($j_0 - i_0 \equiv 1 (\bmod 2)$, $j_0 - (i_0 - 4) \equiv 1 \pmod 2$, and the choice of $\{i_0, j_0\}$). Therefore $(u_{i_0}, u_{j_0} = u_{2n+1}, u_{i_0-4}, u_{i_0-3}, u_{i_0-2}, u_{i_0-1}, u_{i_0})$ is a directed cycle of length 6 (and thus it is monochromatic) coloured 1. Thus $(u = u_0, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, v)$ is a uv-directed path coloured 1, whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

When $i_0 \leq 2$, we have $i_0 = 2$ (as $j_0 - i_0 \equiv 1 (\bmod 2)$, $j_0 = 2n + 1$, and $(u_{2n+1}, u_0) \in A(D)$). Hence $(u_2 = u_{i_0}, u_{j_0} = u_{2n+1}, u_0, u_1, u_2)$ is quasi-monochromatic (as it has length 4). Since (u_{2n+1}, u_0) is not coloured 1, it follows that (u_{i_0}, u_{j_0}) is coloured 1. We conclude that $(u_0, u_1, u_2 = u_{i_0}, u_{j_0} = u_{2n+1}, u_{2n+2} = v)$ is a uv-directed path coloured 1 of length 4.

Case b.1.5. $j_0 = 2n + 2$.
When $i_0 \geq 5$, we have $(u_{2n+2} = u_{j_0}, u_{i_0-4}) \in A(D)$ (arguing as in b.1.4). Thus $(u_{i_0}, u_{j_0}, u_{i_0-4}, u_{i_0-3}, u_{i_0-2}, u_{i_0-1}, u_{i_0})$ is monochromatic (as it is a directed cycle of length 6). Hence $(u, T, u_{i_0}) \cup (u_{i_0}, u_{j_0}) \cup (u_{j_0}, T, v)$ is a uv-monochromatic directed path with length less than $\ell(T)$; and the result follows from the inductive hypothesis.

When $i_0 \leq 3$, we have $i_0 = 3$ (as $j_0 - i_0 \equiv 1 (\bmod 2)$ and $(u_{2n+2}, u_1) \in A(D)$). Hence $(u_3 = u_{i_0}, u_{j_0} = u_{2n+2}, u_1, u_2, u_3)$ is quasi-monochromatic. If (u_{i_0}, u_{2n+2}) is coloured 1, then $(u_0, u_1, u_2, u_3 = u_{i_0}, u_{2n+2} = v)$ is a uv-monochromatic directed path of length 4. So we will assume that (u_{i_0}, u_{2n+2}) is not coloured 1, and hence (u_{2n+2}, u_1) is coloured 1.

If $(u_i, u_0) \in A(D)$ for some $i \in \{3, \ldots, 2n + 1\}$, then (u_i, u_0) is not coloured 1 (otherwise $(v = u_{2n+2}, u_1) \cup (u_1, T, u_i) \cup (u_i, u)$ is a vu-monochromatic directed path, contradicting our hypothesis).

Now observe that $(u_0, u_5) \in A(D)$; otherwise $(u_5, u_0) \in A(D)$ and $(u_0, u_1, u_2, u_3, u_4, u_5, u_0)$ is monochromatic which implies (u_5, u_0) is coloured 1, a contradiction.

Let $k_0 = \max\{i \in \{5, 6, \ldots, 2n - 1\} \mid (u_0, u_i) \in A(D)\}$. Then, we have $(u_0, u_{k_0}) \in A(D)$ and $(u_{k_0+2}, u_0) \in A(D)$; moreover (u_{k_0+2}, u_0) is not coloured 1. Since $(u_0, u_{k_0}, u_{k_0+1}, u_{k_0+2}, u_0)$ is quasi-monochromatic and (u_{k_0+2}, u_0) is not coloured 1, we have (u_0, u_{k_0}) is coloured 1. Thus $(u = u_0, u_{k_0}) \cup (u_{k_0}, T, u_{2(n+1)} = v)$ is a uv-monochromatic directed path whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis.

Case b.2. In view of assertion (a) and case b.1, we may assume that: If $(u_{2(n+1)}, u_i) \in A(D)$ for some $i \in \{1, 2, \ldots, 2(n + 1) - 5\}$ then $(u_{2(n+1)}, u_i)$

is not coloured 1.

— $(u_{2(n+1)}, u_1)$ is not coloured 1: It follows from the fact $(u_{2(n+1)}, u_1) \in A(D)$.

— $(u_{2(n+1)-5}, u_{2(n+1)}) \in A(D)$: Otherwise it follows from Lemma 2.1 that $(u_{2(n+1)}, u_{2(n+1)-5}) \in A(D)$, now $(u_{2(n+1)-5}, T, u_{2(n+1)}) \cup (u_{2(n+1)}, u_{2(n+1)-5})$ is monochromatic coloured 1 (note that it is a directed cycle of length 6 and it has arcs in T), and then $(u_{2(n+1)}, u_{2(n+1)-5})$ is coloured 1, contradicting our assumption.

Let $i_0 = \max\{i \in \{0, 1, 2, \ldots, 2(n+1)-7\} \mid (u_{2(n+1)}, u_i) \in A(D)\}$ (notice that i_0 is well defined as $(u_{2(n+1)}, u_1) \in A(D)$). Therefore $(u_{2(n+1)}, u_{i_0}) \in A(D)$, $(u_{i_0+2}, u_{2(n+1)}) \in A(D)$ and $(u_{2(n+1)}, u_{i_0})$ is not coloured 1. Now we have the directed cycle of length 4 $(u_{2(n+1)}, u_{i_0}, u_{i_0+1}, u_{i_0+2}, u_{2(n+1)})$ which is quasi-monochromatic with $(u_{2(n+1)}, u_{i_0})$ not coloured 1; and (u_{i_0}, u_{i_0+1}), (u_{i_0+1}, u_{i_0+2}) coloured 1; so $(u_{i_0+2}, u_{2(n+1)})$ is coloured 1. Thus, $(u = u_0, T, u_{i_0+2}) \cup (u_{i_0+2}, u_{2(n+1)} = v)$ is a uv-directed path coloured 1 whose length is less than $\ell(T)$; so the assertion follows from the inductive hypothesis. ■

Theorem 2.1. *Let D be an m-coloured bipartite tournament. Assume that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic and D has no subtournament isomorphic to $\widetilde{T}_6$. Then $\mathcal{C}(D)$ is a kernel-perfect digraph.*

Proof. We will prove that every directed cycle contained in $\mathcal{C}(D)$ has at least one symmetrical arc. Then Theorem 2.1 will follow from Theorem 1.1. We proceed by contradiction, suppose that there exists $C = (u_0, u_1, \ldots, u_n, u_0)$ a directed cycle contained in $\mathrm{Asym}(\mathcal{C}(D))$. Therefore, $n \geq 2$. For each $i \in \{0, 1, \ldots, n\}$ there exists a $u_i u_{i+1}$-monochromatic directed path contained in D, and there is no $u_{i+1}u_i$-monochromatic directed path contained in D. Thus, it follows from Lemma 2.3 that at least one of the following assertions hold:

(i) $(u_i, u_{i+1}) \in A(D)$,

(ii) there exists a $u_i u_{i+1}$-directed path of length 2,

(iii) there exists a $u_i u_{i+1}$-monochromatic directed path of length 4. Throughout the proof the indices of the vertices of C are taken $\bmod\, n+1$.

For each $i \in \{0, 1, \ldots, n\}$ let

$$T_i = \begin{cases} (u_i, u_{i+1}) \text{ if } (u_i, u_{i+1}) \in A(D), \\ \text{a } u_i u_{i+1}\text{-directed path of length 2, when } (u_i, u_{i+1}) \notin A(D) \\ \text{and such a path exists,} \\ \text{a } u_i u_{i+1}\text{-monochromatic directed path of length 4, otherwise.} \end{cases}$$

Let $C' = \bigcup_{i=0}^{n} T_i$. Clearly C' is a closed directed walk; let $C' = (z_0, z_1, \ldots, z_k, z_0)$. We define the function φ: $\{0, 1, \ldots, k\} \to V(C)$ as follows: If $T_i = (u_i = z_{i_0}, z_{i_0+1}, \ldots, z_{i_0+r} = u_{i+1})$ with $r \in \{1, 2, 4\}$, then $\varphi(j) = z_{i_0}$ for each $j \in \{i_0, i_0 + 1, \ldots, i_0 + r - 1\}$. We will say that the index i of the vertex z_i of C' is a principal index when $z_i = \varphi(i)$. We will denote by I_p the set of principal indices.

I. First observe that for each $i \in \{0, 1, \ldots, k\}$ we have $\{i, i + 1, i + 2, i + 3\} \cap I_p \neq \emptyset$. Assume w.l.o.g. that $0 \in I_p$ and $z_0 = u_0$. In what follows, the indices of the vertices of C' will be taken modulo $k + 1$.

Case a. $k = 3$.
In this case C' is a directed cycle of length 4 and hence it is quasi-monochromatic. Since $n \geq 2$, then $u_1 \in \{z_1, z_2\}$ and $u_n \in \{z_2, z_3\}$. And it is easy to see that there exists a $u_{i+1}u_i$-monochromatic directed path in D, for some $i \in \{0, 1, \ldots, n\}$, a contradiction.

Case b. $k = 5$.
In this case C' is a directed cycle of length 6, and then it is monochromatic, which clearly implies that there exists a u_1u_0-monochromatic directed path in D, a contradiction.

Case c. $k \geq 7$.
We will prove several assertions.

1(c). For each $i \in \{0, 1, \ldots, k\} \cap I_p$ we have $(z_i, z_{i+5}) \in A(D)$.
Since $(i + 5) - i \equiv 1 (\bmod 2)$, it follows that $(z_i, z_{i+5}) \in A(D)$ or $(z_{i+5}, z_i) \in A(D)$. Assume, for a contradiction that $(z_{i+5}, z_i) \in A(D)$. Therefore $(z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}, z_{i+5}, z_i)$ is monochromatic. Now let $j \in \{0, 1, \ldots, n\}$ be such that $u_j = z_i$, then $u_{j+1} \in \{z_{i+1}, z_{i+2}, z_{i+4}\}$. So there exists a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

2(c). For each $i \in \{0, \ldots, k\}$ such that $i + 5 \in I_p$ we have $(z_i, z_{i+5}) \in A(D)$.

Assume, for a contradiction that $(z_{i+5}, z_i) \in A(D)$. Then $(z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}, z_{i+5}, z_i)$ is monochromatic. Let $j \in \{0, \ldots, n\}$ be such that $u_j = z_{i+5}$, thus $u_{j-1} \in \{z_{i+1}, z_{i+3}, z_{i+4}\}$ and there exists a $u_j u_{j-1}$-monochromatic directed path, a contradiction.

3(c). For each $i \in \{5, \ldots, k-2\}$ such that $i \equiv 1 (\mathrm{mod}\, 4)$ we have $(z_0, z_i) \in A(D)$. We proceed by contradiction; suppose that for some $i \in \{5, \ldots, k-2\}$ we have $i \equiv 1 (\mathrm{mod}\, 4)$ and $(z_i, z_0) \in A(D)$. Since $0 \in I_p$, it follows from 1(c) that $(z_0, z_5) \in A(D)$ and then $i \geq 9$.

Let $i_0 = \min\{j \in \{5, \ldots, k-6\} \mid j \equiv 1 (\mathrm{mod}\, 4)$ and $(z_{j+4}, z_0) \in A(D)\}$ (notice that i_0 is well defined, as $i \geq 9$). Thus $(z_0, z_{i_0-4}) \in A(D)$, $(z_0, z_{i_0}) \in A(D)$ and $(z_{i_0+4}, z_0) \in A(D)$. Now $C^2 = (z_0, z_{i_0}, z_{i_0+1}, z_{i_0+2}, z_{i_0+3}, z_{i_0+4}, z_0)$ is a directed cycle of length 6 and hence it is monochromatic, w.l.o.g we will assume that it is coloured 1. Now we consider two cases:

3(c).1. $i_0 \in I_p$.
In this case $z_{i_0} = u_j$ for some $j \in \{3, \ldots, n\}$ and from the definition of C', $u_{j+1} \in \{z_{i_0+1}, z_{i_0+2}, z_{i_0+4}\}$. In any case, there exists a $u_{j+1}u_j$-monochromatic directed path contained in C, a contradiction.

3(c).2. $i_0 \notin I_p$.
In this case, we have from observation I that $\{i_0-3, i_0-2, i_0-1\} \cap I_p \neq \emptyset$, let $\ell \in \{i_0-3, i_0-2, i_0-1\} \cap I_p$ and $u_j \in V(C)$ such that $u_j = z_\ell$. From 1(c) we have $(z_\ell, z_{\ell+5}) \in A(D)$ and $\ell + 5 \in \{i_0+2, i_0+3, i_0+4\}$ which implies that $C^3 = (z_{i_0-4}, C', z_\ell) \cup (z_\ell, z_{\ell+5}) \cup (z_{\ell+5}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a closed directed walk of length 6 (as $(z_{i_0-4}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a closed directed walk of length 10). It follows from Lemma 2.2 that C^3 is a directed cycle and the hypothesis implies that it is monochromatic. Since $(z_{i_0+4}, z_0) \in A(C^2) \cap A(C^3)$, we have that C^3 is coloured 1. Now from the definition of C' we have $u_{j+1} \in \{z_{\ell+1}, z_{\ell+2}, z_{\ell+4}\} \subseteq \{z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\}$.

When $u_{j+1} \in \{z_{i_0}, z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\}$, we obtain that $(z_{i_0}, C^2, z_0) \cup (z_0, C^3, z_\ell)$ contains a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

When $u_{j+1} \in \{z_{i_0-2}, z_{i_0-1}\}$, we take $i_1 \in \{i_0-2, i_0-1\}$ such that $u_{j+1} = z_{i_1}$. From 1(c) we have $(z_{i_1}, z_{i_1+5}) \in A(D)$ where $z_{i_1+5} \in \{z_{i_0+3}, z_{i_0+4}\}$, thus $C^4 = (z_{i_0-4}, C', z_{i_1}) \cup (z_{i_1}, z_{i_1+5}) \cup (z_{i_1+5}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a directed cycle of length 6 (notice that $(z_{i_0-4}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a closed directed walk of length 10). From the hypothesis we have that C^4 is monochromatic. Since $(z_{i_0+4}, z_0) \in A(C^4) \cap A(C^3)$ we obtain that C^4 is coloured 1.

Finally, since $\{u_j, u_{j+1}\} \subseteq V(C^4)$ we have a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

4(c). For any $i \in \{3, \ldots, k-4\}$ such that $i \equiv k (\bmod 4)$ we have $(z_i, z_0) \in A(D)$. Since $i \equiv k (\bmod 4)$ and $k \equiv 1 (\bmod 2)$ we have $i \equiv 1 (\bmod 2)$, and from Lemma 2.1 we obtain $(z_0, z_i) \in A(D)$ or $(z_i, z_0) \in A(D)$.

Assume, for a contradiction that $(z_0, z_i) \in A(D)$, for some $i \in \{3, \ldots, k-4\}$ such that $i \equiv k (\bmod 4)$.

Since $0 \in I_p$, it follows from 2(c) that $(z_{k-4}, z_0) \in A(D)$, and thus $i \leq k-8$. Let $i_0 = \max\{i \in \{7, \ldots, k-4\} \mid i \equiv k (\bmod 4)$ and $(z_0, z_{i-4}) \in A(D)\}$ therefore $(z_0, z_{i_0-4}) \in A(D)$, $(z_{i_0}, z_0) \in A(D)$ and $(z_{i_0+4}, z_0) \in A(D)$. So $C^2 = (z_0, z_{i_0-4}, z_{i_0-3}, z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_0)$ is a directed cycle of length 6 and hence it is monochromatic, w.l.o.g. assume that it is coloured 1.

When $i_0 \in I_p$, we have $z_{i_0} = u_j$ for some $j \in \{2, \ldots, n-2\}$. From the definition of C' we have $u_{j-1} \in \{z_{i_0-1}, z_{i_0-2}, z_{i_0-4}\}$ and then there exists a $u_j u_{j-1}$-monochromatic directed path contained in C^2, a contradiction.

When $i_0 \notin I_p$, then from I we have $\{i_0-3, i_0-2, i_0-1\} \cap I_p \neq \emptyset$. Let $\ell \in \{i_0-3, i_0-2, i_0-1\} \cap I_p$, so $u_j = z_\ell$ for some $u_j \in V(C)$. From 1(c) it follows $(z_\ell, z_{\ell+5}) \in A(D)$ and $\ell + 5 \in \{i_0+2, i_0+3, i_0+4\}$.

Now $C^3 = (z_{i_0-4}, C', z_\ell) \cup (z_\ell, z_{\ell+5}) \cup (z_{\ell+5}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a directed cycle of length 6 (as $(z_{i_0-4}, C', z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a closed directed walk of length 10), and then it is monochromatic. Since $(z_0, z_{i_0-4}) \in A(C^2) \cap A(C^3)$ we have C^3 is coloured 1. Observe that $u_{j+1} \in \{z_{\ell+1}, z_{\ell+2}, z_{\ell+4}\} \subseteq \{z_{i_0-2}, z_{i_0-1}, z_{i_0}, z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\}$. If $u_{j+1} \in \{z_{i_0-2}, z_{i_0-1}, z_{i_0}\}$ then there exists a $u_{j+1}u_j$-monochromatic directed path contained in C^2, a contradiction.

If $u_{j+1} \in \{z_{i_0+1}, z_{i_0+2}, z_{i_0+3}\}$, then we take $i_1 \in \{i_0+1,\ i_0+2,\ i_0+3\}$ such that $u_{j+1} = z_{i_1}$. From 2(c), $(z_{i_1-5},\ z_{i_1}) \in A(D)$, where $z_{i_1-5} \in \{z_{i_0-4}, z_{i_0-3},\ z_{i_0-2}\}$. Now, $C^4 = (z_{i_0-4}, C', z_{i_1-5}) \cup (z_{i_1-5},\ z_{i_1}) \cup (z_{i_1},\ C',\ z_{i_0+4}) \cup (z_{i_0+4},\ z_0,\ z_{i_0-4})$ is a directed cycle of length 6 (as $(z_{i_0-4}, C',\ z_{i_0+4}) \cup (z_{i_0+4}, z_0, z_{i_0-4})$ is a closed directed walk of length 10), so it is monochromatic and coloured 1 (because $(z_0, z_{i_0-4}) \in A(C^4) \cap A(C^2)$). We conclude that $(u_{j+1} = z_{i_1}, C^4, z_{i_0-4}) \cup (z_{i_0-4}, C^2, z_\ell = u_j)$ contains a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

Now we will analyze the two possible cases:

Case c.1. $k \equiv 1 (\bmod 4)$.
Since $0 \in I_p$, it follows from 2(c) that $(z_{k-4}, z_0) \in A(D)$. On the other hand,

we have $k-4 \equiv 1(\bmod 4)$, and from 3(c), $(z_0, z_{k-4}) \in A(D)$, a contradiction (as D is a bipartite tournament).

Case c.2. $k \equiv 3(\bmod 4)$.
First, we prove several assertions:

5(c.2). For any $i \in \{3, \ldots, k-4\}$ such that $i \equiv 3(\bmod 4)$ we have $(z_i, z_0) \in A(D)$.

This assertion follows from 4(c) as $i \equiv k(\bmod 4)$.

6(c.2). For any $i, j \in \{0, \ldots, k\}$ such that $i \in I_p$ and $j - i \equiv 1(\bmod 4)$, we have $(z_i, z_j) \in A(D)$.

Let $r \in \{0, 1, \ldots, n\}$ be such that $u_r = z_i$, now we rename the vertices of C in such a way that C starts at u_r. Joining the corresponding directed paths (T_i) between the vertices of C, we obtain a closed directed walk $\overline{C}' = (\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_k, \overline{z}_0)$ which is the same as C' where the vertices where renamed as follows: for each $t \in \{0, \ldots, k\}$ $\overline{z}_t = z_{t+i}$, thus $\overline{z}_0 = z_i$. Let $j \in \{0, \ldots, k\}$ be such that $j - i \equiv 1(\bmod 4)$. It follows from 3(c) that $(\overline{z}_0, \overline{z}_{j-i}) \in A(D)$ and that means $(z_i, z_j) \in A(D)$ (as $\overline{z}_0 = z_i$ and $\overline{z}_{j-i} = z_j$).

7(c.2). For any $i, j \in \{0, \ldots, k\}$ such that $i \in I_p$ and $j - i \equiv 3(\bmod 4)$, we have $(z_j, z_i) \in A(D)$.

We proceed as in 6(c.2), to obtain $\overline{C}'$. Taking $j \in \{0, \ldots, k\}$ such that $j - i \equiv 3(\bmod 4)$, we obtain from 5(c.2) that $(\overline{z}_{j-i}, \overline{z}_0) \in A(D)$; i.e., $(z_j, z_i) \in A(D)$.

8(c.2). For any $i \in \{0, \ldots, k\}$ we have $(z_i, z_{i-3}) \in A(D)$.
We proceed by contradiction, suppose that for some $i \in \{0, \ldots, k\}$ we have $(z_{i-3}, z_i) \in A(D)$. Since $i - (i-3) \equiv 3(\bmod 4)$, we have from 7(c.2) that $i-3 \notin I_p$; and since $(i-3)-i \equiv 1(\bmod 4)$, we obtain from 6(c.2) that $i \notin I_p$. From I, $\{i-3, i-2, i-1, i\} \cap I_p \neq \emptyset$. Thus $\{i-2, i-1\} \cap I_p \neq \emptyset$.

And here we consider the two possible cases:

Case 8(c.2) a. $i - 2 \in I_p$.
Let $j \in \{0, \ldots, n\}$ be such that $z_{i-2} = u_j$. We have $(z_{i+1}, z_{i-2} = u_j) \in A(D)$ (this follows directly from 7(c.2), observating that $i+1-(i-2) \equiv 3(\bmod 4)$), also $(z_{i-2}, z_{i-5}) \in A(D)$ (from 6(c.2), just observe that $(i-5)-(i-2) \equiv 1(\bmod 4)$). Now we have $C^2 = (u_j = z_{i-2}, z_{i-5}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i-2} = u_j)$ is a directed cycle of length 6 and from the hypothesis it is monochromatic, assume w.l.o.g. that it is coloured 1. From the definition of C', $u_{j-1} \in \{z_{i-6}, z_{i-4}, z_{i-3}\}$. Since $i-3 \notin I_p$ we obtain $u_{j-1} \in \{z_{i-6}, z_{i-4}\}$.

When $u_{j-1} = z_{i-4}$, we obtain $\{u_{j-1}, u_j\} \subset V(C^2)$. Thus there exists a $u_j u_{j-1}$-monochromatic directed path contained in C^2, a contradiction. When $u_{j-1} = z_{i-6}$, we have $(z_{i+1}, z_{i-6} = u_{j-1}) \in A(D)$ (from 7(c.2) as $(i+1)-(i-6) \equiv 3 (\mathrm{mod}\, 4)$). So $C^3 = (u_{j-1} = z_{i-6}, z_{i-5}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i-6} = u_{j-1})$ is a directed cycle of length 6 and hence it is monochromatic; moreover it is coloured 1 (because $(z_{i-3}, z_i) \in A(C^2) \cap A(C^3)$). Therefore, $(u_j = z_{i-2}, C^2, z_{i+1}) \cup (z_{i+1}, z_{i-6} = u_{j-1})$ is a $u_j u_{j-1}$-monochromatic directed path, a contradiction.

Case 8(c.2) b. $i - 1 \in I_p$.
Let $j \in \{0, \ldots, n\}$ be such that $z_{i-1} = u_j$. We have $(z_{i+2}, z_{i-1} = u_j) \in A(D)$ (this follows from 7(c.2), as $i + 2 - (i - 1) \equiv 3 (\mathrm{mod}\, 4)$), and $(z_{i-1}, z_{i-4}) \in A(D)$ (this follows from 6(c.2), because $(i-4)-(i-1) \equiv 1 (\mathrm{mod}\, 4)$). Therefore $C^2 = (u_j = z_{i-1}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i+2}, z_{i-1} = u_j)$ is a directed cycle of length 6, hence it is monochromatic say coloured 1. From the definition of C', we have $u_{j+1} \in \{z_i, z_{i+1}, z_{i+3}\}$; moreover $u_{j+1} \in \{z_{i+1}, z_{i+3}\}$ because $i \notin I_p$. If $u_{j+1} = z_{i+1}$, then $\{u_j, u_{j+1}\} \subseteq V(C^2)$ and thus there exists a $u_{j+1}u_j$-monochromatic directed path, a contradiction. Hence $u_{j+1} = z_{i+3}$. Now observe that $(u_{j+1} = z_{i+3}, z_{i-4}) \in A(D)$ (this follows from 6(c.2) as $i-4-(i+3) \equiv 1 (\mathrm{mod}\, 4)$). Therefore $C^3 = (u_{j+1} = z_{i+3}, z_{i-4}, z_{i-3}, z_i, z_{i+1}, z_{i+2}, z_{i+3} = u_{j+1})$ is a directed cycle of length 6 and it is coloured 1 (because $(z_{i-3}, z_i) \in A(C^2) \cap A(C^3)$). We conclude that $(u_{j+1} = z_{i+3}, z_{i-4}) \cup (z_{i-4}, C^2, z_{i-1} = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

9(c.2). If for some $i \in \{0, \ldots, k\}$ we have (z_{i-1}, z_i) and (z_i, z_{i+1}) have different colours, then $i \in I_p$.

From I we have $\{i-3, i-2, i-1, i\} \cap I_p \neq \emptyset$. Let $r_0 = \min\{r \in \{0,1,2,3\} \,|\, i - r \in I_p\}$ and let $j \in \{0, 1, \ldots, n\}$ be such that $z_{i-r_0} = u_j$; so we have $u_j \in \{z_{i-3}, z_{i-2}, z_{i-1}, z_i\}$. From the definition of C', $u_{j+1} \in \{z_{i-r_0+1}, z_{i-r_0+2}, z_{i-r_0+4}\} \subseteq \{z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}\}$. Now consider $\ell \in \{i - r_0 + 1, i - r_0 + 2, i - r_0 + 4\}$ such that $u_{j+1} = z_\ell$. From the definition of r_0 and since $\ell \in I_p$, we have $\ell \notin \{i-2, i-1, i\}$, i.e., $u_{j+1} \in \{z_{i+1}, z_{i+2}, z_{i+3}, z_{i+4}\}$.

If T_j has length 4, then T_j is monochromatic; and hence $\{(z_{i-1}, z_i), (z_i, z_{i+1})\} \not\subseteq A(T_j)$, and $z_i = u_j$, $z_{i+4} = u_{j+1}$. Thus $i \in I_p$.

If T_j has length 1, then $z_i = u_j$, i.e., $i \in I_p$.

If T_j has length 2, then $u_j \in \{z_{i-1}, z_i\}$. When $u_j = z_i$ clearly $i \in I_p$.

When $u_j = z_{i-1}$, we have $u_{j+1} = z_{i+1}$. From 8(c.2) we obtain $(z_{i+2}, z_{i-1}) \in A(D)$ and thus $C^2 = (u_j = z_{i-1}, z_i, z_{i+1} = u_{j+1}, z_{i+2}, z_{i-1} = u_j)$ is a directed cycle of length 4 (which from the hypothesis is quasi-monochromatic). Since (z_{i-1}, z_i) and (z_i, z_{i+1}) have different colours, we conclude that (u_{j+1}, C^2, u_j) is a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

10(c.2). There exists a change of colour in C'; i.e., there exists $i \in \{0, \dots, k\}$ such that (z_{i-1}, z_i) and (z_i, z_{i+1}) have different colours.

Otherwise C' is monochromatic, and for any $j \in \{0, \dots, n\}$, there exists a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

We will assume w.l.o.g. that (z_{i-1}, z_i) is coloured 1 and (z_i, z_{i+1}) is coloured 2.

11(c.2). $i \in I_p$.
It follows directly from 9(c.2) and our assumption. Let $j \in \{0, \dots, n\}$ be such that $z_i = u_j$.

12(c.2). $\{(z_{i+2}, z_{i-1}), (z_{i+1}, z_{i-2}), (z_i, z_{i-3}), (z_{i+3}, z_i)\} \subseteq A(D)$.
This follows directly from 8(c.2).

13(c.2). (z_{i+1}, z_{i+2}) and (z_{i+2}, z_{i-1}) have the same colour, say a, with $a \in \{1, 2\}$.

Let $C^2 = (z_{i-1}, z_i = u_j, z_{i+1}, z_{i+2}, z_{i-1})$ from 12(c.2), it is a directed cycle of length 4 and then it is quasi-monochromatic. Since (z_{i-1}, z_i) and (z_i, z_{i+1}) are coloured 1 and 2 respectively, 13(c.2) follows.

14(c.2). (z_{i+1}, z_{i-2}) and (z_{i-2}, z_{i-1}) have the same colour, say b, with $b \in \{1, 2\}$. The proof is similar to that of 13(c.2) by considering the directed cycle of length 4, $C^3 = (z_{i-2}, z_{i-1}, z_i = u_j, z_{i+1}, z_{i-2})$.

15(c.2). $\{i-1, i+1\} \cap I_p = \emptyset$.
First suppose for a contradiction that $i - 1 \in I_p$. From the definition of C', and since $z_i = u_j$, we have $z_{i-1} = u_{j-1}$. From 13(c.2) (z_{i+1}, z_{i+2}) and (z_{i+2}, z_{i-1}) have the same colour $a \in \{1, 2\}$. If $a = 2$, then $(z_i = u_j, z_{i+1}, z_{i+2}, z_{i-1} = u_{j-1})$ is a $u_j u_{j-1}$-monochromatic directed path, a contradiction. If $a = 1$, then from 9(c.2) we have $i + 1 \in I_p$. So, $z_{i+1} = u_{j+1}$ and $(u_{j+1} = z_{i+1}, z_{i+2}, z_{i-1}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

Now, suppose for a by contradiction that $i + 1 \in I_p$. Thus $z_{i+1} = u_{j+1}$. From 14(c.2) we have (z_{i+1}, z_{i-2}) and (z_{i-2}, z_{i-1}) have the same colour b, with $b \in \{1, 2\}$. If $b = 1$ then $(u_{j+1} = z_{j+1}, z_{i-2}, z_{i-1}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path, a contradiction. If $b = 2$ then from 9(c.2) we have $i - 1 \in I_p$, but we have proved that this leads to a contradiction.

16(c.2). (z_{i+1}, z_{i+2}) is coloured 2.
Otherwise (z_i, z_{i+1}) and (z_{i+1}, z_{i+2}) have different colours and from 9(c.2) $i+1 \in I_p$, contradicting 15(c.2).

17(c.2). (z_{i-2}, z_{i-1}) is coloured 1.
Otherwise (z_{i-2}, z_{i-1}) and (z_{i-1}, z_i) have different colours, and from 9(c.2) $i-1 \in I_p$, contradicting 15(c.2).

18(c.2). (z_{i+2}, z_{i-1}) is coloured 2.
This follows directly from 13(c.2) and 16(c.2).

19(c.2). (z_{i+1}, z_{i-2}) is coloured 1.
Follows directly from 14(c.2) and 17(c.2). Now we will analyze the two possible cases: $i+2 \notin I_p$ or $i+2 \in I_p$.

Case c.2.1. $i+2 \notin I_p$.
In this case, we have from the definition of C' that $i+4 \in I_p$ and $z_{i+4} = u_{j+1}$. And we have the following assertions: 1(c.2.1) to 11(c.2.1).

1(c.2.1). (z_{i+2}, z_{i+3}) and (z_{i+3}, z_{i+4}) are coloured 2.
Since $u_{j+1} = z_{i+4}$, then $T_j = (u_j = z_i,\ z_{i+1},\ z_{i+2},\ z_{i+3},\ z_{i+4} = u_{j+1})$ is monochromatic; moreover it is coloured 2 (as (z_i, z_{i+1}) is coloured 2).

2(c.2.1). $(z_{i+4}, z_{i-3}) \in A(D)$.
This follows from 6(c.2) because $i-3-(i+4) \equiv 1 (\bmod 4)$.

3(c.2.1). $(z_{i-1}, z_{i+4}) \in A(D)$.
The assertion follows from 7(c.2) as $i-1-(i+4) \equiv 3 (\bmod 4)$.

4(c.2.1). $\{(z_{i+4}, z_{i+1}), (z_{i+3}, z_i)\} \subseteq A(D)$.
Is a direct consequence of 8(c.2).

5(c.2.1). (z_{i+4}, z_{i+1}) is not coloured 1.
Assuming for a contradiction that (z_{i+4}, z_{i+1}) is coloured 1, we obtain that $(u_{j+1} = z_{i+4}, z_{i+1}, z_{i-2}, z_{i-1}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

6(c.2.1). (z_{i-1}, z_{i+4}) is coloured 1.
We have that $(z_{i+1}, z_{i-2}, z_{i-1}, z_{i+4}, z_{i+1})$ is quasi-monochromatic (because it is a directed cycle of length 4). From 19(c.2) (z_{i+1}, z_{i-2}) is coloured 1, from 17(c.2), (z_{i-2}, z_{i-1}) is coloured 1; and from 5(c.2.1) (z_{i+4}, z_{i+1}) is not coloured 1. So, (z_{i-1}, z_{i+4}) is coloured 1.

7(c.2.1). (z_{i+4}, z_{i+1}) is coloured 2.

We have that: $(z_{i+1}, z_{i+2}, z_{i-1}, z_{i+4}, z_{i+1})$ is quasi-monochromatic (from the hypothesis), (z_{i+1}, z_{i+2}) is coloured 2 (16(c.2)), (z_{i+2}, z_{i-1}) is coloured 2 (18(c.2)) and (z_{i-1}, z_{i+4}) is coloured 1 (6(c.2.1)).

8(c.2.1). $(z_{i-3}, z_{i+2}) \in A(D)$.
Assume, for a contradiction that $(z_{i-3}, z_{i+2}) \notin A(D)$. Then $(z_{i+2}, z_{i-3}) \in A(D)$ and $(z_{i+2}, z_{i-3}, z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2})$ is a directed cycle of length 6. From the hypothesis we have that it must be monochromatic, but it has two arcs coloured 1 ((z_{i-2}, z_{i-1}) and (z_{i-1}, z_i)) and two arcs coloured 2 ((z_i, z_{i+1}) and (z_{i+1}, z_{i+2})), a contradiction.

9(c.2.1). $(z_{i-2}, z_{i+3}) \in A(D)$.
Assuming for a contadiction that $(z_{i-2}, z_{i+3}) \notin A(D)$, we obtain $(z_{i+3}, z_{i-2}) \in A(D)$ and $(z_{i+3}, z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i+2}, z_{i+3})$ is a directed cycle of length 6. It has two arcs coloured 1 ((z_{i-2}, z_{i-1}) and (z_{i-1}, z_i)) and two arcs coloured 2 ((z_i, z_{i+1}) and (z_{i+1}, z_{i+2})), contradicting the hypothesis.

10(c.2.1). (z_{i+3}, z_i) is not coloured 2.
Assume, for a contradiction that (z_{i+3}, z_i) is coloured 2, then $(u_{j+1} = z_{i+4}, z_{i+1}, z_{i+2}, z_{i+3}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path, a contradiction.

11(c.2.1). The arcs (z_{i-2}, z_{i+3}) and (z_{i+3}, z_i) are coloured 1.
We have $(z_{i+3}, z_i, z_{i+1}, z_{i-2}, z_{i+3})$ a directed cycle of length 4, thus it is quasi-monochromatic. Since (z_i, z_{i+1}) is coloured 2 and (z_{i+1}, z_{i-2}) is coloured 1 (19(c.2)), then (z_{i-2}, z_{i+3}) and (z_{i+3}, z_i) are both coloured 1 or are both coloured 2. And from 10(c.2.1) (z_{i+3}, z_i) is not coloured 2.

12(c.2.1). (z_{i+4}, z_{i-3}) and (z_{i-3}, z_{i-2}) are both coloured 1 or are both coloured 2.

We have $(z_{i-2}, z_{i+3}, z_{i+4}, z_{i-3}, z_{i-2})$ is quasi-monochromatic; (z_{i-2}, z_{i+3}) is coloured 1 (11(c.2.1)) and (z_{i+3}, z_{i+4}) is coloured 2 (1(c.2.1)).

If (z_{i+4}, z_{i-3}) and (z_{i-3}, z_{i-2}) are both coloured 1, then $(u_{j+1} = z_{i+4}, z_{i-3}, z_{i-2}, z_{i-1}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path (coloured 1), a contradiction. If (z_{i+4}, z_{i-3}) and (z_{i-3}, z_{i-2}) are both coloured 2, then $(z_{i-1}, z_{i+4}, z_{i-3}, z_{i-2}, z_{i-1})$ is a directed cycle of length 4 with two arcs coloured 1 and two arcs coloured 2, a contradiction to the hypothesis. So case (c.2.1) is not possible.

Case c.2.2. $i + 2 \in I_p$.
Since $i + 1 \notin I_p$, then $z_{i+2} = u_{j+1}$. We have the following assertions:

1(c.2.2). $(z_{i+2}, z_{i-5}) \in A(D)$.
This follows from 6(c.2), as $(i-5)-(i+2) \equiv 1 (\bmod 4)$.

2(c.2.2). $(z_{i-3}, z_{i+2}) \in A(D)$.
Since $(i-3)-(i+2) \equiv 3 (\bmod 4)$, the assertion follows from 7(c.2).

3(c.2.2). $(z_{i-4}, z_{i+1}) \in A(D)$.
Assume, for a contradiction that $(z_{i-4}, z_{i+1}) \notin A(D)$. Then $(z_{i+1}, z_{i-4}) \in A(D)$ and $(z_{i-4}, z_{i-3}, z_{i-2}, z_{i-1}, z_i, z_{i+1}, z_{i-4})$ is monochromatic (as it is a directed cycle of length 6), but (z_{i-1}, z_i) is coloured 1 and (z_i, z_{i+1}) is coloured 2, a contradiction.

4(c.2.2). $(z_{i-1}, z_{i-4}) \in A(D)$.
It follows from 8(c.2).

5(c.2.2). $(z_{i-5}, z_i) \in A(D)$.
Since $(i-5)-i \equiv 3 (\bmod 4)$ then the assertion follows from 7(c.2).

6(c.2.2). $(z_{i-2}, z_{i-5}) \in A(D)$.
This follows from 8(c.2).

7(c.2.2). The arcs (z_i, z_{i-3}) and (z_{i-3}, z_{i+2}) are both coloured 2.
We have $(z_{i-1}, z_i, z_{i-3}, z_{i+2}, z_{i-1})$ a directed cycle of length 4, thus it is quasi-monochromatic. Since (z_{i-1}, z_i) is coloured 1 and (z_{i+2}, z_{i-1}) is coloured 2 then (z_i, z_{i-3}) and (z_{i-3}, z_{i+2}) are both coloured 1 or are both coloured 2. If they are both coloured 2, then we are done.

Now suppose that (z_i, z_{i-3}) and (z_{i-3}, z_{i+2}) are both coloured 1. Therefore $(z_{i+2}, z_{i-1}, z_{i-4}, z_{i-3}, z_{i+2})$ is quasi-monochromatic. Since (z_{i+2}, z_{i-1}) is coloured 2 and (z_{i-3}, z_{i+2}) is coloured 1, then (z_{i-1}, z_{i-4}) and (z_{i-4}, z_{i-3}) are both coloured 1 or are both coloured 2.

We will analyze the two possible cases:

Case 7(c.2.2)a. The arcs (z_{i-1}, z_{i-4}) and (z_{i-4}, z_{i-3}) are both coloured 2. In this case we have (z_{i-3}, z_{i-2}) is coloured 2 because $(z_{i-1}, z_{i-4}, z_{i-3}, z_{i-2}, z_{i-1})$ is quasi-monochromatic, (z_{i-2}, z_{i-1}) is coloured 1 and (z_{i-1}, z_{i-4}) and (z_{i-4}, z_{i-3}) are both coloured 2.

So, it follows from 9(c.2) that $i-2 \in I_p$. Since $i-1 \notin I_p$ (15(c.2)) then $z_{i-2} = u_{j-1}$. Thus $(u_j = z_i, z_{i+1}, z_{i+2}, z_{i-1}, z_{i-4}, z_{i-3}, z_{i-2} = u_{j-1})$ is a $u_j u_{j-1}$-directed path coloured 2, a contradiction. So case 7(c.2.2)a is not possible.

Case 7(c.2.2)b. The arcs (z_{i-1}, z_{i-4}) and (z_{i-4}, z_{i-3}) are both coloured 1. In this case we have (z_{i-3}, z_{i-2}) is not coloured 1 (otherwise $(u_j = z_i,\ z_{i-3},\ z_{i-2},\ z_{i-1},\ z_{i-4})$ is a directed walk coloured 1 which contains $\{z_{i-2}, z_{i-4}\}$; and from the definition of C', $u_{j-1} \in \{z_{i-2}, z_{i-4}\}$ thus there exists a $u_j u_{j-1}$-monochromatic directed path; a contradiction). Now from 9(c.2) we have $\{i-3, i-2\} \subseteq I_p$. Since $i-1 \notin I_p$ we have $z_{i-2} = u_{j-1}$ and $z_{i-3} = u_{j-2}$. Therefore $(u_{j-1} = z_{i-2}, z_{i-1}, z_{i-4}, z_{i-3} = u_{j-2})$ is a $u_{j-1}u_{j-2}$-monochromatic directed path (coloured 1), a contradiction.

We conclude that the arcs (z_i, z_{i-3}) and (z_{i-3}, z_{i+2}) are both coloured 2.

8(c.2.2). (z_{i-3}, z_{i-2}) is coloured 1.
We have $(z_{i-2}, z_{i-1}, z_i, z_{i-3}, z_{i-2})$ which is quasi-monochromatic; (z_i, z_{i-3}) coloured 2 and $((z_{i-2}, z_{i-1})$ and $(z_{i-1}, z_i))$ coloured 1.

9(c.2.2). (z_{i-2}, z_{i-5}) and (z_{i-5}, z_i) are both coloured 1.
(z_{i-2}, z_{i-5}) and (z_{i-5}, z_i) are both coloured 1 or are both coloured 2: this is because $(z_i, z_{i-3}, z_{i-2}, z_{i-5}, z_i)$ is quasi-monochromatic with (z_i, z_{i-3}) coloured 2 and (z_{i-3}, z_{i-2}) coloured 1.

Assume, for a contradiction that (z_{i-2}, z_{i-5}) and (z_{i-5}, z_i) are both coloured 2.

Denote by a the colour of the arc (z_{i+2}, z_{i-5}). We have $a \neq 2$ (otherwise $(u_{j+1} = z_{i+2}, z_{i-5}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path, a contradiction). Now, (z_{i-5}, z_{i-4}) and (z_{i-4}, z_{i-3}) are both coloured b with $b \in \{1, 2\}$ (this is because $(z_{i-5}, z_{i-4},\ z_{i-3}, z_{i-2}, z_{i-5})$ is quasi-monochromatic with (z_{i-3}, z_{i-2}) coloured 1 and (z_{i-2}, z_{i-5}) coloured 2). If $b = 1$ then $a = 1$ (notice that $(z_{i+2}, z_{i-5}, z_{i-4}, z_{i-3}, z_{i+2})$ is quasi-monochromatic; with (z_{i-3}, z_{i+2}) coloured 2 and $((z_{i-5}, z_{i-4})$ and $(z_{i-4}, z_{i-3}))$ coloured 1; so $a = 1$). Thus $(u_{j+1} = z_{i+2}, z_{i-5}, z_{i-4}, z_{i-3}, z_{i-2}, z_{i-1}, z_i = u_j)$ is a $u_{j+1}u_j$-monochromatic directed path (coloured 1), a contradiction. If $b = 2$, then $i-3 \in I_p$ (from 9(c.2)) and from the definition of C', $i-2 \in I_p$. Thus $z_{i-2} = u_{j-1}$, $z_{i-3} = u_{j-2}$ and $(u_{j-1} = z_{i-2},\ z_{i-5},\ z_{i-4},\ z_{i-3} = u_{j-2})$ is a $u_{j-1}u_{j-2}$-monochromatic directed path (coloured 2), a contradiction.

10(c.2.2). (z_{i+2}, z_{i-5}) is coloured 2.
$(z_i, z_{i+1}, z_{i+2}, z_{i-5}, z_i)$ is quasi-monochromatic with (z_{i-5}, z_i) coloured 1 and $((z_i, z_{i+1})$ and $(z_{i+1}, z_{i+2}))$ coloured 2.

11(c.2.2). (z_{i-4}, z_{i-3}) is not coloured 2.
Assume, for a contradiction that (z_{i-4}, z_{i-3}) coloured 2. Then $i-3 \in I_p$. On the other hand we have $i-4 \in I_p$ (because $(z_{i-5}, z_{i-4}, z_{i-3}, z_{i-2}, z_{i-5})$

is quasi-monochromatic with (z_{i-4}, z_{i-3}) coloured 2 and $((z_{i-3}, z_{i-2})$ and $(z_{i-2}, z_{i-5}))$ coloured 1; so (z_{i-5}, z_{i-4}) is coloured 1 and then (from 9(c.2)) $i-4 \in I_p$). Now, from the definition of C', we have $z_{i-3} = u_r$ and $z_{i-4} = u_{r-1}$ for some $r \in \{1, 2, \ldots, n\}$. Thus $(u_r = z_{i-3}, z_{i-2}, z_{i-5}, z_{i-4} = u_{r-1})$ is a $u_r u_{r-1}$-monochromatic directed path (coloured 1), a contradiction.

12(c.2.2). (z_{i-5}, z_{i-4}) is coloured 2.
$(z_{i-5}, z_{i-4}, z_{i-3}, z_{i+2}, z_{i-5})$ is quasi-monochromatic; with $((z_{i-3}, z_{i+2})$ and $(z_{i+2}, z_{i-5}))$ coloured 2 and (z_{i-4}, z_{i-3}) not coloured 2.

13(c.2.2). (z_{i-4}, z_{i+1}) is coloured 1.
$(z_{i+1}, z_{i-2}, z_{i-5}, z_{i-4}, z_{i+1})$ is quasi-monochromatic with (z_{i-5}, z_{i-4}) coloured 2 and $((z_{i+1}, z_{i-2})$ and $(z_{i-2}, z_{i-5}))$ coloured 1.

14(c.2.2). $D[\{z_i, z_{i+1}, z_{i+2}, z_{i-5}, z_{i-4}, z_{i-2}\}]$ is isomorphic to $\widetilde{T}_6$.
Let $f : \{z_i, z_{i+1}, z_{i+2}, z_{i-5}, z_{i-4}, z_{i-2}\} \to V(\widetilde{T}_6)$ defined as follows: $f(z_i) = x$, $f(z_{i+1}) = y$, $f(z_{i+2}) = v$, $f(z_{i-5}) = w$, $f(z_{i-4}) = z$, $f(z_{i-2}) = u$ is an isomorphism.

Assertion 14(c.2.2) contradicts the hypothesis, so case c(2.2) is not possible; also case c.2 is not possible. ∎

As a direct consequence of Theorem 2.1, we have the following result:

Theorem 2.2. *Let D be an m-coloured bipartite tournament. Assume that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic and D has no subtournament isomorphic to $\widetilde{T}_6$. Then D has a kernel by monochromatic paths.*

Remark 2.1. The hypothesis that every directed cycle of length 6 is monochromatic in Theorem 2.1 is tight.

Let D be the 3-coloured bipartite tournament defined in [8] as follows: $V(D) = \{u, v, w, x, y, z\}$, $A(D) = \{(u, x), (x, v), (v, y), (y, w), (w, z), (z, u), (x, w), (y, u), (z, v)\}$; the arcs $(x, w), (w, z)$ and (z, u) coloured 1; the arcs $(y, u), (u, x)$ and (x, v), coloured 2; and the arcs $(z, v), (v, y)$ and (y, w) coloured 3. D has a directed cycle of length 6 which is not monochromatic, every directed cycle of length 4 in D is quasi-monochromatic, D has no subtournament isomorphic to $\widetilde{T}_6$ and $\mathcal{C}(D)$ is a complete multidigraph which has no kernel.

Remark 2.2. The hypothesis that every directed cycle of length 6 in a bipartite tournament D is monochromatic, does not imply that every directed cycle of length 4 in D is quasi-monochromatic.

Proof. Let $T = (U, W)$ be the 2-coloured bipartite tournament defined as follows: $U = \{u, v, w, x, y\}$ and $W = \{a, b, c, d, e\}$. In T, $C_1 = (u, a, v, b, w, c, u)$ is a directed cycle of length 6 coloured 1, $C_2 = (x, d, y, e, x)$ is a directed cycle of length 4 coloured 2. T has arcs from $U \cap V(C_1)$ to $W \cap V(C_2)$ coloured 1 and finally T contains the arcs (u, b), (a, w), (c, w) coloured 1 (see Figure 2). C_1 is the only directed cycle of length 6 contained in T, and it is monochromatic. And C_2 is a directed cycle of length 4 that is not quasi-monochromatic. ■

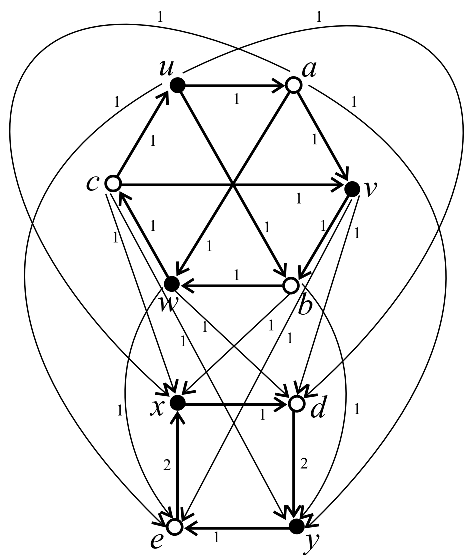

Figure 2

Remark 2.3. For each m there exists an m-coloured Hamiltonian bipartite tournament such that: every directed cycle of length 4 is quasi-monochromatic; every directed cycle of length 6 is monochromatic and D has no subtournament isomorphic to $\widetilde{T}_6$.

Proof. Let $D = (V_1, V_2)$ be the m-coloured bipartite tournament defined as follows:

$$V(D) = \bigcup_{i=1}^{6} X_i \text{ where } X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,m}\},$$

$$V_1 = X_1 \cup X_3 \cup X_5, \quad V_2 = X_2 \cup X_4 \cup X_6,$$

$$A(D) = \bigcup_{i=1}^{5} X'_i \bigcup_{\ell \in \{1,2,3\}} X_\ell^3 \cup X_6^0 \text{ where } X_i^{'} = \{(x_{i,j}, x_{i+1,j}) \,|\, j \in \{1, \ldots, m\}\},$$

$$X_\ell^3 = \{(x_{\ell,j}, x_{\ell+3,j}) \,|\, j \in \{1, \ldots, m\}\}, X_6^0$$

$$= \{(x_{6,i}, x_{1,i+1}) \,|\, i \in \{1, \ldots, m-1\}\} \cup \{(x_{6,m}, x_{1,1})\},$$

where $(x_{1,i}, x_{2,i})$ is coloured i; and any other arc of D is coloured 1 and in any direction. ■

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