# MAXIMAL $k$-INDEPENDENT SETS IN GRAPHS 

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#### Abstract

A subset of vertices of a graph $G$ is $k$-independent if it induces in $G$ a subgraph of maximum degree less than $k$. The minimum and maximum cardinalities of a maximal $k$-independent set are respectively denoted $i_{k}(G)$ and $\beta_{k}(G)$. We give some relations between $\beta_{k}(G)$ and $\beta_{j}(G)$ and between $i_{k}(G)$ and $i_{j}(G)$ for $j \neq k$. We study two families of extremal graphs for the inequality $i_{2}(G) \leq i(G)+\beta(G)$. Finally we give an upper bound on $i_{2}(G)$ and a lower bound when $G$ is a cactus.


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## 1. Introduction

For notation and graph theory terminology, we in general follow [6, 7]. In a graph $G=(V, E)$ of order $n(G)=n$, the neighborhood of a vertex $v \in V$ is $N_{G}(v)=\{u \in V \mid u v \in E\}$. If $X$ is a subset of vertices, then $N_{G}(X)=$ $\cup_{v \in X} N_{G}(v)$. The closed neighbohoods of $v$ and $X$ are respectively $N_{G}[v]=$ $N(v) \cup\{v\}$ and $N[X]=N(X) \cup X$. The degree of a vertex $v$ of $G$, denoted by $d_{G}(v)$, is the order of its neighborhood. For a subset $A$ of $V$, let us
denote by $G[A]$ the subgraph induced in $G$ by $A$. If $x$ is a vertex of $V$, then $d_{A}(x)=|N(x) \cap A|$ and $\Delta(A)=\max \left\{d_{A}(x) \mid x \in A\right\}$. A vertex of degree one is called a leaf and its neighbor is called a support vertex. We denote the set of leaves of a graph $G$ by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)|=\ell(G),|S(G)|=s(G)$. If $T=P_{2}$, then $\ell\left(P_{2}\right)=s\left(P_{2}\right)=2$. A double star $S_{p, q}$ is obtained by attaching $p$ leaves at an endvertex of a path $P_{2}$ and $q$ leaves at the second one. A cactus is a graph in which every edge is contained in at most one cycle. A graph is called trivial if its order is $n=1$.

An independent set is a set of vertices whose induced subgraph has no edge. The independence number $\beta(G)$ is the maximum cardinality of an independent set in $G$. The independence domination number $i(G)$ is the minimum cardinality of a maximal independent set in $G$.

In [5] Fink and Jacobson generalized the concepts of independent and dominating sets. A subset $X$ of $V$ is $k$-independent if the maximum degree of the subgraph induced by the vertices of $X$ is less or equal to $k-1$. The subset $X$ is $k$-dominating if every vertex of $V-X$ is adjacent to at least $k$ vertices in $X$. The lower $k$-independence number $i_{k}(G)$ is the minimum cardinality of a maximal $k$-independent set in $G$, the $k$-independence number $\beta_{k}(G)$ is the maximum cardinality of a maximal $k$-independent set, and the $k$-domination number $\gamma_{k}(G)$ is the minimum cardinality of a $k$ dominating set of $G$. A $k$-independent set with maximum cardinality of a graph $G$ is called a $\beta_{k}(G)$-set. Similarly we define a $i_{k}(G)$-set and a $\gamma(G)$ set. For $k=1$, the 1 -independent and 1 -dominating sets are the classical independent and dominating sets and so $i_{1}(G)=i(G), \beta_{1}(G)=\beta(G)$, and $\gamma_{1}(G)=\gamma(G)$.

Note that Borowiecki and Michalak [2] gave a generalization of the concept of $k$-independence by considering other hereditary-induced properties than the property for a subgraph to have maximum degree at most $k-1$.

On the same way that the minmax parameter $i$ is more difficult to study than $\beta$, very few results are known on $i_{k}$ while the literature on $\beta_{k}$, and even more on $\gamma_{k}$, is rather copious. The irregularity of the behaviour of $i_{k}$ is shown for instance by the followings two facts. The well-known inequalities $\gamma(G) \leq i(G) \leq \beta(G)$ only extend to $\gamma_{k}(G) \leq \beta_{k}(G)$ [3] but $i_{k}(G)$ may be smaller than $\gamma_{k}(G)$. The sequence $\left(\beta_{k}(G)\right)$ is always non-decreasing while the sequence $\left(i_{k}(G)\right)$ is not necessarily monotone. In this paper we show some properties related to $\beta_{k}$ and $i_{k}$.

A matching in a graph $G$ is a collection of pairwise non-adjacent edges.

The matching is called induced if no two edges of the matching are joined by an edge in $G$.

## 2. Bounds on $\beta_{k}$ and $i_{k}$.

Theorem 1. For every graph $G$ and integers $j, k$ with $1 \leq j \leq k, \beta_{k+1}(G) \leq$ $\beta_{j}(G)+\beta_{k-j+1}(G)$.

Proof. Let $T$ be a maximum $(k+1)$-independent set of $G$ and $X$ both a $j$-independent and $j$-dominating set of $G[T]$. Such a set $X$ exists by [3]. Thus $\beta_{j}(G) \geq|X|$. Let $Y=T-X$. Since $X$ is $j$-dominating in $G[T]$, $\Delta(G[Y]) \leq k-j$. Hence $Y$ is a $(k-j+1)$-independent set and therefore $\beta_{k-j+1}(G) \geq|Y|=|T|-|X| \geq \beta_{k+1}(G)-\beta_{j}(G)$.

Corollary 2. For every graph $G$ and every integer $k \geq 1$,
(a) $\beta_{k+1}(G) \leq \beta_{k}(G)+\beta(G)$,
(b) $\beta_{k+1}(G) \leq 2 \beta_{\lceil k+1 / 2\rceil}(G)$,
(c) $\beta_{k+1}(G) \leq(k+1) \beta(G)$.

The next theorem gives a structural property of the graphs satisfying (c).
Theorem 3. Let $k \geq 2$ be an integer and $G$ a graph such that $\beta_{k}(G)=$ $k \beta(G)$. Then every $\beta_{k}(G)$-set $T$ is the disjoint union of $\beta(G)$ cliques $U^{j}$, $1 \leq j \leq \beta$, of order $k$ and every vertex $v \in V \backslash T$ has at least one clique $U^{j}$ entirely contained in its neighborhood.

Proof. Since $T$ is a $k$-independent set, $\Delta(T) \leq k-1$. Let $X_{1}$ be a maximal independent set of $G[T]$. Every vertex of $T \backslash X_{1}$ has at least one neighbor in $X_{1}$ and thus, $\Delta\left(T \backslash X_{1}\right) \leq k-2$. Let $X_{2}$ be a maximal independent set of $G\left[T \backslash X_{1}\right]$. Every vertex of $T \backslash\left(X_{1} \cup X_{2}\right)$ has at least one neighbor in $X_{1}$ and one in $X_{2}$, and thus $\Delta\left(T \backslash\left(X_{1} \cup X_{2}\right)\right) \leq k-3$. We continue the process until the choice of a maximal independent set $X_{k-1}$ of $G\left[T \backslash\left(X_{1} \cup \cdots \cup X_{k-2}\right)\right]$. Then $\Delta\left(T \backslash\left(X_{1} \cup \cdots \cup X_{k-1}\right) \leq 0\right.$ and thus the set $X_{k}=T \backslash\left(X_{1} \cup \cdots \cup X_{k-1}\right)$ is independent. Therefore every set $X_{i}$ is independent in $G$ and $\left|X_{i}\right| \leq \beta(G)$ for $1 \leq i \leq k$. Hence $|T|=\sum_{i=1}^{k}\left|X_{i}\right| \leq k \beta(G)$ and since $T=\beta_{k}(G)=k \beta(G),\left|X_{i}\right|=\beta(G)(=\beta$ for short) for $1 \leq i \leq k$. Let $X_{1}=\left\{u_{1}^{1}, u_{1}^{2}, \ldots, u_{1}^{\beta}\right\}$. Then $k \beta=|T|=\mid N_{T}\left[u_{1}^{1}\right] \cup N_{T}\left[u_{1}^{2}\right] \cup \cdots \cup$ $N_{T}\left[u_{1}^{\beta}\right]\left|\leq \sum_{j=1}^{\beta}\right| N_{T}\left[u_{1}^{j}\right] \mid \leq k \beta$ since $d_{T}\left(u_{1}^{j}\right) \leq k-1$ for $1 \leq j \leq \beta$.

Therefore the sets $N_{T}\left[u_{1}^{j}\right]$ are disjoint and $\left|N\left[u_{1}^{j}\right]\right|=k$ for $1 \leq j \leq \beta$. If one of the sets $N_{T}\left[u_{1}^{j}\right]$, say $N_{T}\left[u_{1}^{1}\right]$, does not induce a clique, let $a$ and $b$ two non-adjacent vertices of $N_{T}\left(u_{1}^{1}\right)$. Then $\left\{a, b, u_{1}^{2}, \ldots, u_{1}^{\beta}\right\}$ is an independent set of $\beta+1$ elements of $G$, a contradiction. Hence each $U^{j}=N\left[u_{1}^{j}\right]$ is a clique and $G$ is the disjoint unions of $\beta$ cliques of order $k$. Let now $v$ be any vertex in $V \backslash T$. If every clique $U^{j}$ contains a vertex which is not adjacent to $v$, say $u_{1}^{j} v \notin E(G)$ for $1 \leq j \leq \beta$, then $\left\{v, u_{1}^{1}, u_{1}^{2}, \ldots, u_{1}^{\beta}\right\}$ is an independent set of $G$ of $\beta+1$ elements, a contradiction which completes the proof.

Corollary 4. Every connected graph with order $n$ and clique number $\omega<n$ satisfies $\beta_{\omega}(G)<\omega \beta(G)$.

Proof. If $\beta_{\omega}(G)=\omega \beta(G)$, then every $\beta_{\omega}(G)$-set $T$ consists of disjoint cliques $K_{\omega(G)}$. Since $G$ is connected and different from $K_{\omega}, V \backslash T$ is not empty and every vertex $v \in V \backslash T$ forms with one of these cliques a clique of order $\omega+1$, a contradiction.

Theorem 5. For every graph $G$ and integers $j, k$ with $1 \leq j \leq k, i_{k+1}(G) \leq$ $(k-j+2) i_{j}(G)$. Equality can occur only when $j=1$ or $j=k$.

Proof. Let $S$ be a $i_{j}(G)$-set, $X=\left\{x \in S \mid d_{S}(x)=j-1\right\}$ and $Y=S \backslash X=$ $\left\{y \in S \mid d_{S}(y)<j-1\right\}$. Since $j<k+1$, the set $S$ is a $(k+1)$-independent set of $G$. Let $I$ be a maximal $(k+1)$-independent set of $G$ containing $S, A=N_{I \backslash S}(X)$ and $B=I \backslash(A \cup S)$. Since $I$ is $(k+1)$-independent, $d_{A}(x) \leq k-j+1$ for every $x \in X$, which implies $|A| \leq(k-j+1)|X|$, and $d_{I}(y) \leq k$ for every $y \in Y$. Since the $j$-independent set $S$ is maximal in $G, d_{Y}(v) \geq j$ for every $v \in B$. Hence the number $m(Y, B)$ of edges of $G$ between $Y$ and $B$ satisfies $j|B| \leq m(Y, B) \leq k|Y|$, which implies $|B| \leq k|Y| / j \leq(k-j+1)|Y|$. Therefore
$i_{k+1}(G) \leq|I|=|A|+|B|+|S| \leq(k-j+1)(|X|+|Y|)+|S|=(k-j+2) i_{j}(G)$.
If $i_{k+1}(G)=(k-j+2) i_{j}(G)$, then $i_{k+1}(G)=|I|,|A|=(k-j+1)|X|$, and $|B|=k|Y| / j=(k-j+1)|Y|$. Equality $|A|=(k-j+1)|X|$ implies $d_{A}(x)=k-j+1$ for every $x \in X$. Equality $|B|=k|Y| / j$ implies $d_{B}(y)=k$ for every $y \in Y$ and $d_{Y}(v)=j$ for every $v \in B$. Finally, $k / j=k-j+1$ if and only if $j=1$ or $j=k$.

Case $j=1$. If $i_{k+1}(G)=(k+1) i(G)$ for some $k \geq 1$, then $Y=\emptyset$, $X=S$ is an independent set, $A=I \backslash S,|A|=k|X|$, the neighborhoods in $A$ of the $i(G)$ vertices of $S$ are disjoint and each of order $k$, and $G[A]$ has maximum degree at most $k-1$.

Case $j=k$. If $i_{k+1}(G)=2 i_{k}(G)$ for some $k \geq 1$, then $|A|=|X|$, $|B|=|Y|$, the edges of $G$ between $A$ and $X$ form a perfect matching $M$ and the edges of $G$ between $B$ and $Y$ form a $k$-regular bipartite graph.

Corollary 6. For every graph $G$ of order $n$ and maximum degree $\Delta, i_{\Delta}(G) \geq$ $n / 2$, and this bound is sharp.

Proof. Obvious consequence of $i_{k+1}(G) \leq 2 i_{k}(G)$ obtained from Theorem 5 when $j=k$ and $i_{\Delta+1}(G)=n$.

Let $G$ be obtained by attaching one pendant vertex at each vertex of a clique $K_{k}$. Then $n=2 k, \Delta=k$ and $i_{k}(G)=k$. Hence $i_{\Delta}=n / 2$.

Corollary 7. If a graph $G$ of order $n \geq 2$ satisfies $i_{2}(G)=2 i(G)$, then $G$ contains an induced matching of size $i(G)$.

Proof. In the equality case $k=j=1$ in Theorem $5, S=X, G$ contains a perfect matching between $A$ and $X$, and this matching is induced of size $S=i(G)$ since $G[A]$ has maximum degree 0 .
The converse of Corollary 7 is not true. For instance the cycle $C_{6}$ admits an induced mathing $M$ of size $i\left(C_{6}\right)=2$ but $i_{2}\left(C_{6}\right)=3<2 i(G)$.

The inequality $i_{2}(G) \leq 2 i(G)$ cannot be improved to $i_{2}(G) \leq 2 \gamma(G)$, even for trees, as shown by the caterpillar obtained by adding $k \geq 5$ pendant vertices at each vertex of a path $P_{3}$. However the next theorem improves it to $i_{2}(G) \leq \gamma(G)+i(G)$ in the class of trees and unicyclic graphs.

Theorem 8. If the graph $G$ contains at most one cycle, then $i_{2}(G) \leq$ $\gamma(G)+i(G)$.

Proof. Let $S$ be a $i(G)$-set and $I$ a maximal 2-independent set of $G$ containing $S$. With the notation of Theorem $5, X=S$ is independent, $A=N_{I}(S)=I \backslash S$, and the edges of $G[I]$ form an induced matching $M$ between $A$ and a subset $A^{\prime}$ of $S$. Let $Z$ be a $\gamma(G)$-set, $M_{1}$ the edges of $M$ with no endvertex in $Z$, and $A_{1}$ ( $A_{1}^{\prime}$ respectively) the set of the endvertices of the edges of $M_{1}$ in $A$ ( $A^{\prime}$ respectively). If $\gamma(G)<|M|$, then $M_{1} \neq \emptyset$
and since $M$ is induced, the vertices of $A_{1} \cup A_{1}^{\prime}$ cannot be dominated by vertices in $Z \cap\left(A \cup A^{\prime}\right)$. Hence the set $W=Z \backslash\left(A \cup A^{\prime}\right)$ is not empty and dominates $A_{1} \cup A_{1}^{\prime}$. Therefore the induced subgraph $G\left[W \cup A_{1} \cup A_{1}^{\prime}\right]$ of order $|W|+2\left|M_{1}\right|$ contains at least $3\left|M_{1}\right|$ edges. Moreover, since $Z$ contains at least one endvertex of each edge in $M \backslash M_{1},|W| \leq|Z|-\left|M \backslash M_{1}\right|=$ $(\gamma(G)-|M|)+\left|M_{1}\right|<\left|M_{1}\right|$. Thus $3\left|M_{1}\right|>|W|+2\left|M_{1}\right|$, which contradicts the assumption that $G$ contains at most one cycle. Therefore $\gamma(G) \geq|M|=|A|$ and $i_{2}(G) \leq|S|+|A| \leq i(G)+\gamma(G)$.
The result of Theorem 8 is not valid for all graphs as shown by the following example. We consider eight disjoint triangles $x_{i} y_{i} z_{i}$ and identify the vertex $x_{i}$ with $x_{i+1}$ for $i=1,3,5,7$. Let $w_{1}, w_{2}, w_{3}, w_{4}$ denote the resulting new vertices. To complete $G$, we add the edges $w_{1} w_{2}, w_{1} w_{3}$ and $w_{1} w_{4}$. Then $\left\{w_{2}, y_{3}, w_{3}, y_{5}, w_{4}, y_{7}, y_{1}, z_{1}, y_{2}, z_{2}\right\}$ is a $i_{2}(G)$-set and thus $i_{2}(G)=$ $10, \gamma(G)=4$, and $i(G)=5$. By attaching $q$ triangles at each vertex $w_{i}$ instead of $2, \gamma(G)$ does not change while now $i(G)=3+q$ and $i_{2}(G)=6+2 q$. Therefore the difference $i_{2}(G)-(i(G)+\gamma(G))$ can be done arbitrarily large and the ratio $i_{2}(G) /(i(G)+\gamma(G))$ arbitrarily close to 2 .

The next corollary is another consequence of Theorem 5. A graph $G$ is well-covered if $i(G)=\beta(G)$ and well-k-covered if $i_{k}(G)=\beta_{k}(G)$.

Corollary 9. For any $k \geq 1, i_{k+1}(G) \leq(k+1) i(G) \leq k i(G)+\beta(G)$ and if $i_{k+1}(G)=k i(G)+\beta(G)$, then $G$ is well-covered and well- $(k+1)$-covered.

Proof. The inequality comes from Theorem 5 with $j=1$. If $i_{k+1}(G)=$ $k i(G)+\beta(G)$ then $i(G)=\beta(G)$, that is $G$ is well-covered, and thus $i_{k+1}(G)=$ $(k+1) \beta(G)$. Therefore $\beta_{k+1}(G) \leq i_{k+1}(G)$ from Corollary 2 , which implies $\beta_{k+1}(G)=i_{k+1}(G)$ and proves that $G$ is well- $(k+1)$-covered.

## 3. Graphs with $i_{2}=i+\beta$

In this section we are interested in graphs $G$ satisfying the equality in Corollary 9 when $k=1$. We describe two particular classes of them defined by forbidden subgraphs.

## Definition 10

- The graphs $F$ of the family $\mathcal{F}$ are formed by five disjoint cliques $X_{i}$ of cardinality at least 2 together with all the edges between $X_{i}$ and $X_{i+1}$ for $1 \leq i \leq 5(\bmod 5)$.
- The graphs $C_{4}$ and $g$ are shown in Figure 1.


Figure 1
Clearly every non-trivial clique and every graph of $\mathcal{F}$ satisfies $i_{2}(G)=i(G)+$ $\beta(G)$ with $i(G)=\beta(G)=1$ for a clique, $i(G)=\beta(G)=2$ for a graph of $\mathcal{F}$.

Theorem 11. Let $G$ be a graph such that $i_{2}(G)=i(G)+\beta(G)$. Then

1. If $G$ is $g$-free, the component of $G$ are $\lambda_{1}$ non-trivial cliques with $\lambda_{1}=$ $i(G)$.
2. If $G$ is $C_{4}$-free, the components of $G$ are $\lambda_{1} \geq 0$ non-trivial cliques and $\lambda_{2} \geq 0$ graphs of $\mathcal{F}$ with $\lambda_{1}+2 \lambda_{2}=i(G)$.

Proof. From Corollary $9, i(G)=\beta(G)$ and $i_{2}(G)=\beta_{2}(G)$. The maximal independent sets of $G$ have all the same cardinality and each maximal 2independent set induces a matching $M$ of size $i(G)$. In particular, $G$ has no isolated vertex. We make an induction on the common value $\lambda$ of $i(G)$ and $\beta(G)$. If $\lambda=1$, then $G$ is a clique of cardinality at least 2 . For $\lambda>1$, suppose the property true when $i_{2}(G)=i(G)+\beta(G)<2 \lambda$ and let $G$ be a $g$ free or $C_{4}$-free graph such that $i_{2}(G)=i(G)+\beta(G)=\beta_{2}(G)=2 \lambda$. Let $a$ be a vertex of $G$ such that $\beta(N[a])$ is maximum and let $G^{\prime}=G-N[a]$. Every maximal independent set of $G^{\prime}$ can be completed to a maximal independent set of $G$ by adding $a$. Hence $i\left(G^{\prime}\right)=\beta\left(G^{\prime}\right)=\lambda-1$. If $S^{\prime}$ is a $i_{2}\left(G^{\prime}\right)$-set then, by Theorem $5,\left|S^{\prime}\right|=i_{2}\left(G^{\prime}\right) \leq 2 i\left(G^{\prime}\right)=2 \lambda-2$. The 2 -independent set $S^{\prime} \cup\{a\}$ of $G$ can be completed to a maximal 2-independent set of $G$ by adding at most one vertex of $N(a)$. Hence $i_{2}(G) \leq\left|S^{\prime}\right|+2 \leq 2 \lambda$. Since $i_{2}(G)=2 \lambda, i_{2}\left(G^{\prime}\right)=2 \lambda-2=i\left(G^{\prime}\right)+\beta\left(G^{\prime}\right)$. By the induction hypothesis applied to $G^{\prime}$, which is $g$-free or $C_{4}$-free as $G$, the components of $G^{\prime}$ are $\mu_{1}$ non-trivial cliques if $G$ is $g$-free, $\mu_{1}$ non-trivial cliques and $\mu_{2}$ graphs of $\mathcal{F}$ if $G$ is $C_{4}$-free, with $\mu_{1}+2 \mu_{2}=\lambda-1$. To continue, we distinguish two cases.

Case 1. $N[a]$ is a clique of $G$. By the choice of $a, N[x]$ is a clique for every vertex $x$ of $G$. Therefore the components of $G$ are $\lambda$ non-trivial cliques.

Case 2. $N[a]$ is not a clique. Let $b$ and $c$ be two non-adjacent vertices of $N(a)$. If for each component $H$ of $G^{\prime}, V(H) \backslash(N(b) \cup N(c)) \neq \emptyset$ if $H$ is a clique and $\beta(V(H) \backslash N(b) \cup N(c))=2$ if $H \in \mathcal{F}$, then $\beta(G) \geq \beta\left(G^{\prime}\right)+2=\lambda+1$ which is impossible. Therefore there exists a component $H$ of $G^{\prime}$ such that either $H$ is a non-trivial clique and $V(H) \subseteq N(b) \cup N(c)$ or $H \in \mathcal{F}$ and $\beta(V(H) \backslash N(b) \cup N(c))<2$.

Subcase 2.1. Suppose first that $H \in \mathcal{F}$ and $\beta(V(H) \backslash(N(b) \cup N(c)) \leq 1$. Then $G$ is not $g$-free and thus is $C_{4}$-free. We will prove that this subcase is impossible. Let $X_{i}, 1 \leq i \leq 5$ be the five cliques of $H$ as described in the definition of $\mathcal{F}$. Then $V(H) \backslash(N(b) \cup N(c))$ is a (possibly empty) clique $U$ and since $G$ is $C_{4}$-free, $N(b) \cap N(c) \cap V(H)=\emptyset$. If $V(H) \subseteq N(b)$, then a maximal 2-independent set $I$ of $G$ containing $\{a, b\}$ contains no other vertex in $N[a] \cup V(H)$ and at most $2 \mu_{1}+4\left(\mu_{2}-1\right)=2 \lambda-6$ vertices in $G^{\prime}-V(H)$, that is $|I| \leq 2 \lambda-4$ which is impossible. Hence $V(H)$ is not contained in $N(b)$, neither in $N(c)$ by symmetry. If $V(H) \cap N(b)$ and $V(H) \cap N(c)$ are cliques, then $U \neq \emptyset$ and $H$ contains an edge $u v$ with $u \in U$ and $v \in N(c)$. The set $S=\{u, v, a, b\}$ dominates $V(H) \cup N[a]$. If $N(b) \cap V(H)$ is not a clique, say $b$ is adjacent to $x_{1} \in X_{1}$ and to $x_{3} \in X_{3}$, then by $C_{4}$-freeness, $b$ is adjacent to every vertex $x_{2}$ of $X_{2}$ and to no vertex of $X_{4} \cup X_{5}$ for otherwise $V(H) \subseteq N(b)$. Similarly, if $V(H) \cap N(c)$ is neither a clique then $N(c)$ entirely contains $X_{4}$ or $X_{5}$, say $N(c)$ contains vertices in $X_{3}$ and in $X_{5}$ and entirely contains $X_{4}$. The set $U$ is contained in $X_{1} \cup X_{5}$. If $U \neq \emptyset$, then $U$ contains a vertex $u$ adjacent to some vertex $v$ in $X_{2}$ or in $X_{4}$, say $v \in X_{4}$. The set $S=\{u, v, a, b\}$ dominates $V(H) \cup N[a]$. If $U=\emptyset$, then $b$ (respectively $c$ ) entirely dominates $X_{1}$ (respectively $X_{5}$ ). Let $u$ and $v$ be vertices in $X_{4}$. Again $S=\{u, v, a, b\}$ dominates $V(H) \cup N[a]$. Finally, if $V(H) \cap N(b)$ is not a clique and $V(H) \cap N(c)$ is a clique $C$, we can find two adjacent vertices $u$ and $v$ with $u$ in $C$ or in $U$, depending on whether $U$ is or not equal to $\emptyset$, and $v$ in $C$. The set $S=\{u, v, a, b\}$ dominates $V(H) \cup N[a]$. In any case, the set $S$ is 2 -independent and a maximal 2-independent set of $G$ containing $S$ contains no other vertex in $V(H) \cup N[a]$ and at most $2 \mu_{1}+4\left(\mu_{2}-1\right)=2 \lambda-6$ vertices in $G^{\prime}-V(H)$. Hence $i_{2}(G) \leq 2 \lambda-2$, a contradiction. Therefore Subcase 2.1 is impossible.

Subcase 2.2. $H$ is a non-trivial clique contained in $N(b) \cup N(c)$. Since every maximal 2 -independent set $S$ of $G$ contains at most two vertices in each clique-component and four vertices in each $\mathcal{F}$-component of $G^{\prime}$, $2 \lambda=i_{2}(G) \leq 2\left(\mu_{1}-1\right)+4 \mu_{2}+i_{2}(V(H) \cup N[a])=2 \lambda-4+i_{2}(V(H) \cup N[a])$, which gives $i_{2}(V(H) \cup N[a]) \geq 4$. Hence if $S$ is a maximal 2-independent set of $G[V(H) \cup N[a]]$ containing $\{a, c\}$, then $\mid S \cap(V(H) \backslash N(c) \mid \geq 2$. Let $x$ and $x^{\prime}$ be two vertices in $V(H) \backslash N(c) \subseteq V(H) \cap N(b)$. Similarly, $V(H) \backslash N(b)$ contains at least two vertices $y$ and $y^{\prime}$ which are adjacent to $c$. The induced subgraph $G\left[\left\{a, b, c, x, x^{\prime}, y, y^{\prime}\right\}\right]$ is equal to $g$. Hence $G$ is $C_{4}$-free and $V(H)$ is partitioned into $V(H) \cap N(b)$ and $V(H) \cap N(c)$.

If $\beta(N[a])>2$, let $\left\{a_{1}, a_{2}, \ldots, a_{\ell}\right\}$ be a $\beta(N[a])$-set with $\ell \geq 3$. For each pair $\left\{a_{i}, a_{j}\right\}$ of nonadjacent vertices of $N(a)$, there exists, by Subcase 2.1, a non-trivial clique-component $H_{i j}$ of $G^{\prime}$ contained in $N\left(a_{i}\right) \cap N\left(a_{j}\right)$. Since $G$ is $C_{4}$-free, $V\left(H_{i j}\right) \cap N\left(a_{i}\right)$ and $V\left(H_{i j}\right) \cap N\left(a_{j}\right)$ partition $V\left(H_{i j}\right)$ and the $\ell(\ell-1) / 2$ cliques $H_{i j}$ are different. Then any maximal independent set $I$ of $G$ containing $\left\{a_{1}, a_{2}, \ldots, a_{\ell}\right\}$ satisfies $\beta(G)-|I| \geq \ell(\ell-1) / 2+1-\ell>0$, contradicting $i(G)=\beta(G)$. Therefore $\beta(N[a])=2$.

Let $u$ be a vertex of $N(a)$ adjacent to $c$ but not to $b$, if any. As above, let $H^{\prime}$ be the clique-component of $G^{\prime}$ contained in $N(b) \cup N(u)$. If $H^{\prime} \neq H$, then $N(b)$ contains at least one vertex in $V(H)$, one vertex in $V\left(H^{\prime}\right)$ and $a$, that is $\beta(N[b]) \geq 3$, in contradiction to the choice of $a$. Therefore $H^{\prime}=H$ and $N(u) \cap V(H)=N(c) \cap V(H)$. Similarly, $N(v) \cap V(H)=N(b) \cap V(H)$ for every vertex $v$ of $N(a)$ adjacent to $b$ but not to $c$. Hence every vertex $z$ in $N[a] \cup H$ satisfies $\beta(N[z] \cap(V(H) \cup N[a]))=2$ and by the choice of $a, N[z] \subseteq V(H) \cup N[a]$ and $N[a] \cup V(H)$ forms a component $L$ of $G$. For each vertex $z$ of $L$, the vertices of $L$ which are not adjacent to $z$ form a clique. The clique $V(L) \backslash N[a]$ is $H$. Let $\mathcal{B}, \mathcal{C}, \mathcal{X}, \mathcal{Y}$ be respectively the cliques $V(L) \backslash N[b], V(L) \backslash N[c], V(L) \backslash N[x], V(L) \backslash N[y]$. Let $B=\mathcal{C} \cap \mathcal{Y}$, $A=\mathcal{Y} \cap \mathcal{X}, C=\mathcal{X} \cap \mathcal{B}, Y=\mathcal{B} \cap H, X=H \cap \mathcal{C}$. Then $a \in A, b \in B$, $c \in C, x \in X$ and $y \in Y$. Since $G$ is $C_{4}$-free, $A \cap B=\emptyset$ and $(A, B)$ form a partition of $\mathcal{Y}$. Similarly $(A, C),(C, Y),(Y, X)$ and $(X, B)$ respectively form a partition of $\mathcal{X}, \mathcal{B}, V(H)$ and $\mathcal{C}$. Finally if, say, $|A|=1$, then every maximal 2-independent set of $G$ containing $\{x, y, a\}$ contains no other vertex in $L$ and thus at most $2\left(\mu_{1}-1\right)+4 \mu_{2}+3=2(\lambda-1)+1$ vertices, in contradiction to $i_{2}(G)=2 \lambda$. Therefore each of the five cliques $A, B, X, Y, C$ has at least two vertices and $L$ is a graph of $\mathcal{F}$. The components of $G^{\prime}$ are $\mu_{1}$ non-trivial cliques and $\mu_{2}$ graphs of $\mathcal{F}$ with $\mu_{1}+2 \mu_{2}=\lambda-1$. Hence the components
of $G$ are $\lambda_{1}=\mu_{1}-1$ non-trivial cliques and $\lambda_{2}=\mu_{2}+1$ graphs of $\mathcal{F}$ with $\lambda_{1}+2 \lambda_{2}=\mu_{1}+2 \mu_{2}+1=\lambda$. This completes the proof.

## 4. Bounds on $i_{2}$

In [4], it is proved that every graph $G$ of maximum degree $\Delta \geq 1$ satisfies $i_{k}(G) \geq(n+k-1) /(\Delta+1)$ for $1 \leq k \leq n-1$ and examples of extremal graphs are given for $k \geq 3$. Here we slightly improve the bound when $k=2$ and characterize the extremal graphs.

Theorem 12. Let $G$ be a connected graph of order $n \geq 2$ and maximum degree $\Delta$. Then $i_{2}(G) \geq(n+2) /(\Delta+1)$, with equality if and only if $G=P_{2}$ or $G$ is obtained from a double star $S_{\Delta-1, \Delta-1}$ by adding zero or more edges between its leaves without creating a vertex of degree larger than $\Delta$.

Proof. If $n=2$, then $i_{2}\left(P_{2}\right)=2=(n+2) /(\Delta+1)$. If $n=3$ then $G=P_{3}$ or $C_{3}$ and $i_{2}(G)=2>(n+2) /(\Delta+1)$. So assume that $n \geq 4$ and $\Delta \geq 2$ since $G$ is connected. Let $S$ be a $i_{2}(G)$-set, $p$ be the number of edges in $G[S]$ and $t$ the number of edges joining the vertices in $S$ and $V-S$. Assume first that $p \geq 1$. Then since the $p$ edges are independent, $t \leq 2 p(\Delta-1)+(|S|-2 p) \Delta$. Also since $S$ dominates $V-S, t \geq|V-S|$. It follows that $|V-S| \leq t \leq 2 p(\Delta-1)+(|S|-2 p) \Delta$. Thus

$$
i_{2}(G)=|S| \geq(n+2 p) /(\Delta+1) \geq(n+2) /(\Delta+1)
$$

If further $i_{2}(G)=(n+2) /(\Delta+1)$, then we must have equality throughout the above inequality chain, in particular we have $p=1$, every vertex of $\langle S\rangle$ has degree $\Delta$ and every vertex of $V-S$ is adjacent to exactly one vertex of $S$. If $\langle S\rangle$ contains an isolated vertex say $u$, then $S \cup\{v\}$ is a 2 independent set of $G$, where $v \in V-S$ is any neighbor of $u$, contradicting the maximality of $S$. Therefore $S$ contains only two adjacent vertices, each of degree $\Delta$, and $G$ has the structure described in the theorem. The converse is easy to show.

Now assume that $p=0$. Then $t \leq \Delta|S|$. If $V-S$ contains any vertex, say $w$, that has only one neighbor in $S$ then $S \cup\{w\}$ is a 2-independent set of $G$, a contradiction with the maximality of $S$. Thus each vertex of $V-S$ has at least two neighbors in $S$ and hence $t \geq 2|V-S|$. It follows that $\Delta|S| \geq t \geq 2|V-S|$ and so $i_{2}(G) \geq 2 n /(\Delta+2)$. Notice that $2 n /(\Delta+2) \geq$ $(n+2) /(\Delta+1)$ for $n \geq 4$ with equality if and only if $n=4$ and $\Delta=2$.

If further $i_{2}(G)=(n+2) /(\Delta+1)$ then $n=4, \Delta=2$ and every vertex of $V-S$ has exactly two neighbors in $S$. Thus $G$ is a cycle $C_{4}$ which is obtained from a double star $S_{1,1}$ by adding an edge joining the two leaves.

In [1], Blidia et al. have given an upper bound on $i_{2}(G)$ for every nontrivial connected bipartite graph.

Theorem 13. If $G$ is a connected nontrivial bipartite graph with $s(G)$ support vertices, then $i_{2}(G) \leq(n+s(G)) / 2$.

When $G$ is a cactus, this upper bound can be extended to non-bipartite graphs. First we give a lemma related to matchings in cactus.

Lemma 14. In every cactus $G$ with $k$ odd cycles, there exists a matching of size $k$ containing exactly one edge in each odd cycle of $G$.

Proof. We proceed by induction on the number of odd cycles. Clearly the property is true for $k=0$ and $k=1$. Let $k \geq 2$. Assume the property true for cactus with less than $k$ odd cycles and let $G$ be a cactus with $k$ odd cycles. Let $C=x_{1} x_{2} \cdots x_{2 p+1}$ with $p \geq 1$ be an odd cycle of $G$. For each $x_{i} \in V(C)$, let $A_{i}=N\left(x_{i}\right) \backslash V(C)$. By the definition of cactus, all the sets $A_{i}$ are disjoint. Let $G^{\prime}$ be the graph obtained from $G$ by contracting the cycle $C$ into one vertex $c$. More precisely, $V\left(G^{\prime}\right)=(V(G) \backslash V(C)) \cup\{c\}$ and for $1 \leq i \leq 2 p+1$, the edges between $x_{i}$ and $A_{i}$ are replaced by the edges between $c$ and $A_{i}$. Every cycle $\mathcal{C} \neq C$ of $G$ is unchanged in $G^{\prime}$ if $V(\mathcal{C}) \cap V(C)=\emptyset$ or is replaced by a cycle $\mathcal{C}^{\prime}$ of same length and containing $c$ if $|V(\mathcal{C}) \cap V(C)|=1$. Hence $G^{\prime}$ is a cactus with $k-1$ odd cycles and by the inductive hypothesis, contains a matching $M^{\prime}$ of size $k-1$ with exactly one edge in each of its odd cycles. All the edges of $M^{\prime}$ are edges of $G$ except possibly one, say $c y_{1}$ with $y_{1} \in A_{1}$. In this case, the edge $c y_{1}$ belongs to an odd cycle $\mathcal{C}_{1}^{\prime}$ of $G^{\prime}$ corresponding to an odd cycle $\mathcal{C}_{1}$ of $G$ containing $x_{1}$. The set $M=M^{\prime} \cup\left\{x_{2} x_{3}\right\}$ if $M^{\prime} \subseteq E(G), M=\left(M^{\prime} \backslash\left\{c y_{1}\right\}\right) \cup\left\{x_{1} y_{1}, x_{2} x_{3}\right\}$ if $c y_{1} \in M^{\prime}$, is a matching of $G$ containing exactly one edge in each of its odd cycles.

Theorem 15. If $G$ is a connected nontrivial cactus graph with $k$ odd cycles and $s(G)$ support vertices, then $i_{2}(G) \leq(n+s(G)+k) / 2$ and this bound is sharp.

Proof. Let $G$ be a connected nontrivial cactus graph with $k$ odd cycles and $s(G)$ support vertices. If $k=0$, then $G$ is a bipartite graph and hence by Theorem 13 the result is valid. So assume that $G$ contains at least one odd cycle. By Lemma 14, there exists in $G$ a matching $M$ of size $k$ containing one edge in each odd cycle of $G$. We subdivide each edge of $M$ by exactly one vertex. Let $D$ be the set of such vertices and $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ the resulting graph. Then every vertex of $D$ has degree two and $G^{\prime}$ is a connected bipartite graph of order $n+k$ with $s\left(G^{\prime}\right)=s(G)$ and different from a tree. Let $C$ be a set of leaves of $G^{\prime}$ so that every support vertex has exactly one leaf in $C$. Clearly $|C|=s(G)$. Let $A$ and $B$ be the two classes of the bipartition of $G^{\prime}\left[V^{\prime} \backslash C\right]$ with $|A| \leq|B|$. Then $|B| \geq\left(n+k-s\left(G^{\prime}\right)\right) / 2 \geq|A|>0$. Let $S_{A}, C_{A}$ denote the set of support vertices and leaves of $G^{\prime}$ belonging to $A$, respectively, and let $A^{\prime}=A \backslash\left(S_{A} \cup C_{A}\right)$. Likewise, we define $S_{B}, C_{B}$ and $B^{\prime}$. The 2-independent set $S^{\prime}=A \cup C$ is maximal in $G^{\prime}$ since every leaf of $B$ is adjacent to a support vertex of $A$, which has degree one in $G^{\prime}\left[S^{\prime}\right]$, and the other vertices of $B$ have at least two neighbors in $A$. Its order satisfies

$$
\left|S^{\prime}\right|=|A \cup C| \leq\left(n+k-s\left(G^{\prime}\right)\right) / 2+|C|=(n+s(G)+k) / 2 .
$$

We shall construct a maximal 2-independent set $S$ of $G$ with $|S| \leq(n+$ $s(G)+k) / 2$. Let $D_{A}=D \cap A, D_{B}=D \cap B, F_{B}=N\left(D_{A}\right) \cap B$ and $F_{A}=N\left(D_{B}\right) \cap A$. Note that each of $G\left[F_{A}\right]$ and $G\left[F_{B}\right]$ consists of disjoint copies of $P_{2}$. Then $F_{B} \subset B \backslash\left(C_{B} \cup D_{B}\right), F_{A} \subset A \backslash\left(C_{A} \cup D_{A}\right)$. Thus each component in $G\left[A \backslash D_{A}\right]$ is either an isolated vertex or a path $P_{2}$. So $A \backslash D_{A}$ is a 2-independent set but $\left(A \backslash D_{A}\right) \cup C$ may be not 2-independent. This occurs if $F_{A} \cap S_{A} \neq \emptyset$. In that case delete from $C$ all leaves adjacent to $F_{A} \cap S_{A}$ and let $C^{\prime} \subseteq C$ be the resulting set. Thus $\left(A \backslash D_{A}\right) \cup C^{\prime}$ is a 2 -independent set. To extend it to a maximal 2 -independent set of $G$, we can only add vertices of $B \backslash D_{B}$ having only one neighbor in $A \backslash D_{A}$ and this neighbor must be isolated in $G\left[\left(A \backslash D_{A}\right) \cup C^{\prime}\right]$. Hence we add at most one endvertex of each edge of $G\left[F_{B}\right]$, that is at most $\left|D_{A}\right|$ vertices. Thus $i_{2}(G) \leq\left|\left(A \backslash D_{A}\right) \cup C^{\prime}\right|+\left|D_{A}\right| \leq|A \cup C|=\left|S^{\prime}\right|$. This completes the proof. Odd cycles are examples of graphs attaining the bounds.

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