Discussiones Mathematicae
Graph Theory 28 (2008) 179–184

A CANCELLATION PROPERTY FOR THE DIRECT PRODUCT OF GRAPHS

Richard H. Hammack

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284–2014, USA

e-mail: rhammack@vcu.edu

Abstract

Given graphs A, B and C for which $A \times C \cong B \times C$, it is not generally true that $A \cong B$. However, it is known that $A \times C \cong B \times C$ implies $A \cong B$ provided that C is non-bipartite, or that there are homomorphisms from A and B to C. This note proves an additional cancellation property. We show that if B and C are bipartite, then $A \times C \cong B \times C$ implies $A \cong B$ if and only if no component of B admits an involution that interchanges its partite sets.

Keywords: graph products, graph direct product, cancellation.

2000 Mathematics Subject Classification: 05C60.

1. Introduction

Denote by Γ_0 the class of graphs for which vertices are allowed to have loops. The *direct product* of two graphs A and B in Γ_0 is the graph $A \times B$ whose vertex set is the Cartesian product $V(A) \times V(B)$ and whose edges are all pairs $(a,b)(a',b')$ with $aa' \in E(A)$ and $bb' \in E(B)$. By interpreting aa', bb' and $(a,b)(a',b')$ as directed arcs from the left to the right vertex, the direct product can also be understood as a product on digraphs. In fact, since any graph can be identified with a symmetric digraph (where each edge is replaced by a double arc) the direct product of graphs is a special case of the direct product of digraphs. However, except where digraphs are needed in one proof, we restrict our attention to graphs.

The direct product obeys a limited cancellation property. Lovász [4] proved that if C is not bipartite, then $A \times C \cong B \times C$ if and only if $A \cong B$. He also proved cancellation holds if C is arbitrary but there are homomorphisms $A \to C$ and $B \to C$. Since such homomorphisms exist if both A and B are bipartite (and C has at least one edge) then cancellation can fail only if C is bipartite and A and B are not both bipartite. Failure of cancellation can thus be divided into two cases, both involving a bipartite factor C. On one hand it is possible for cancellation to fail if A and B are both non-bipartite. For example, if $A = K_3$ and B is the path of length two with loops at each end, then $A \times K_2$ and $B \times K_2$ are both isomorphic to the 6-cycle, but $A \not\cong B$. On the other hand, cancellation can fail if only one of A and B is bipartite. Figures 1(a) and 1(b) show an example. In those figures, A consists of two copies of an edge with loops at both ends, B is the four-cycle, and C is the path of length 2. The figures show that $A \times C \cong B \times C$, but clearly $A \not\cong B$.

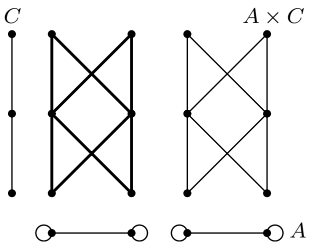

Figure 1(a)

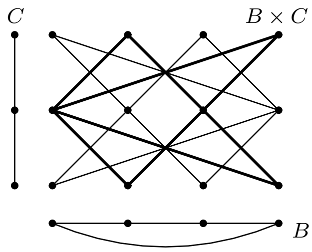

Figure 1(b)

This note is concerned with the second case. We describe the exact conditions a bipartite graph B must meet in order for $A \times C \cong B \times C$ to imply $A \cong B$. Specifically, we prove that if B and C are both bipartite, then $A \times C \cong B \times C$ necessarily implies that $A \cong B$ if and only if no component of B admits an involution (that is an automorphism of order two) that interchanges its partite sets. Figure 1 can be taken as an illustration of this. The 4-cycle B in Figure 1(b) has an involution that interchanges its partite sets (reflection across the vertical axis) and indeed cancellation fails. Our result will imply that if a bipartite graph B does not have this kind of symmetry (or more precisely if no component of B has such symmetry) then $A \times C \cong B \times C$ will guarantee that $A \cong B$. Conversely, if some component of B has a bipartition-reversing involution, then there is a graph A with $A \times C \cong B \times C$ but $A \not\cong B$.

The reader is assumed to be familiar with the basic properties of direct products, including Weichsel's theorem on connectivity. See Chapter 5 of [3] for an excellent survey.

2. Results

In what follows, let $V(K_2) = \{0, 1\}$. For $\varepsilon \in V(K_2)$, set $\overline{\varepsilon} = 1 - \varepsilon$, so $\overline{1} = 0$ and $\overline{0} = 1$. An *involution* of a graph is an automorphism β for which β^2 is the identity. Recall that if G is a connected non-bipartite graph, then $G \times K_2$ is a connected bipartite graph, and $(g, \varepsilon) \mapsto (g, \overline{\varepsilon})$ is an involution of $G \times K_2$ that interchanges the partite sets $V(G) \times \{0\}$ and $V(G) \times \{1\}$. By contrast, if G is bipartite, then $G \times K_2 \cong 2G$, where $2G$ designates the disjoint union of two copies of G. We will need the following lemma. It appeared in [1], but it is included here for completeness.

Lemma 1. *Suppose A, B and C are graphs and C has at least one edge. Then $A \times C \cong B \times C$ implies $A \times K_2 \cong B \times K_2$.*

Proof. Given digraphs X and Y, let $\hom(X, Y)$ be the number of homomorphisms from X to Y. We will use the following theorem of Lovász: If D and D' are digraphs, then $D \cong D'$ if and only if $\hom(X, D) = \hom(X, D')$ for all digraphs X ([2], Theorem 2.11). We will also use the fact that $\hom(X, A \times B) = \hom(X, A)\hom(X, B)$ for all digraphs X, A and B. ([2], Corollary 2.3).

Identify A, B, C and K_2 with their symmetric digraphs (i.e., each edge is replaced with a double arc). If we can show $A \times C \cong B \times C$ implies $A \times K_2 \cong B \times K_2$ for the symmetric digraphs, then certainly this holds for the underlying graphs as well.

From $A \times C \cong B \times C$ we get $(A \times K_2) \times C \cong (B \times K_2) \times C$. Let X be a digraph. Then

$$\begin{aligned}\hom(X, A \times K_2)\hom(X, C) &= \hom(X, (A \times K_2) \times C) \\ &= \hom(X, (B \times K_2) \times C) \\ &= \hom(X, B \times K_2)\hom(X, C).\end{aligned}$$

If X is bipartite (i.e., if its underlying graph is bipartite) then $\hom(X, C) \neq 0$ because the map sending two partite sets to the two endpoints of a double arc of C is a homomorphism. Thus $\hom(X, A \times K_2) = \hom(X, B \times K_2)$.

On the other hand, if X is not bipartite, then there can be no homomorphism from X to a bipartite graph, and hence $\hom(X, A \times K_2) = 0 = \hom(X, B \times K_2)$. Thus $\hom(X, A \times K_2) = \hom(X, B \times K_2)$ for any X, so Lovász's theorem gives $A \times K_2 \cong B \times K_2$. ■

We are now in a position to prove our main result.

Proposition 1. *Suppose A, B and C are graphs for which B and C are bipartite and C has at least one edge. If $A \times C \cong B \times C$ and no component of B admits an involution that interchanges its partite sets, then $A \cong B$. Conversely, if some component of B admits an involution that interchanges its partite sets, then there is a graph A for which $A \times C \cong B \times C$ and $A \ncong B$.*

Proof. Let A, B and C be as stated. Suppose $A \times C \cong B \times C$, and no component of B admits an involution that interchanges its partite sets. From $A \times C \cong B \times C$, the lemma yields $A \times K_2 \cong B \times K_2$. List the components of A as $A_1, A_2, \dots A_m$, and those of B as $B_1, B_2, \dots B_n$, so that $A = \sum_{i=1}^m A_i$ and $B = \sum_{i=1}^n B_i$, where the sums indicate disjoint union. Then

$$
\begin{aligned}
A \times K_2 &\cong B \times K_2,\\
\left(\sum_{i=1}^m A_i\right) \times K_2 &\cong \left(\sum_{j=1}^n B_j\right) \times K_2,\\
\sum_{i=1}^m (A_i \times K_2) &\cong \sum_{j=1}^n 2B_j.
\end{aligned}
$$

From this last equation we see that if A had a component A_i that was not bipartite, then some component B_j of B would be isomorphic to $A_i \times K_2$. But $A_i \times K_2$ has a bipartition-reversing involution $(a, \varepsilon) \mapsto (a, \bar{\varepsilon})$, contradicting the fact that no component of B has such an involution. Therefore every component A_i of A is bipartite, so A is bipartite. Then $A \times K_2 \cong B \times K_2$ implies $2A \cong 2B$, whence $A \cong B$.

Conversely, suppose B has a component B_1 for which there is an involution $\beta : B_1 \to B_1$ that interchanges the partite sets of B_1. We need to produce a graph A with $A \ncong B$, but $A \times C \cong B \times C$.

Say the partite sets of B_1 are X and Y, so $\beta(X) = Y$. Define a graph B_1' as $V(B_1') = V(B_1)$ and $E(B_1') = \{b\beta(b') : bb' \in E(B_1)\}$. Notice that for each edge bb' of B_1, the graph B_1' has edges $b\beta(b')$ and $\beta(b)b'$, and conversely every edge of B_1' has such a form. It follows that every edge of B_1' has both endpoints in X or both endpoints in Y, so B_1' is disconnected. (Example: Let B_1 be the graph B in Figure 1(b), and let β be reflection across the vertical axis. Then B_1' is the graph A in Figure 1(a).)

Let $A = B_1' + B_2 + B_3 + \cdots + B_n$. In words, A is identical to B except the component B_1 of B is replaced with B_1'. Then $A \ncong B$ because A has more components than B.

However, we claim $A \times C \cong B \times C$. To prove this, it suffices to show $B_1' \times C \cong B_1 \times C$. (For A and B are identical except for B_1' and B_1.) Select a bipartition $V(C) = C_0 \cup C_1$ of C. Define a map $\theta : B_1 \times C \to B_1' \times C$ as

$$\theta(b, c) = \begin{cases} (b, c) & \text{if } c \in C_0, \\ (\beta(b), c) & \text{if } c \in C_1. \end{cases}$$

Certainly this is a bijection of vertex sets. But it is an isomorphism as well, as follows. Suppose $(b, c)(b', c') \in E(B_1 \times C)$. Then $bb' \in E(B_1)$ and $cc' \in E(C)$. We may assume $c \in C_0$ and $c' \in C_1$, so $\theta(b, c)\theta(b', c') = (b, c)(\beta(b'), c')$. But $b\beta(b') \in E(B_1')$, by definition of B_1', so it follows $\theta(b, c)\theta(b', c') \in E(B_1' \times C)$. In the other direction, suppose $\theta(b, c)\theta(b', c') \in E(B_1' \times C)$. From this and by definition of θ, it follows that $cc' \in E(C)$, so we may assume $c \in C_0$ and $c' \in C_1$. Then we have $\theta(b, c)\theta(b', c') = (b, c)(\beta(b'), c') \in E(B_1' \times C)$. In particular, $b\beta(b') \in E(B_1')$, and by definition of the edge set of B_1', this means that either $bb' \in E(B_1)$ or $\beta^{-1}(b)\beta(b') \in E(B_1)$. In the latter case, since β is an involution we have $\beta(b)\beta(b') \in E(B_1)$, so $bb' \in E(B_1)$. Either way, $bb' \in E(B_1)$, so $(b, c)(b', c') \in E(B_1 \times C)$. Thus θ is an isomorphism.

Consequently, $A \times C \cong B \times C$, but $A \ncong B$. ■

To conclude, we mention one open question suggested by our result. In the introduction we noted that cancellation of $A \times C \cong B \times C$ can fail only if C is bipartite and at least one of A or B is not bipartite. (We assume, as always, that C has at least one edge.) Given that C is bipartite, our result completely characterizes whether or not cancellation holds in the case that B is bipartite. It does not address the situation in which neither A nor B is bipartite. Thus, to complete the picture we would need to understand structural properties of non-bipartite graphs A and B that characterize whether or not cancellation of $A \times C \cong B \times C$ holds.

Here is one perspective on this question. The article [1] introduces an equivalence relation on graphs as $A \sim B$ if and only if $A \times K_2 \cong B \times K_2$.

It is proved that if C is bipartite (and has an edge), then $A \times C \cong B \times C$ if and only if $A \sim B$. Let $[A] = \{G \in \Gamma_0 : G \sim A\}$ be the equivalence class containing A. Then for bipartite C, cancellation in $A \times C \cong B \times C$ holds if and only if the class $[A]$ (hence also $[B]$) contains only one graph. The present note implies that for a bipartite graph B, we have $[B] = \{B\}$ if and only if no component of B admits a bipartition-reversing involution. It remains to characterize which classes contain a single non-bipartite graph.

References

[1] R. Hammack, *A quasi cancellation property for the direct product*, Proceedings of the Sixth Slovenian International Conference on Graph Theory, under review.

[2] P. Hell and J. Nešetřil, Graphs and Homomorphisms, Oxford Lecture Series in Mathematics (Oxford U. Press, 2004).

[3] W. Imrich and S. Klavžar, Product Graphs; Structure and Recognition (Wiley Interscience Series in Discrete Mathematics and Optimization, 2000).

[4] L. Lovász, *On the cancellation law among finite relational structures*, Period. Math. Hungar. **1** (1971) 59–101.

Received 13 August 2007
Revised 5 December 2007
Accepted 5 December 2007