Discussiones Mathematicae
Graph Theory 28 (2008) 165–178

RADIO k-LABELINGS FOR CARTESIAN PRODUCTS OF GRAPHS

Mustapha Kchikech, Riadh Khennoufa
and Olivier Togni

LE2I, UMR CNRS 5158
Université de Bourgogne, 21078 Dijon cedex, France

e-mail: {kchikech, khennoufa, otogni}@u-bourgogne.fr

Abstract

Frequency planning consists in allocating frequencies to the transmitters of a cellular network so as to ensure that no pair of transmitters interfere. We study the problem of reducing interference by modeling this by a radio k-labeling problem on graphs: For a graph G and an integer $k \geq 1$, a radio k-labeling of G is an assignment f of non negative integers to the vertices of G such that

$$|f(x) - f(y)| \geq k + 1 - d_G(x, y),$$

for any two vertices x and y, where $d_G(x, y)$ is the distance between x and y in G. The *radio k-chromatic number* is the minimum of $\max\{f(x) - f(y) : x, y \in V(G)\}$ over all radio k-labelings f of G. In this paper we present the radio k-labeling for the Cartesian product of two graphs, providing upper bounds on the radio k-chromatic number for this product. These results help to determine upper and lower bounds for radio k-chromatic numbers of hypercubes and grids. In particular, we show that the ratio of upper and lower bounds of the radio number and the radio antipodal number of the square grid is asymptotically $\frac{3}{2}$.

Keywords: graph theory, radio channel assignment, radio k-labeling, Cartesian product, radio number, antipodal number.

2000 Mathematics Subject Classification: 05C15, 05C78.

1. Introduction

In wireless networks, an important task is the management of the radio spectrum, that is the assignment of radio frequencies to transmitters in a way that avoid interferences. Interferences can occur if transmitters with close locations receive close frequencies. The problem, often modeled as a coloring problem on the graph where vertices represent by transmitters and edges indicate closeness of the transmitters, has been studied by several authors under different scenarios.

In this paper, we study the radio k-labeling problem defined by Chartrand *et al.* [2, 3]. Formally, for a graph $G = (V, E)$, we denote by $d_G(x, y)$ the distance between two vertices x and y, and by $D(G)$ the diameter of G. A *radio k-labeling* of G is a function $f : V \to \mathbb{N}$ such that for every two distinct vertices x and y of G the following is satisfied:

$$|f(x) - f(y)| \geq k + 1 - d_G(x, y).$$

The *span* of the function f denoted by $\lambda^k(f)$, is $\max\{f(x) - f(y) : x, y \in V(G)\}$. The *radio k-chromatic number* $\lambda^k(G)$ of G is the minimum span of all radio k-labelings of G.

Determining the radio k-chromatic number seems to be a difficult task, even for particular graphs. For instance, the radio k-chromatic number for paths was studied in [3, 6], where lower and upper bounds were given. Radio labelings for particular values of k were also considered: the radio k-chromatic number of paths and cycles is studied for $k = D(G) - 1$ in [1, 2] known as *radio antipodal number* and for $k = D(G)$ in [12] known as *radio number*. The radio number of trees and square of paths has been studied in [10, 11]. Recently, the radio antipodal number of the hypercube was determined [8].

Notice that, although the authors in [3] only consider radio k-labelings for $k \leq D(G)$, one can also consider the case $k > D(G)$. The motivation behind the study of the case $k > D(G)$ is of two kinds: first this case seems less difficult to study than the case $k \leq D(G)$ and secondly, computing the radio k-chromatic number of a graph for $k \geq D(G)$ can help to compute the radio k-labeling number of other graphs with larger diameter, as it is done in [6].

For the Cartesian product, we use the notation from [5]: The Cartesian product $G \square G'$ of two graphs G and G' is the graph with vertex set $V(G) \times V(G')$ and edge set $\{(x_i, u_j)(x_{i'}, u_{j'}) | i = i'$ and $u_j u_{j'} \in E(G')$, or

$x_i x_{i'} \in E(G)$ and $j = j'\}$. Therefore, to each vertex u_j of G' corresponds a copy G_j of G in $G \square G'$, with $1 \leq j \leq |V(G)|$.

As shown in [6], the radio k-chromatic number is related to two other graph parameters: the *upper Hamiltonian number* [9] of a graph G, denoted by $h^+(G)$, is the maximum of $\sum_{i=0}^{n-1} d_G(\pi(i+1), \pi(i))$, over all cyclic permutations π of the vertices of G. The *upper traceable number*, denoted by $t^+(G)$, is obtained from $h^+(G)$ by ignoring the distance between the first and the last vertex: $t^+(G) = \max_\pi \sum_{i=0}^{n-2} d_G(\pi(i+1), \pi(i))$.

Král *et al.* in [9] showed that the problem of determining the upper Hamiltonian number of a graph is NP-hard. The same method can be used to prove that computing the upper traceable number is also an NP-hard problem.

We shall use the following results from [6], which present a lower bound and an upper bound on the radio k-chromatic number of a graph G in terms of the parameter $t^+(G)$ and of the chromatic number $\chi(G^k)$ of the k^{th} power G^k of G (i.e., the graph with the same vertex set as G and with edges between vertices at distance at most k in G).

Theorem 1 ([6]). *Let G be a graph of order n, then for any positive integer k,*

$$\lambda^k(G) \geq (n-1)(k+1) - t^+(G).$$

Moreover, if $k \geq 2D(G) - 2$, then

$$\lambda^k(G) = (n-1)(k+1) - t^+(G).$$

Theorem 2 ([6]). *For any graph G and any integer $k \geq 1$,*

$$\lambda^k(G) \leq k(\chi(G^k) - 1).$$

The aim of this paper is to find relations between the radio k-chromatic number of the Cartesian products of two graphs, and some (other) coloring parameters on the factors. In Section 2, we propose general upper bounds for the radio k-chromatic number for the product of two graphs. In Section 3, we find more refined results for the product of a graph and a path. Applying these results, we present in Section 4 upper and lower bounds for the radio k-chromatic number of the hypercube and of the grid. In particular, for the radio antipodal and radio numbers of the grid, the ratio of the upper and lower bounds is small (asymptotically equal to $\frac{3}{2}$).

2. Radio k-Labelings of the Cartesian Product of Two Graphs

In this section we give general bounds for the radio k-chromatic number of the Cartesian product of two graphs G and G'.

Theorem 3. *For any two graphs G and G' of order $n \geq 2$ and m respectively, and for any integer $k \geq D(G\Box G') - 1$,*

$$\lambda^k(G\Box G') \leq m\lambda^k(G) + (m-1)k - t^+(G').$$

Proof. Let f be a radio k-labeling of G with $\lambda^k(f) = \lambda^k(G)$.

Let $x_0, x_1, \ldots, x_{n-1}$ be an ordering of the vertices of G such that $f(x_i) \leq f(x_{i+1})$ and let $u_0, u_1, \ldots, u_{m-1}$ be an ordering of the vertices of G' such that $\sum_{i=0}^{m-2} d_{G'}(u_{i+1}, u_i) = t^+(G')$.

For each vertex u_i of G' we associate a copy G_i of G in $G\Box G'$, where $V(G_i) = \{X_i^0, X_i^1, \ldots, X_i^{n-1}\}$ with $X_i^j = (x_j, u_i)$. With this notation, we have $d_{G\Box G'}(X_i^j, X_{i'}^{j'}) = d_G(x_j, x_{j'}) + d_{G'}(u_i, u_{i'})$.

Therefore, we can define a labeling g of $G\Box G'$ by setting

$$\begin{cases} g(X_0^j) = f(x_j), & \text{for } 0 \leq j \leq n-1, \\ g(X_i^0) = g(X_{i-1}^{n-1}) + k + 1 - d_{G\Box G'}(X_i^0, X_{i-1}^{n-1}), & \text{for } 1 \leq i \leq m-1, \\ g(X_i^j) = g(X_i^0) + g(X_0^j), & \text{for } 1 \leq j \leq n-1, \\ & \quad 1 \leq i \leq m-1. \end{cases}$$

The maximum label used is

$$\begin{aligned} g(X_{m-1}^{n-1}) = g(X_{m-1}^0) + g(X_0^{n-1}) &= g(X_{m-2}^{n-1}) + \lambda^k(G) + k + 1 \\ &\quad - d_{G\Box G'}(X_{m-1}^0, X_{m-2}^{n-1}), \end{aligned}$$

and thus

$$g(X_{m-1}^{n-1}) = m\lambda^k(G) + (m-1)(k+1) - \sum_{p=0}^{m-2} d_{G\Box G'}(X_{p+1}^0, X_p^{n-1}).$$

As $n \geq 2$, then $\sum_{p=0}^{m-2} d_{G\Box G'}(X_{p+1}^0, X_p^{n-1}) \geq \sum_{p=0}^{m-2}(1 + d_{G'}(u_{p+1}, u_p)) = m - 1 + t^+(G')$ because $x_0 \neq x_{n-1}$. Consequently

$$g(X_{m-1}^{n-1}) \leq m\lambda^k(G) + (m-1)k - t^+(G').$$

Now, we show that g is a radio k-labeling of $G\Box G'$ by checking the distance condition for each pair of vertices in $G\Box G'$: we want

$$|g(X_i^j) - g(X_{i'}^{j'})| \geq k + 1 - d_{G\Box G'}(X_i^j, X_{i'}^{j'}).$$

Case 1. If the two vertices are in the same copy G_i of $G\Box G'$, then the difference between their labels given by g is the same as that between the corresponding two vertices in G_0:

$$\begin{aligned}|g(X_i^j) - g(X_i^{j'})| &= |g(X_i^0) + g(X_0^j) - (g(X_i^0) + g(X_0^{j'}))|\\ &= |g(X_0^j) - g(X_0^{j'})|\\ &= |f(x_j) - f(x_{j'})|\\ &\geq k + 1 - d_G(x_j, x_{j'}).\end{aligned}$$

As $d_{G\Box G'}(X_i^j, X_i^{j'}) = d_G(x_j, x_{j'})$, we obtain

$$|g(X_i^j) - g(X_i^{j'})| \geq k + 1 - d_{G\Box G'}(X_i^j, X_i^{j'}).$$

Case 2. If the two vertices are not in the same copy of $G\Box G'$, then we just check the distance condition between two vertices X_{i+1}^j and $X_i^{j'}$ which are in two successive copies G_{i+1} and G_i respectively. We have

$$\begin{aligned}&|g(X_{i+1}^j) - g(X_i^{j'})| =\\ &= |g(X_{i+1}^0) + g(X_0^j) - g(X_i^{j'})|\\ &= |g(X_i^{n-1}) + k + 1 - d_{G\Box G'}(X_{i+1}^0, X_i^{n-1}) + g(X_0^j) - g(X_i^{j'})|\\ &\geq g(X_i^{n-1}) - g(X_i^{j'}) + g(X_0^j) \quad \text{(because we have } k \geq D(G\Box G') - 1)\\ &\geq g(X_0^{n-1}) - g(X_0^{j'}) + g(X_0^j)\\ &\geq g(X_0^j) - g(X_0^{j'})\\ &\geq f(x_j) - f(x_{j'})\\ &\geq k + 1 - d_G(x_j, x_{j'})\\ &\geq k + 1 - d_{G\Box G'}(X_{i+1}^j, X_i^{j'}).\end{aligned}$$

Thus g is a radio k-labeling and $\lambda^k(G\Box G') \leq m\lambda^k(G) + (m-1)k - t^+(G')$.

■

We now give another upper bound for the radio k-chromatic number of the Cartesian product which is valid for any $k \geq 2$.

Theorem 4. *For any two graphs G and G' of order n and m respectively and for any integer $k \geq 2$,*

$$\lambda^k(G \square G') \leq \chi(G'^k)(\lambda^k(G) + k - 1) - k + 1.$$

Proof. Let f be a radio k-labeling of G with $\lambda^k(f) = \lambda^k(G)$ and let $x_0, x_1, \ldots, x_{n-1}$ be an ordering of the vertices of G such that $f(x_i) \leq f(x_{i+1})$.

Let c be a proper vertex-coloring of G'^k with colors from $\{0, 1, \ldots, \chi(G'^k) - 1\}$. Denote the vertices of G' by u_j, $0 \leq j \leq m-1$.

Consider the labeling g of $G \square G'$ given by

$$g((x_i, u_j)) = f(x_i) + c(u_j)(\lambda^k(G) + k - 1).$$

The maximal label used by g is $\chi(G'^k)(\lambda^k(G) + k - 1) - k + 1$. To show that g is a radio k-labeling of $G \square G'$, we have to check that the distance condition is satisfied.

Notice that for any two vertices in the same copy G_j of G, the condition can be easily verified.

Next, two vertices (x_i, u_j) and $(x_{i'}, u_{j'})$ with $c(u_j) = c(u_{j'})$ are at distance at least $k+1$ in $G \square G'$ (since c is a proper coloring of G'^k), thus the condition is also verified for these two vertices.

Finally, for two vertices (x_i, u_j) and $(x_{i'}, u_{j'})$ with $c(u_j) \neq c(u_{j'})$ and $i > i'$, we have

$$|g((x_i, u_j)) - g((x_{i'}, u_{j'}))| = |f(x_i) - f(x_{i'}) + (c(u_j) - c(u_{j'}))(\lambda^k(G) + k - 1)|.$$

If $c(u_j) > c(u_{j'})$ then

$$\begin{aligned}
&|f(x_i) - f(x_{i'}) + (c(u_j) - c(u_{j'}))(\lambda^k(G) + k - 1)| \\
&\geq f(x_i) - f(x_{i'}) \\
&\geq k + 1 - d_G(x_i, x_{i'}) \\
&\geq k + 1 - d_{G \square G'}((x_i, u_j), (x_{i'}, u_{j'})).
\end{aligned}$$

If $c(u_j) < c(u_{j'})$ then $|f(x_i) - f(x_{i'}) + (c(u_j) - c(u_{j'}))(\lambda^k(G) + k - 1)| = (c(u_{j'}) - c(u_j))(\lambda^k(G) + k - 1) - f(x_i) + f(x_{i'})$, because $\lambda^k(G) + k - 1 - f(x_i) + f(x_{i'}) \geq 0$.

Thus,

$$
\begin{aligned}
&|g((x_i, u_j)) - g((x_{i'}, u_{j'}))| \\
&\geq \lambda^k(G) + k - 1 - (f(x_i) - f(x_{i'})) \\
&\geq k + 1 - d_{G\Box G'}((x_i, u_j), (x_{i'}, u_{j'})) + d_{G\Box G'}((x_i, u_j), (x_{i'}, u_{j'})) - 2 \\
&+ \lambda^k(G) - (f(x_i) - f(x_{i'})) \\
&\geq k + 1 - d_{G\Box G'}((x_i, u_j), (x_{i'}, u_{j'}))
\end{aligned}
$$

since $d_{G\Box G'}((x_i, u_j), (x_{i'}, u_{j'})) \geq 2$ and $f(x_i) \leq \lambda^k(G)$. ∎

3. Radio k-Labelings of the Products of a Graph and a Path

In [12], the value of $t^+(G)$ was determined for the path.

Lemma 1 ([12])**.** *For any integer $n \geq 2$,*

$$
t^+(P_n) = \begin{cases} \frac{1}{2}n^2 - 1 & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 1) - 1 & \text{if } n \text{ is odd.} \end{cases}
$$

Then, applying Theorem 3, we obtain:

Theorem 5. *For any graph G of order n and for any integers m and k, with $k \geq D(G) + m - 2$,*

$$
\lambda^k(G\Box P_m) \leq \begin{cases} m\lambda^k(G) + (m-1)k - \frac{1}{2}(m^2 - 1) + 1 & \text{if } m \text{ is odd,} \\ m\lambda^k(G) + (m-1)k - \frac{1}{2}m^2 + 1 & \text{if } m \text{ is even.} \end{cases}
$$

Another bound for the product of a graph and a path can be determined directly:

Theorem 6. *For any graph G and for any integer $k \geq 2$,*

$$
\lambda^k(G\Box P_m) \leq k(\lambda^k(G) + \min\{k, m - 1\}).
$$

Proof. Let f be a radio k-labeling of G such that $\lambda^k(G) = \lambda^k(f)$. Let $(x_0, x_1, ..., x_{n-1})$ be an ordering of the vertices of G such that $f(x_i) \leq f(x_{i+1})$ and let $V(P_m) = \{0, 1, \dots, m-1\}$.

Now, we construct a labeling g of $G \Box P_m$:

$$g((x_i, j)) = k(f(x_i) + (j \bmod k)).$$

In order to show that g is a radio k-labeling, first note that for any j, j' and for any x_i, $x_{i'}$, we have

$$d_{G\Box P_m}((x_i, j), (x_{i'}, j')) = d_G(x_i, x_{i'}) + |j - j'|.$$

Hence, if $|j - j'| \geq k$, then the condition $|g((x_i, j)) - g((x_i, j'))| \geq k + 1 - d_{G\Box P_m}((x_i, j), (x_i, j'))$ is satisfied provided that $(x_i, j) \neq (x_i, j')$.

- For any $j \neq j'$, $|j - j'| < k$ and for any x_i, we have

$$\begin{aligned} |g((x_i, j)) - g((x_i, j'))| = k|(j \bmod k) - (j' \bmod k)| \geq k \geq k + 1 \\ - d_{G\Box P_m}((x_i, j), (x_i, j')). \end{aligned}$$

- For any j and for any $i > i'$, we have

$$\begin{aligned} |g((x_i, j)) - g((x_{i'}, j))| &= k|f(x_i) - f(x_{i'})| \\ &\geq |f(x_i) - f(x_{i'})| \\ &\geq k + 1 - d_G(x_i, x_{i'}) \\ &\geq k + 1 - d_{G\Box P_m}((x_i, j), (x_{i'}, j)). \end{aligned}$$

- For any $j \neq j'$, $|j - j'| < k$, let $j_k = (j \bmod k)$ and $j'_k = (j' \bmod k)$. For any $i > i'$, we have

$$|g((x_i, j)) - g((x_{i'}, j'))| = k|f(x_i) - f(x_{i'}) + j_k - j'_k|.$$

  If $j_k > j'_k$ then,

$$\begin{aligned} |g((x_i, j)) - g((x_{i'}, j'))| &= k(f(x_i) - f(x_{i'}) + j_k - j'_k) \\ &\geq f(x_i) - f(x_{i'}) \\ &\geq k + 1 - d_G(x_i, x_{i'}) \\ &\geq k + 1 - d_{G\Box P_m}((x_i, j), (x_{i'}, j')). \end{aligned}$$

If $j_k < j'_k$ then, if $f(x_i) - f(x_{i'}) \geq j'_k - j_k$, we have,

$$\begin{aligned}|g((x_i, j)) - g((x_{i'}, j'))| &= k(f(x_i) - f(x_{i'}) - (j'_k - j_k)) \\ &\geq f(x_i) - f(x_{i'}) - (j'_k - j_k) \\ &\geq k + 1 - d_G(x_i, x_{i'}) - (j'_k - j_k) \\ &\geq k + 1 - d_G(x_i, x_{i'}) - (j' - j) \\ &\geq k + 1 - d_{G\Box P_m}((x_i, j), (x_{i'}, j')).\end{aligned}$$

If $f(x_i) - f(x_{i'}) \leq j'_k - j_k$, then

$$\begin{aligned}|g((x_i, j)) - g((x_{i'}, j'))| &= k(j'_k - j_k - (f(x_i) - f(x_{i'}))) \\ &\geq j'_k - j_k - (f(x_i) - f(x_{i'})) \\ &\geq f(x_i) - f(x_{i'}) - (j'_k - j_k) \\ &\geq k + 1 - d_G(x_i, x_{i'}) - (j'_k - j_k) \\ &\geq k + 1 - d_G(x_i, x_{i'}) - (j' - j) \\ &\geq k + 1 - d_{G\Box P_m}((x_i, j), (x_{i'}, j')).\end{aligned}$$

Thus, for any j, j' and for any x_i, $x_{i'}$, we have

$$|g((x_i, j)) - g((x_{i'}, j'))| \geq k + 1 - d_{G\Box P_m}((x_i, j), (x_{i'}, j')).$$

Consequently, g is a radio k-labeling of $G\Box P_m$ and

$$\lambda^k(G\Box P_m) \leq \lambda^k(g) \leq k(\lambda^k(G) + m - 1). \qquad \blacksquare$$

4. Radio k-Labelings for Hypercubes and Grids

4.1. Hypercubes

Let H_n be the hypercube of dimension n ($H_n = \underbrace{P_2\Box \cdots \Box P_2}_{n\ times}$).

Lemma 2. *For any integer $n \geq 2$,*

$$t^+(H_n) = 2^{n-1}(2n - 1) - (n - 1).$$

Proof. First, observe that there are no three vertices x, y, z in H_n such that $d_{H_n}(x,y) = d_{H_n}(y,z) = n$. Thus, the best we can do in order to maximize the sum of distances between consecutive vertices is to find an ordering of the vertices such that the distance between consecutive vertices is alternately n and $n-1$.

This can be easily done by considering an Hamiltonian path P of a subgraph H of H_n isomorphic to H_{n-1} (it is straightforward that such a path exists). The ordering is obtained by taking the first vertex of P, then its antipodal vertex, then the second vertex of P, its antipodal, and so on.

Therefore,

$$t^+(H_n) = \sum_{i=1}^{2^{n-1}} n + \sum_{i=1}^{2^{n-1}-1} (n-1) = 2^{n-1}(2n-1) - (n-1).$$

■

Theorem 7. *For the hypercube H_n of dimension $n \geq 2$ and for any $k \geq 2$,*

$$(2^n - 1)k - 2^{n-1}(2n-3) + n - 2 \leq \lambda^k(H_n) \leq (2^n-1)k - 2^{n-1} + 1.$$

Moreover, for $k \geq 2n-2$,

$$\lambda^k(H_n) = (2^n-1)k - 2^{n-1}(2n-3) + n - 2.$$

Proof. The lower bound is a direct consequence of Theorem 1 and Lemma 2. To show the upper bound, observe that $H_n = H_{n-1} \Box P_2$. Thus, applying Theorem 3 inductively, we obtain

$$\begin{aligned}
\lambda^k(H_n) &= \lambda^k(H_{n-1} \Box P_2) \\
&\leq 2\lambda^k(H_{n-1}) + k - 1 \\
&\leq 2^2\lambda^k(H_{n-2}) + (k-1)(2+1) \\
&\dots \\
&\leq 2^{n-1}\lambda^k(P_2) + (k-1)\sum_{i=0}^{n-2} 2^i \\
&\leq (2^n-1)k - 2^{n-1} + 1.
\end{aligned}$$

■

4.2. Grids

Let $M_{m,n} = P_m \Box P_n$ denote the 2-dimensional grid.

In this section, we provide upper and lower bounds for the radio k-chromatic number for the grid $M_{m,n}$ only in terms of k as given in [3, 6] for the path.

The k-distance chromatic number of a 2-dimensional grid (or equivalently, the chromatic number of the k^{th} power of the 2-dimensional grid) was determined in [4]. Using Theorem 2, we obtain an upper bound as given in the result below:

Theorem 8. *For the grid $M_{m,n}$ and for any integer $k > 2$,*

$$\lambda^k(M_{m,n}) \leq \begin{cases} \frac{1}{2}(k^3 + 2k^2 - k) & \text{if } k \text{ is odd,} \\ \frac{1}{2}(k^3 + 2k^2) & \text{if } k \text{ is even.} \end{cases}$$

In order to present a lower bound, we need the following lemma.

Lemma 3. *For any integer $p \geq 1$,*

$$h^+(M_{2p,2p}) \leq 8p^3.$$

Proof. Let $V(M_{2p,2p}) = \{(i,j) | 0 \leq i,j \leq 2p-1\}$. Let $N = 4p^2$ and let $X_0, \ldots, X_{N-1}$ be an ordering of $V(M_{2p,2p})$ such that $h^+(M_{2p,2p}) = \sum_{J=0}^{N-1} d_{M_{2p,2p}}(X_J, X_{J+1})$. Remark that $X_{J+1} = (i,j)$ and $X_J = (i',j')$ for some $0 \leq i,i',j,j' \leq 2p-1$. Thus, $d_{M_{2p,2p}}(X_J, X_{J+1}) = |i-i'| + |j-j'|$. Therefore, the summation $\sum_{J=0}^{N-1} d_{M_{2p,2p}}(X_J, X_{J+1})$ consists in $4p^2$ distances, each distance corresponding to 4 numbers (two with a positive sign and two with a negative sign). Moreover, each number $i \in \{0, 1, \ldots, 2p-1\}$ occurs $8p$ times in the summation.

Consequently, the configuration achieving the maximum summation is when each number $i \in \{0, 1, \ldots, p-1\}$ occurs $8p$ times as $-i$ and each of $\{p, p+1, \ldots, 2p-1\}$ occurs $8p$ times as i. In that case we obtain

$$\begin{aligned} \sum_{J=0}^{N-1} d_{M_{2p,2p}}(X_J, X_{J+1}) &\leq 8p\Big(\sum_{i=p}^{2p-1} i - \sum_{i=0}^{p-1} i\Big) \\ &= 8p\Big(\sum_{i=0}^{p-1} p\Big) \\ &= 8p^3. \end{aligned}$$

■

Theorem 9. *For any positive integer k and for any integer $p \geq 1$,*

$$\lambda^k(M_{2p,2p}) \geq 4p^2(k - 2p + 1) - k.$$

Proof. The result follows by combining Theorem 1 and Lemma 3, using the fact that $t^+(M_{2p,2p}) \leq h^+(M_{2p,2p}) - 1$. ■

Theorem 10. *For the grid $M_{m,n}$ and for any integer $k \geq 2$ with $\min\{m,n\} \geq 2\lceil \frac{k}{3} \rceil$,*

$$\lambda^k(M_{m,n}) \geq \frac{4}{27}(kl + 3k) - k - \frac{16}{27}.$$

Proof. It is easily seen that for any integers m, n, m', n' such that $m' \geq m$ and $n' \geq n$ then $\lambda^k(M_{m',n'}) \geq \lambda^k(M_{m,n})$ since any radio k-labeling of $M_{m',n'}$ provides a radio k-labeling of $M_{m,n}$.

The lower bound is obtained by setting $p = \lceil \frac{k}{3} \rceil$ in Theorem 9 (we chose this value of p in order to maximize the expression $4p(k - 2p + 1) - k$).

If $k \equiv 0 \bmod 3$ then $p = \frac{k}{3}$ and $\lambda^k(M_{2p,2p}) \geq \frac{4}{27}k(k+3) - k \geq \frac{4}{27}(k + 3k) - k - \frac{16}{27}$.

If $k \equiv 1 \bmod 3$ then $p = \frac{k+2}{3}$ and $\lambda^k(M_{2p,2p}) \geq \frac{4}{27}(k+2)(k-1) - k = \frac{4}{27}(kl + 3k) - k - \frac{16}{27}$.

If $k \equiv 2 \bmod 3$ then $p = \frac{k+1}{3}$ and $\lambda^k(M_{2p,2p}) \geq \frac{4}{27}(k+1) - k \geq \frac{4}{27}(k + 3k) - k - \frac{16}{27}$. ■

Using Theorem 5 and the following theorem in [6], we obtain another upper bound for the grid $M_{n,n}$.

Theorem 11 ([6])**.** *For the path P_n of order $n \geq 2$,*

$$\lambda^k(P_n) = \begin{cases} (n-1)k - \frac{1}{2}(n-1)^2 + 1 & \textit{if } n \textit{ is odd and } k \geq n-2, \\ (n-1)k - \frac{1}{2}n(n-2) & \textit{if } n \textit{ is even and } k \geq n-1. \end{cases}$$

Proposition 1. *For the grid $M_{n,n} = P_n \Box P_n$ of order n^2 and for any $k \geq 2n-3$,*

$$\lambda^k(M_{n,n}) \leq \begin{cases} (n^2-1)k - \frac{1}{2}(n-1)(n^2-1) + 2 & \textit{if } n \textit{ is odd,} \\ (n^2-1)k - \frac{1}{2}n^2(n-1) + 1 & \textit{if } n \textit{ is even.} \end{cases}$$

In particular, for $k = D(M_{n,n}) - 1$ and $k = D(M_{n,n})$ the above results give bounds for the radio antipodal number and radio number of $M_{n,n}$. The ratio of the upper and lower bounds is asymptotically equal to $\frac{3}{2}$.

Proposition 2. *The radio antipodal number of the mesh $M_{n,n}$ satisfies*

$$n^3 - 3n^2 + n + 2 \leq \lambda^{2n-3}(M_{n,n}) \leq \frac{3}{2}n^3 - \frac{5}{2}n^2 - \frac{3}{2}n + \frac{9}{2}.$$

Proposition 3. *The radio number of the mesh $M_{n,n}$ satisfies*

$$n^3 - 2n^2 - n + 2 \leq \lambda^{2n-2}(M_{n,n}) \leq \frac{3}{2}\Big(n^3 - n^2 - n + \frac{7}{3}\Big).$$

Conclusion

We have presented several bounds for the radio k-chromatic number of the Cartesian product of graphs. Although it seems difficult to judge sharpness of the bounds, we have shown in Section 4 that for some values of k near the diameter of the graph, some of the bounds proposed are relatively close to the optimal. Moreover, this study is among the first to consider radio k-labeling of graphs different from a path or a cycle. An interesting question is to know if the radio k-chromatic number of a graph G is closer to the chromatic number of the graph G^k than to k times this number.

Acknowledgements

We wish to thank Daphne Der Fan Liu for helpful discussions and for providing us the notion of upper Hamiltonian number of a graph.

References

[1] G. Chartrand, D. Erwin and P. Zhang, *Radio antipodal colorings of cycles*, Congr. Numer. **144** (2000) 129–141.

[2] G. Chartrand, D. Erwin and P. Zhang, *Radio antipodal colorings of graphs*, Math. Bohem. **127** (2002) 57–69.

[3] G. Chartrand, L. Nebeský and P. Zhang, *Radio k-colorings of paths*, Discuss. Math. Graph Theory **24** (2004) 5–21.

[4] G. Fertin, E. Godard and A. Raspaud, *Acyclic and k-distance coloring of the grid*, Inform. Process. Lett. **87** (2003) 51–58.

[5] W. Imrich and S. Klavžar, Product graphs, Structure and recognition, With a foreword by Peter Winkler, Wiley-Interscience Series in Discrete Mathematics and Optimization (Wiley-Interscience, New York, 2000).

[6] M. Kchikech, R. Khennoufa and O. Togni, *Linear and cyclic radio k-labelings of trees*, Discuss. Math. Graph Theory **27** (2007) 105–123.

[7] R. Khennoufa and O. Togni, *A note on radio antipodal colourings of paths*, Math. Bohemica **130** (2005) 277–282.

[8] R. Khennoufa and O. Togni, The Radio Antipodal Number of the Hypercube, submitted, 2007.

[9] D. Král, L.-D. Tong and X. Zhu, *Upper Hamiltonian numbers and Hamiltonian spectra of graphs*, Australasian J. Combin. **35** (2006) 329–340.

[10] D. Liu, *Radio Number for Trees*, manuscript, 2006.

[11] D. Liu and M. Xie, *Radio Number for Square Paths*, Discrete Math., to appear.

[12] D. Liu and X. Zhu, *Multi-level distance labelings for paths and cycles*, SIAM J. Discrete Math. **19** (2005) 610–621.

Received 28 March 2007
Revised 24 September 2007
Accepted 24 September 2007