SECURE DOMINATION AND SECURE TOTAL DOMINATION IN GRAPHS

WILLIAM F. KLOSTERMEYER

School of Computing
University of North Florida
Jacksonville, FL 32224–2669

e-mail: wkloster@unf.edu

AND

CHRISTINA M. MYNHARDT*

Department of Mathematics and Statistics
University of Victoria, P.O. Box 3060 STN CSC
Victoria, BC, Canada V8W 3R4

e-mail: mynhardt@math.uvic.ca

Abstract

A *secure (total) dominating set* of a graph $G = (V, E)$ is a (total) dominating set $X \subseteq V$ with the property that for each $u \in V - X$, there exists $x \in X$ adjacent to u such that $(X - \{x\}) \cup \{u\}$ is (total) dominating. The smallest cardinality of a secure (total) dominating set is the *secure (total) domination number* $\gamma_s(G)$ ($\gamma_{\mathrm{st}}(G)$). We characterize graphs with equal total and secure total domination numbers. We show that if G has minimum degree at least two, then $\gamma_{\mathrm{st}}(G) \leq \gamma_s(G)$. We also show that $\gamma_{\mathrm{st}}(G)$ is at most twice the clique covering number of G, and less than three times the independence number. With the exception of the independence number bound, these bounds are sharp.

Keywords: secure domination, total domination, secure total domination, clique covering.

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1. Introduction

We consider finite, simple graphs, and unless stated otherwise, denote the number of vertices of the graph $G = (V, E)$ by n. We study the problem of using guards to defend the vertices of G against an attacker. At most one guard is located at each vertex. A guard can protect the vertex at which it is located and can move to a neighboring vertex to defend an attack there. At most one guard is allowed to move in order to defend an attack (other papers have studied the model in which multiple guards can move simultaneously when an attack occurs [13, 21]). This paper deals with the "secure" version of the problem in which the configuration of guards induces a dominating set before and after the attack has been defended.

Several variations of this graph protection problem have been studied, including Roman domination [8, 17], weak Roman domination [9], k-secure sets [5], and eternal m-secure sets [13]. The term Roman domination stems from the problem's ancient origins in Emperor Constantine's efforts to defend the Roman Empire from attackers [18, 23]. Secure domination has been studied previously in [7, 9, 11, 15, 22], for example. Roman domination [8, 17, 23] and weak Roman domination [9, 19] are previously studied protection strategies that allow up to two guards at each vertex to deal with a single attack. The work of [4, 5, 6, 13, 14, 20, 21] (respectively [2, 18]) considers sequences of attacks (as opposed to single attacks) with at most two guards (respectively multiple guards) permitted at a vertex. The term *eternal domination* refers to guarding against infinitely long sequences of attacks and is considered in, for example, [1, 6, 13, 14, 20, 21].

In this paper we compare the sizes of smallest secure dominating sets, secure total dominating sets (in which each guard is adjacent to another guard), and other graph parameters such as the independence number and clique covering number. We formally define these concepts now.

A *total dominating set* (*TDS*) of G is a set $X \subseteq V$ with the property that for each $u \in V$, there exists $x \in X$ adjacent to u. The minimum cardinality amongst all total dominating sets is the *total domination number* $\gamma_t(G)$. Note that this parameter is only defined for graphs without isolated vertices.

A *secure dominating set* (*SDS*) of G is a set $X \subseteq V$ with the property that for each $u \in V - X$,

(1)   there exists $x \in X$ adjacent to u such that $(X - \{x\}) \cup \{u\}$ is dominating.

A *secure total dominating set* (*STDS*) of G is a total dominating set $X \subseteq V$ with the property that for each $u \in V - X$,

(2)   there exists $x \in X$ adjacent to u such that $(X - \{x\}) \cup \{u\}$
         is total dominating.

The minimum cardinality amongst all SDSs (STDSs) is the *secure domination number* $\gamma_s(G)$ (*secure total domination number* $\gamma_{st}(G)$) of G. Note that $\gamma_{st}(G)$ is defined if and only if G has no isolated vertices, and for such graphs, $\gamma_s(G) \le \gamma_{st}(G)$ since any STDS of G is an SDS.

If (1) holds, we say that x *protects* (or X-*protects*, to emphasize X) u, and if (1) holds for all $u \in V - X$, we say that X *protects* G. Similarly, if (2) holds, then x *totally protects* (or *totally X-protects*) u, and if X is a total dominating set and (2) holds for all $u \in V - X$, then X *totally protects* G. The term "defends" is sometimes used in the literature instead of "protects".

In Section 2 we define key terms and outline some of the relevant previous results. Section 3 shows that the secure total domination number of a graph with minimum degree at least two, is at most twice its secure domination number. In Section 4 we first characterize graphs with equal total and secure total domination numbers. We then compare the secure total domination number of a graph with its *clique covering number* $\theta(G)$ (the chromatic number of the complement of G) and its independence number, showing that $\gamma_{st}(G)$ is at most twice the clique covering number and less than three times the independence number $\alpha(G)$. Some open problems are listed in Section 5.

2.   Definitions and Earlier Results

We follow the notation and terminology of [16]. We denote the open and closed neighborhoods of $X \subseteq V$ by $N(X)$ and $N[X]$, respectively, and abbreviate $N(\{x\})$ and $N[\{x\}]$ to $N(x)$ and $N[x]$. The

$$\left. \begin{array}{l} \textit{private neighborhood } \ \mathrm{pn}(x, X) \\ \textit{external private neighborhood } \ \mathrm{epn}(x, X) \\ \textit{internal private neighborhood } \ \mathrm{ipn}(x, X) \end{array} \right\}$$

of $x \in X$ relative to X is defined by

$$\begin{cases} \mathrm{pn}(x,X) &= N[x] - N[X - \{x\}] \\ \mathrm{epn}(x,X) &= \mathrm{pn}(x,X) - \{x\} \\ \mathrm{ipn}(x,X) &= \{w \in X : N(w) \cap X = \{x\}\} \end{cases}$$

and the vertices in these sets are called, respectively, the

$$\left. \begin{array}{l} \textit{private neighbors} \\ \textit{external private neighbors} \\ \textit{internal private neighbors} \end{array} \right\} \text{ of } x \text{ relative to } X.$$

An SDS of G with cardinality $\gamma_s(G)$ is called a γ_s-*set*; a γ_{st}-*set* is defined similarly. The subgraph of G induced by X is denoted by $\langle X \rangle$. When we consider the union of disjoint sets A and B, we sometimes write $A \uplus B$ to emphasize this property.

If $u \in V - X$ is protected by $x \in X$ and by no other vertex in X, then x *uniquely protects* u; otherwise we also say that x *jointly protects* u. If x X-protects $u \in V - X$, it is easy to see that $\mathrm{pn}(x,X) \subseteq N[u]$. For $x \in X$ and $y \in \mathrm{epn}(x,X)$, if x protects y then x uniquely protects y. This leads to the following result of [11].

Proposition 1 [11]. *Let* $X \subseteq V$. *The vertex* $u \in V - X$ *is* X-*protected by* x *if and only if* $\mathrm{epn}(x,X) \cup \{x\} \subseteq N[u]$, *and* X *is an SDS of* G *if and only if for each* $u \in V - X$ *there exists* $x \in X$ *such that*

$$\langle \{u,x\} \cup \mathrm{epn}(x,X) \rangle \text{ is complete.}$$

A similar result from [3] holds for secure total domination.

Proposition 2 [3]. *Let* $Z \subseteq V$ *such that* $\langle Z \rangle$ *has no isolated vertices. The vertex* z *totally* Z-*protects* $u \in V - Z$ *if and only if*

(3) $$\mathrm{epn}(z,Z) = \emptyset \text{ and}$$

(4) $$\{z\} \cup \mathrm{ipn}(z,Z) \subseteq N(u),$$

and Z *is an STDS of* G *if and only if* (3) *holds for each* $z \in Z$, *and for each* $u \in V - Z$ *there exists* $z \in Z$ *such that* (4) *holds.*

There exist graphs whose only STDS is the vertex set of the graph. These graphs were characterized in [3]. Denote the set of leaves of a graph by L, and the set of support vertices (vertices adjacent to leaves) by S.

Theorem 3 [3]. *For any graph G, $\gamma_{\mathrm{st}}(G) = n$ if and only if $V - (L \cup S)$ is independent.*

Let $\alpha(G)$ denote the independence number of G. As shown in [9], if G is claw-free, then $\gamma_s(G) \leq 3\alpha(G)/2$, and if, in addition, G is C_5-free, then $\gamma_s(G) \leq \alpha(G)$. In general, though, $\gamma_s(G) \leq 2\alpha(G)$. To see this, let A be a maximum independent set of G. For each $a \in A$, $\langle \mathrm{pn}(a, A) \rangle$ is complete, otherwise G has a larger independent set than A. If $\mathrm{epn}(a, A) \neq \emptyset$, choose arbitrary $x_a \in \mathrm{epn}(a, A)$. Define $X = A \cup \{x_a : a \in A \text{ and } \mathrm{epn}(a, A) \neq \emptyset\}$. Since A is a dominating set, $\mathrm{epn}(x, X) = \emptyset$ for all $x \in X$, and X is an SDS by Proposition 1.

We state this result for referencing and note that we do not know of any graphs attaining this bound.

Proposition 4. *For any graph G, $\gamma_s(G) \leq 2\alpha(G)$.*

A graph G is γ_s-*edge-removal-critical*, abbreviated to γ_s-*ER-critical*, if $\gamma_s(G - e) > \gamma_s(G)$ for all edges e of G. Any graph G contains a γ_s-ER-critical graph H with $\gamma_s(H) = \gamma_s(G)$ as spanning subgraph — simply remove edges from the graph until the removal of any further edge changes the secure domination number. This class of graphs was characterized in [15]. To state this characterization, which we use in Section 3 to establish a bound for the ratio $\gamma_{\mathrm{st}}/\gamma_s$, we define notation that is used throughout. For any $X \subseteq V$, define

$$P = \bigcup_{x \in X} \mathrm{epn}(x, X) \quad \text{and}$$
$$Y = V - (X \cup P).$$

Also define the subsets $X_1, \dots, X_4$ of X as follows:

$X_1 = \{x \in X : N(x) \cap Y = \emptyset\}$,
$X_2 = \{x \in X - X_1 : x \text{ does not } X\text{-protect any vertex in } Y\}$,
$X_3 = \{x \in X - X_1 : x \ X\text{-protects some but not all vertices in } N(x) \cap Y\}$,
$X_4 = \{x \in X - X_1 : x \ X\text{-protects all vertices in } N(x) \cap Y\}$.

Then $X = \biguplus_{i=1}^{4} X_i$; possibly $X_i = \emptyset$ for some i. All this is depicted in Figure 1. Define

$$U_x = \{y \in Y : x \text{ uniquely } X\text{-protects } y\},$$

$$Y_{xx'} = \{y \in Y : x, x' \in X \text{ jointly protect } y\},$$

and note that $U_x \cap U_{x'} = \emptyset$ if $x \neq x'$. The characterization of γ_s-ER-critical graphs follows.

Theorem 5 [15]. *The graph G is γ_s-ER-critical if and only if for every γ_s-set X of G,*

 (i) *X and Y are independent,*

 (ii) *every $y \in Y$ has precisely two neighbors in X,*

(iii) *if $x \in X$ protects a vertex in Y (i.e., $x \in X_3 \cup X_4$), then $|U_x| \geq 2$,*

(iv) *the only edges in P are in $\langle \operatorname{epn}(x, X) \rangle$, $x \in X$,*

 (v) *the only edges between Y and P are between $\operatorname{epn}(x, X)$ and $y \in Y$ protected by x,*

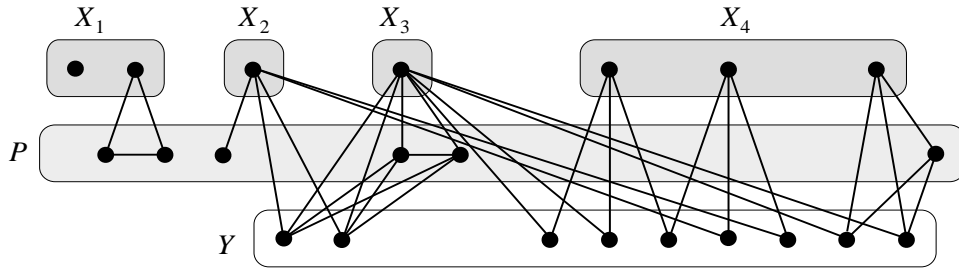(vi) *if $x \in X$ jointly protects a vertex in Y, then $\operatorname{epn}(x, X) = \emptyset$.*

Figure 1. A γ_s-set $X = X_1 \cup X_2 \cup X_3 \cup X_4$ in a γ_s-ER-critical graph.

Let X be a γ_s-set of a γ_s-ER-critical graph. We state some other properties of the sets X_i for future reference. See Figure 1.

Remark 6 [15].

 (i) If $x \in X_1$, then $\langle N[x] \rangle$ is a complete component of G.

 (ii) If $x \in X_2$, then $\operatorname{epn}(x, X) \neq \emptyset$ and $\langle \{x\} \cup \operatorname{epn}(x, X) \rangle$ is complete and an end-block of G.

(iii) If $x \in X_3$, then $\mathrm{epn}(x, X) \neq \emptyset$ and every vertex in U_x is adjacent to every vertex in $\mathrm{epn}(x, X)$.

(iv) If $y \in Y_{xx'}$, then $x, x' \in X_4$ and $\deg_G y = 2$.

3.   SECURE DOMINATION VERSUS SECURE TOTAL DOMINATION

For graphs containing leaves, the ratio $\gamma_{\mathrm{st}}(G)/\gamma_s(G)$ can be arbitrarily large. For example, let G be the graph obtained from $K_{2,n}$, $n \geq 1$, by adding a pendant edge at each vertex of degree n (or at two nonadjacent vertices of C_4 if $n = 2$). Then $\gamma_s(G) = 3$ (use one leaf and the two vertices of $K_{2,n}$ of degree n) and $\gamma_{\mathrm{st}}(G) = |V(G)| = n + 4$ by Theorem 3. The ratio $\gamma_{\mathrm{st}}(G)/\gamma_s(G)$ can also be arbitrarily close to 1, because $\gamma_{\mathrm{st}}(K_{1,n}) = n + 1$ and $\gamma_s(K_{1,n}) = n$. However, if $\delta(G) = 1$, then these two parameters are never equal, as we show next.

Proposition 7. *If $\delta(G) = 1$, then $\gamma_s(G) < \gamma_{\mathrm{st}}(G)$.*

Proof. Let ℓ and s be a leaf and adjacent support vertex, respectively, of G, and Z any STDS of G. Then $\ell, s \in Z$ and s does not totally protect any vertex of G, otherwise ℓ is isolated in the resulting set. Therefore each $v \in N(s) - Z$ is adjacent to some vertex in $Z - \{s\}$. Let $X = Z - \{\ell\}$. Each vertex in $V - Z$ is X-protected by the same vertex that totally Z-protects it, and ℓ is X-protected by s. Therefore X is an SDS of G and $\gamma_s(G) \leq \gamma_{\mathrm{st}}(G) - 1$. ∎

The situation for graphs without leaves is completely different. We now use the properties of γ_s-ER-critical graphs to bound $\gamma_{\mathrm{st}}(G)$ in terms of $\gamma_s(G)$ for graphs with minimum degree at least two.

Theorem 8. *For any graph G with $\delta(G) \geq 2$, $\gamma_s(G) \leq \gamma_{\mathrm{st}}(G) \leq 2\gamma_s(G)$, and both bounds are sharp.*

Proof. As mentioned before, the lower bound holds. To prove the upper bound, let H' be a spanning γ_s-ER-critical subgraph of G with $\gamma_s(H') = \gamma_s(G)$ and note that $\gamma_{\mathrm{st}}(H') \geq \gamma_{\mathrm{st}}(G)$. Let H be the subgraph of H' consisting of all nontrivial components and consider any SDS X of H. We define a set Z with $X \subseteq Z$ below and show that Z is an STDS of H.

If $\mathrm{epn}(x, X) \neq \emptyset$, let $u_x \in \mathrm{epn}(x, X)$ and $u_x, x \in Z$. If $\mathrm{epn}(x, X) = \emptyset$, then by Remark 6 and the construction of H, $x \in X_4$ and x uniquely protects

some $y_x \in Y$. Let $x, y_x \in Z$ to complete the set Z. See Figure 2. Note that $|Z| = 2|X|$. We use Proposition 2 to show that Z is an STDS of H.
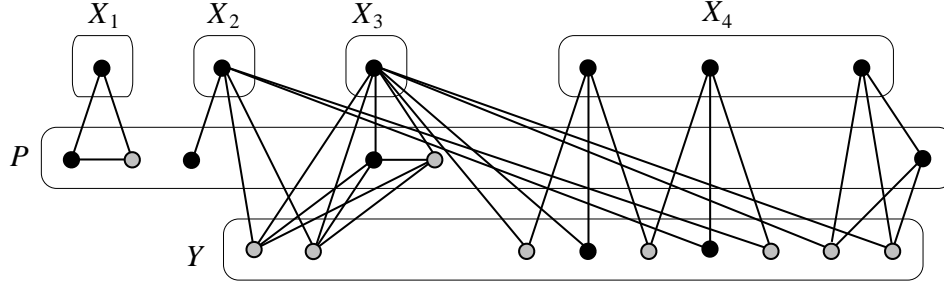

Figure 2. The set Z (denoted by black vertices).

Since X dominates H, Z dominates H, and since $\langle Z \rangle$ has no isolated vertices, Z is a TDS of H. Consider any $z \in Z$. If $z \in X$, then either $\mathrm{epn}(z, X) = \emptyset$ and thus $\mathrm{epn}(z, Z) = \emptyset$, or there exists $u_z \in \mathrm{epn}(z, X) \cap Z$, and again $\mathrm{epn}(z, Z) = \emptyset$. If $z \in V - X$, then $\mathrm{epn}(z, Z) = \emptyset$ because X dominates H. Thus (3) of Proposition 2 holds for each $z \in Z$.

Consider $v \in V - Z$. If $v \in \mathrm{epn}(x, X)$, then $\mathrm{ipn}(x, Z) = \{u_x\}$, where $u_x \in \mathrm{epn}(x, X)$. But $\langle \{x\} \cup \mathrm{epn}(x, X) \rangle$ is complete by Proposition 1. Therefore $\{x\} \cup \mathrm{ipn}(x, Z) \subseteq N(v)$.

Suppose $v \in Y$. By Theorem 5(ii), y is adjacent to exactly two vertices $x, x' \in X$, at least one of which, say x, X-protects v. If $\mathrm{epn}(x, X) \neq \emptyset$, then there exists $u_x \in \mathrm{epn}(x, X) \cap Z$, in which case $\mathrm{ipn}(x, Z) = \{u_x\}$, $u_x v \in E(H)$ and $\{x\} \cup \mathrm{ipn}(x, Z) \subseteq N(v)$. If $\mathrm{epn}(x, X) = \emptyset$, then $N(x) \cap Z = \{y_x\}$. But y_x is also adjacent to another vertex in X, hence $\mathrm{ipn}(x, Z) = \emptyset$. Obviously, $\{x\} \cup \mathrm{ipn}(x, Z) \subseteq N(v)$. Thus (4) of Proposition 2 holds for each $v \in V - Z$. It follows that Z is a STDS of H, and so $\gamma_{\mathrm{st}}(H) \leq |Z| = 2\gamma_s(H)$.

We now return to the graph G. If $H' = H$, then $\gamma_{\mathrm{st}}(G) \leq \gamma_{\mathrm{st}}(H) = 2\gamma_s(H) = 2\gamma_s(G)$ and we are done. Hence suppose $W = \{w : w$ is isolated in $H'\} \neq \emptyset$ and note that $X' = X \cup W$ is an SDS of H'. Since $\delta(G) \geq 2$, any $w \in W$ is adjacent to at least two vertices of G. Construct the set Z' as follows. Let $Z \cup W \subseteq Z'$, and for any $w \in W$, if w is adjacent to at most one vertex in Z, choose any $r_w \in N_G(w) - Z$ and let $r_w \in Z'$. We show that Z' is an STDS of G.

For any $z \in Z$, $\mathrm{epn}_G(z, Z') = \emptyset$. Also, each neighbor of $w \in W$ in $G - Z'$ is adjacent to a vertex in X, hence $\mathrm{epn}_G(w, Z') = \emptyset$, and similarly, if $r_w \in Z'$, then $\mathrm{epn}_G(r_w, Z') = \emptyset$. Hence $\mathrm{epn}_G(z', Z') = \emptyset$ for each $z' \in Z'$.

If w is adjacent to $z \in Z$, then $w \notin \mathrm{ipn}_G(z, Z')$ because w is adjacent either to some other vertex in Z or to $r_w \in Z'$. Moreover, $r_w \notin \mathrm{ipn}_G(z, Z')$ for any $z \in Z'$ because r_w is adjacent to w as well as to some vertex $x \in X$. It follows that for each $z \in Z$, $\mathrm{ipn}_G(z, Z') \subseteq \mathrm{ipn}_H(z, Z)$ and thus each vertex in $G - Z'$ is totally Z'-protected by the same vertex that totally Z-protects it in H. Hence Z' is an STDS of G. Since $|Z'| \leq 2|X'|$, the bound follows.

To show equality in the lower bound, consider the even cycle C_{2m} with vertex sequence $W = \{1, 2, \ldots, 2m\}$. For each odd integer $i \in W$, join vertices i and $i + 1$ to two new vertices u_i and v_i to form the graph G. See Figure 3.
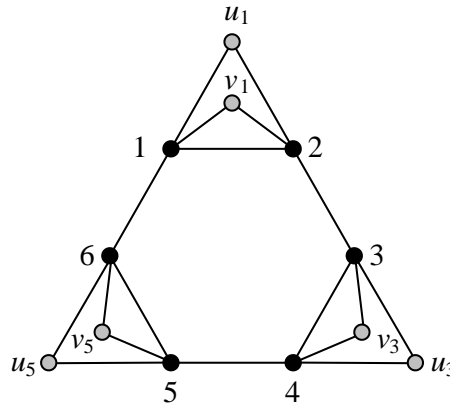

Figure 3. A graph with $\gamma_s = \gamma_{\mathrm{st}} = 6$.

It is easy to see that W is an STDS of G, hence $\gamma_{\mathrm{st}}(G) \leq 2m$. Suppose G has an SDS X with $|X| < 2m$. By the pigeonhole principle, $|\{i, i+1, u_i, v_i\} \cap X| \leq 1$ for some odd i, and since u_i is dominated, $|\{i, i+1, u_i, v_i\} \cap X| = 1$. We may then assume that $\{i, i+1, u_i, v_i\} \cap X = \{i\}$. But $(X - \{i\}) \cup \{u_i\}$ does not dominate v_i, a contradiction. Thus $\gamma_s(G) = \gamma_{\mathrm{st}}(G) = 2m$. (Note: to construct a graph with $\gamma_s = \gamma_{\mathrm{st}} = 2$, use K_2 instead of C_{2m}.)

To show equality in the upper bound, let G be an isolate-free γ_s-ER-critical graph with SDS X such that no vertex in X jointly protects any vertex in Y. (These graphs are easy to construct from Theorem 5; a constructive characterization is also given in [15].) By Theorem 5(i), X is

independent. For each $x \in X$, define $A_x = \text{pn}(x, X) \cup U_x$ and note that $V(G) = \uplus_{x \in X} A_x$.

Suppose G has an STDS Z with $|Z| \leq 2\gamma_s(G) - 1 = 2|X| - 1$. Then $|Z \cap A_x| \leq 1$ for some $x \in X$. By Remark 6(i), $x \notin X_1$.

Suppose $x \in X_i$, $i \neq 1$, and $\text{epn}(x, X) \neq \emptyset$. By Theorem 5 and Remark 6, for each $v \in \text{epn}(x, X)$, $N[v] = \text{pn}(x, X) \cup \{y \in Y : x \text{ } X\text{-protects } y\}$. Since no vertex in X jointly protects any vertex in Y, $\{y \in Y : x \text{ } X\text{-protects } y\} = U_x$ and thus $N[v] = A_x$. To dominate v, $|Z \cap A_x| = 1$. But if $v \in Z$, then v is isolated in $\langle Z \rangle$, and if $u \in Z$ for some $u \in A_x - \{v\}$, then $v \in \text{epn}(u, Z)$, both of which are contradictions.

Thus assume $\text{epn}(x, X) = \emptyset$. By Remark 6, $x \in X_4$ and by definition of X_4, x X-protects all its neighbors in Y, i.e., $N[x] \subseteq A_x$. Either $x \in Z$, in which case x is isolated in $\langle Z \rangle$, or $u \in Z$ for some $u \in A_x - \{x\}$, in which case $x \in \text{epn}(u, Z)$, contradicting (3) of Proposition 2.

Therefore $\gamma_{\text{st}}(G) \geq 2\gamma_s(G)$ and equality follows from the upper bound. $\blacksquare$

4.   Total Domination, Secure Total Domination, Clique Covers and Independence

Obviously, $\gamma_t(G) \leq \gamma_{\text{st}}(G)$ for all graphs G without isolated vertices. The two parameters can differ considerably. For example, $\gamma_t(K_{1,m}) = 2$ and $\gamma_{\text{st}}(K_{1,m}) = m + 1$. We begin this section by characterizing graphs for which $\gamma_t = \gamma_{\text{st}}$. For $n \geq 1$, let $J_{2,n}$ be the graph obtained from $K_{2,n}$ by joining the two vertices of degree n (or two nonadjacent vertices of C_4 if $n = 2$).

Theorem 9. *If G is connected, then $\gamma_{\text{st}}(G) = \gamma_t(G)$ if only if $\gamma_{\text{st}}(G) = 2$, i.e., if and only if $G = K_2$ or $J_{2,n}$ is a spanning subgraph of G for some $n \geq 1$.*

Proof. If $G = K_2$ or G has $J_{2,n}$ as spanning subgraph, then obviously $\gamma_{\text{st}}(G) = \gamma_t(G) = 2$.

Suppose $\gamma_{\text{st}}(G) = 2$ and let $X = \{x, y\}$ be an STDS of G. Then $xy \in E(G)$, and by Proposition 2, $\text{epn}(x, X) = \text{epn}(y, X) = \emptyset$. Hence each vertex in $V - X$ is adjacent to both x and y, from which it follows that $G = K_2$ or $J_{2,n}$ is a spanning subgraph of G.

Now suppose $|V| \geq 4$, $\gamma_{\text{st}}(G) = \gamma_t(G)$ but $J_{2,n}$ is not a spanning subgraph of G. Then $\gamma_{\text{st}}(G) \geq 3$. Assume without loss of generality that G is

edge-removal critical with respect to secure total domination, i.e., removing any edge increases γ_{st}. Let X be an STDS of G. By Proposition 2, each vertex in $Y = V - X$ is adjacent to at least two vertices in X. Since X is a TDS, the subgraph $\langle X \rangle$ of G induced by X has no isolated vertices. Furthermore, by the criticality of G, Y is independent.

If $\langle X \rangle$ has a component C of size at least three, let w be a vertex of C which is not a cutvertex of C. Then $\langle X - \{w\} \rangle$ has no isolated vertices. Moreover, since each vertex in Y is adjacent to at least two vertices in X, $X - \{w\}$ is a TDS of G, contradicting $\gamma_t(G) = \gamma_{\mathrm{st}}(G)$. Hence we may assume that $\langle X \rangle = mK_2$ for some $m \geq 2$.

Let $x_i y_i$, $i = 1, 2$, be two edges of $\langle X \rangle$ such that some vertex $u \in Y$ is adjacent to (without loss of generality) x_1 and x_2. These edges exist because G is connected. Note that $\mathrm{ipn}(y_i, X) = \{x_i\}$. If there exists a vertex $u' \in Y$ such that $N(u') \cap X = \{y_1, y_2\}$, then neither y_1 nor y_2 satisfies (4), contradicting Proposition 2. Hence

(5)   each vertex in Y is adjacent to some vertex in $X - \{y_1, y_2\}$.

Let $X' = (X - \{y_1, y_2\}) \cup \{u\}$. Then $\langle X' \rangle$ has no isolated vertices. Moreover, x_i dominates y_i, and by (5), X' dominates Y. Hence X' is a TDS of G with $|X'| < |X|$. The result follows from this final contradiction. ∎

We now focus on comparing γ_t and γ_{st} to the clique covering number $\theta(G)$ and the independence number $\alpha(G)$, as these parameters are useful upper and lower bounds, respectively, on the eternal domination number [13]. It follows that $\gamma_s(G) \leq \theta(G)$ for all graphs G and there are many graphs for which this bound is sharp, such as $K_n, K_{1,n}, P_4, C_4$, to name just a few.

The total domination number has been related to clique covers in [10] and [12]. A graph G is K_r-*covered* if every vertex of G is contained in a clique of size r. It was conjectured in [10] that $\gamma_t(G) \leq \frac{2n}{r+1}$ whenever G is K_r-covered, and the conjecture was proved there for $r = 3$ and 4, and in [12] for $r = 5$ and 6.

We begin by relating γ_t and θ. Note that $\gamma_t(G) = 2$ when $\theta(G) = 1$. For any graph G, fix a minimum clique cover $\mathcal{C}$ of G. Construct the *clique cover graph* $\mathcal{C}(G)$ *of* G *with respect to* $\mathcal{C}$ by mapping each clique in $\mathcal{C}$ to a corresponding vertex in $\mathcal{C}(G)$ such that two vertices in $\mathcal{C}(G)$ are adjacent if and only if the corresponding cliques in G have adjacent vertices.

Proposition 10. *If G is connected and $\theta(G) \geq 2$, then $\gamma_t(G) \leq 2\theta(G) - 2$, and the bound is sharp.*

Proof. The case $\theta(G) = 2$ is trivial, so assume $\theta(G) \geq 3$. Fix a minimum clique cover $\mathcal{C}$ of G and consider a spanning tree T of $\mathcal{C}(G)$. Note that the vertex set of T is also a TDS of $\mathcal{C}(G)$ since $\theta(G) > 1$. Map $V(T)$ to a dominating set D of G as follows. Fix a root r of T to be a vertex of degree greater than one (such a vertex exists since T has at least three vertices). Let d be the maximum distance from any vertex in T to r. First process the vertices of distance d from r, followed by vertices of distance $d - 1$ and so on until the root is reached.

If a vertex $v \neq r$ is a vertex of T with parent u, it is mapped to two vertices in D: a vertex v_1 in the clique of G corresponding to v and a vertex v_2 in the clique corresponding to u, where $v_1 v_2 \in E(G)$. Note that, depending on the number of children a vertex has, it is possible that all vertices in the corresponding clique in G are included in D (or a particular vertex of G might be mapped to more than once). When r is processed, no additional vertices are added to D. It is easy to see that $|D| \leq 2\theta - 2$. A simple induction on the height of T shows that D is a TDS of G.

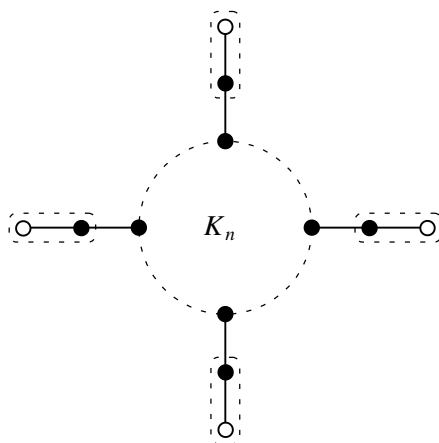The graph in Figure 4 shows that the bound is sharp for all $\theta \geq 2$.   ∎

Figure 4. A graph with $\theta = n + 1$ and $\gamma_t = 2n$.

Since the subgraph induced by the TDS D constructed in the proof of Proposition 10 is connected, the bound also holds for the connected domination number (see [16]).

It is clear that $\gamma_{st}(G) - \theta(G)$ can also be arbitrarily large. For example, if $P_{n,1}$ is the caterpillar obtained from P_n by attaching a pendant vertex to each

vertex of P_n, then $\theta(P_{n,1}) = \gamma_t(P_{n,1}) = n$ and $\gamma_{st}(P_{n,1}) = 2n$ (Theorem 3). Another example is the graph in Figure 4 with $\theta = n + 1$, $\gamma_t = 2n$ and $\gamma_{st} = 2n + 1$.

The difference $\gamma_{st}(G) - \theta(G)$ may even be negative. Let W_n denote the wheel with $n \geq 4$ spokes (obtained by joining a new vertex to every vertex of C_n). It is easy to see that $\theta(W_n) = \theta(C_n) = \left\lceil \frac{n}{2} \right\rceil$, while $\gamma_{st}(W_n) = \left\lceil \frac{n}{3} \right\rceil + 1$ (use the hub vertex and every third vertex on C_n).

We show next that the ratio γ_{st}/θ is bounded.

Theorem 11. *For all graphs G without isolated vertices, $\gamma_{st}(G) \leq 2\theta(G)$, and the bound is sharp.*

Proof. The result is obvious if $\theta(G) = 1$, so assume $\theta(G) \geq 2$. Also assume without loss of generality that G is connected. Construct an STDS D of G as follows. Fix a minimum clique cover $\mathcal{C}$. We process the cliques of $\mathcal{C}$ in two phases: (1) cliques of size one and (2) cliques of size greater than one. Note that if $\{v\}$ and $\{w\}$ are cliques of size one in $\mathcal{C}$, then $vw \notin E(G)$, for otherwise $(\mathcal{C} - \{\{v\}, \{w\}\}) \cup \{v, w\}$ is a smaller clique cover of G.

For each vertex v such that $\{v\} \in \mathcal{C}$, add v to D. Since $\theta(G) \geq 2$ and G is connected, v is adjacent to a vertex w in a clique of size greater than one in $\mathcal{C}$; add w to D, if possible (w may already have been added to D when another clique of size one was considered).

For the second phase, place two additional vertices from each clique in $\mathcal{C}$ of size at least two, in D, if possible. [If C_i is a clique of size $r \geq 2$, it is possible that r or $r - 1$ of its vertices were added to D during the first phase. Then during the second phase we add no vertex or the last remaining vertex, respectively, of C_i to D.] It follows that $|D| \leq 2\theta$. It is easy to see that D is a TDS.

To see that D is an STDS, let $u \in V - D$ and say $u \in C_i \in \mathcal{C}$. By the construction of D, C_i contains at least two vertices $z, z' \in D$ that are not adjacent to cliques of size one in $\mathcal{C}$. Then $\text{ipn}(z, D) \subseteq \{z'\}$ and $\{z\} \cup \text{ipn}(z, D) \subseteq N(u)$, hence (4) holds. Since (3) holds for each $z \in D$, the result follows from Proposition 2.

The caterpillar $P_{n,1}$ mentioned above shows that the bound is sharp for all θ.   ∎

We now compare γ_{st} and the independence number α. Observe that $\gamma_{st}(K_{n,n}) = 4 < \alpha(K_{n,n}) = n$ when $n > 4$. Of course, $\gamma_{st}(K_n) = 2 > \alpha(K_n) = 1$, and P_4 is another example of a small graph G with $\alpha(G) = 2$ and $\gamma_{st}(G) = 4 >$

$\alpha(G)$. It is an easy corollary of Proposition 4 and Theorem 8 that $\gamma_{\mathrm{st}}(G) \leq 4\alpha(G)$ for graphs G with $\delta(G) \geq 2$. Furthermore, since $\gamma_s(G) \leq 3\alpha(G)/2$ for claw-free graphs [9], Theorem 8 also implies that $\gamma_{\mathrm{st}}(G) \leq 3\alpha(G)$ for claw-free graphs with $\delta(G) \geq 2$. We now extend a slight improvement of the latter bound to all graphs.

Theorem 12. *For all graphs G without isolated vertices, $\gamma_{\mathrm{st}}(G) \leq 3\alpha(G)-1$.*

Proof. Assume without loss of generality that G is connected. Let X be a maximum independent set (and thus a dominating set) that contains all end-vertices of G, and note that $\langle \mathrm{pn}(x,X) \rangle$ is complete for each $x \in X$ (otherwise there is a larger independent set than X).

Let $X' = \{x \in X : \mathrm{epn}(x,X) = \emptyset\}$ and define the set X^* by

$$X^* = \begin{cases} X - X' & \text{if } X' \neq \emptyset, \\ X - \{x^*\} & \text{for an arbitrary, fixed } x^* \in X \text{ if } X' = \emptyset. \end{cases}$$

Construct an STDS Z as follows. Let $X \subseteq Z$. For each $x \in X'$, add one neighbor y_x of x to Z; note that $y_x \in Y = V - X$. For each $x \in X^*$, if x has at least two external private neighbors, then add two private neighbors u_x, v_x of x to Z. If $x \in X^*$ has one external private neighbor u_x, add u_x to Z. By the choice of X, $\deg u_x \geq 2$. Choose $w_x \in N(u_x) - \{x\}$ and let $w_x \in Z$. Finally, if x^* is defined, add any $u_{x^*} \in \mathrm{epn}(x^*, X)$ to Z. Clearly, $|Z| < 3|X|$.

By construction, $\mathrm{epn}(z, Z) = \emptyset$ for each $z \in Z$. Thus (3) in Proposition 2 holds for each $z \in Z$. We also assert that for each $x \in X - \{x^*\}$ and each vertex $z \in Z - X$ adjacent to x, z is adjacent to some vertex in $Z - \{x\}$: if $z = v_x$, then z is adjacent to u_x; if $z = u_x$, then z is adjacent to v_x or w_x; if $z = w_{x'}$ for some $x' \in X$, then z is adjacent to $u_{x'}$; and if $z = y_{x'}$ for some $x' \in X$, then z is adjacent to at least two vertices in X. It follows that

(6) $$\mathrm{ipn}(x, Z) = \emptyset \text{ for each } x \in X - \{x^*\},$$

while $\mathrm{ipn}(x^*, Z) \subseteq \{u_{x^*}\}$.

Consider any $a \in V - Z$. Since X is a dominating set, a is adjacent to some $x \in X$. If $a \in \mathrm{epn}(x^*, X)$, then x is adjacent to both x^* and u_{x^*} and (4) holds for a and x^*. If $a \notin \mathrm{epn}(x^*, X)$, then either $a \in \mathrm{epn}(x, X)$ for $x \in X - \{x^*\}$, or $a \in Y$, in which case a has at least two neighbors in X, one of which is $x \neq x^*$, and thus (6) implies that (4) holds for a and x. By Proposition 2, Z is an STDS and the result follows. ∎

For $\alpha = 2$, the bound in Theorem 12 can be improved.

Theorem 13. *Let G be an isolate-free graph with $\alpha(G) = 2$. Then $\gamma_{st}(G) \leq 4$.*

Proof. Let $X = \{x, y\}$ be an independent set (and thus a dominating set) of G and let $W = N(x) \cap N(y)$. If $epn(x, X) = epn(y, X) = \emptyset$, then $W = V - X$. If $G = P_3$, let $Z = V(G)$; otherwise, let u and v be any vertices in $V - X$ and $Z = \{u, v, x, y\}$. It is easy to see that Z is an STDS.

Suppose $epn(x, X) = \emptyset$ and $epn(y, X) \neq \emptyset$. Then $\langle epn(y, X) \rangle$ is complete. If x and y have only one common neighbor, then $\theta(G) = 2$ and the result follows from Theorem 11. Assume they have at least two common neighbors. Let $u \in N(x)$, $v \in epn(y, X)$ and $Z = \{u, v, x, y\}$. Then $epn(z, Z) = \emptyset$ for each $z \in Z$. Consider $w \in V - Z$. If $w \in epn(y, X)$, then y totally Z-protects w, so assume $w \in W$. If $uw \in E(G)$, then u totally protects w. If $uw \notin E(G)$, then at least one of u and w is adjacent to v, since $\alpha(G) = 2$. If $vw \in E(G)$, then v totally protects w, and if $uv \in E(G)$, then y totally protects w. Hence Z is an STDS.

Suppose $epn(x, X) \neq \emptyset$ and $epn(y, X) \neq \emptyset$. Then both these sets induce complete subgraphs of G. Let $u \in epn(x, X)$, $v \in epn(y, X)$ and $Z = \{u, v, x, y\}$, and consider $w \in V - Z$. If $w \in epn(x, X)$, then x totally Z-protects w; likewise if $w \in epn(y, X)$. Hence assume $w \in W$ and note that u, x, w, y, v is a path in G. If, without loss of generality, $uw \in E(G)$, then u totally protects w. If w is not adjacent to either u or v, then $uv \in E(G)$ because $\alpha(G) = 2$. In this case, x totally protects w and we are done.   ∎

Corollary 14. *There exists a graph G with $\gamma_{st}(G) < \theta(G) - c$, for any constant c.*

Proof. Consider a graph G with $\alpha(G) = 2$, $\delta(G) > 1$, and $\theta(G) = t$, for sufficiently large $t > 2$ (for example, the complement of a triangle-free graph of large chromatic number).   ∎

5.   Problems for Future Research

1. It was shown in [9] that if G is claw-free, then $\gamma_s(G) \leq 3\alpha(G)/2$, and if, in addition, G is C_5-free, then $\gamma_s(G) \leq \alpha(G)$. As stated in Proposition 4,

in general $\gamma_s(G) \leq 2\alpha(G)$. Find graphs for which this bound is exact, or improve the bound.

2. By Proposition 10, $\gamma_t(G) \leq 2\theta(G) - 2$ if G is connected and $\theta(G) \geq 2$. Although this bound is sharp, the bound is not very good in general. For example, $\theta(K_{1,n}) = n$, but $\gamma_t(K_{1,n}) = 2$.

If G is the graph in Figure 4, then G has a unique minimum clique cover, the clique cover graph of G is the star $K_{1,n}$, and $\gamma_t(G) = 2\theta(G) - 2$. On the other hand, the spider $S(2, \ldots, 2)$ obtained by subdividing each edge of $K_{1,n}$ once, also has a unique minimum clique cover, and its clique cover graph is also $K_{1,n}$, but $\gamma_t(G) = \theta(G) = n + 1$. Compare this with the caterpillar $P_{n,1}$, which also has a unique minimum clique cover. Its clique cover graph is P_n, and $\gamma_t(P_{n,1}) = \theta(P_{n,1}) = n$.

Is there a relationship between the properties of the clique cover graph of G (number of vertices corresponding to cliques of size 1, connectivity, number of leaves, toughness, (total) domination number) and the difference $2\theta(G) - \gamma_t(G)$? For which classes of graphs can this bound be improved?

3. Similarly, for which classes of graphs can the bound in Theorem 11 be improved? For which classes of graphs is the bound close to γ_{st}?

4. The bound in Theorem 12 is probably weak in general and needs to be improved. Are there any graphs that achieve the bound of $3\alpha - 1$? We suspect the answer is negative and that 2 is the maximum ratio that can be attained, as in the case of the caterpillar $P_{n,1}$ and when $\alpha = 2$ (Theorem 13).

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