MAXIMAL k-INDEPENDENT SETS IN GRAPHS

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Abstract

A subset of vertices of a graph G is k-independent if it induces in G a subgraph of maximum degree less than k. The minimum and maximum cardinalities of a maximal k-independent set are respectively denoted $i_k(G)$ and $\beta_k(G)$. We give some relations between $\beta_k(G)$ and $\beta_j(G)$ and between $i_k(G)$ and $i_j(G)$ for $j \neq k$. We study two families of extremal graphs for the inequality $i_2(G) \leq i(G) + \beta(G)$. Finally we give an upper bound on $i_2(G)$ and a lower bound when G is a cactus. **Keywords:** k-independent, cactus.

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1. INTRODUCTION

For notation and graph theory terminology, we in general follow [6, 7]. In a graph G = (V, E) of order n(G) = n, the *neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$. If X is a subset of vertices, then $N_G(X) = \bigcup_{v \in X} N_G(v)$. The closed neighborhoods of v and X are respectively $N_G[v] = N(v) \cup \{v\}$ and $N[X] = N(X) \cup X$. The degree of a vertex v of G, denoted by $d_G(v)$, is the order of its neighborhood. For a subset A of V, let us denote by G[A] the subgraph induced in G by A. If x is a vertex of V, then $d_A(x) = |N(x) \cap A|$ and $\Delta(A) = \max\{d_A(x) \mid x \in A\}$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. We denote the set of leaves of a graph G by L(G), the set of support vertices by S(G), and let $|L(G)| = \ell(G), |S(G)| = s(G)$. If $T = P_2$, then $\ell(P_2) = s(P_2) = 2$. A *double star* $S_{p,q}$ is obtained by attaching p leaves at an endvertex of a path P_2 and q leaves at the second one. A *cactus* is a graph in which every edge is contained in at most one cycle. A graph is called *trivial* if its order is n = 1.

An independent set is a set of vertices whose induced subgraph has no edge. The independence number $\beta(G)$ is the maximum cardinality of an independent set in G. The independence domination number i(G) is the minimum cardinality of a maximal independent set in G.

In [5] Fink and Jacobson generalized the concepts of independent and dominating sets. A subset X of V is k-independent if the maximum degree of the subgraph induced by the vertices of X is less or equal to k - 1. The subset X is k-dominating if every vertex of V - X is adjacent to at least k vertices in X. The lower k-independence number $i_k(G)$ is the minimum cardinality of a maximal k-independent set in G, the k-independence number $\beta_k(G)$ is the maximum cardinality of a maximal k-independent set, and the k-domination number $\gamma_k(G)$ is the minimum cardinality of a kdominating set of G. A k-independent set with maximum cardinality of a graph G is called a $\beta_k(G)$ -set. Similarly we define a $i_k(G)$ -set and a $\gamma(G)$ set. For k = 1, the 1-independent and 1-dominating sets are the classical independent and dominating sets and so $i_1(G) = i(G), \beta_1(G) = \beta(G),$ and $\gamma_1(G) = \gamma(G)$.

Note that Borowiecki and Michalak [2] gave a generalization of the concept of k-independence by considering other hereditary-induced properties than the property for a subgraph to have maximum degree at most k - 1.

On the same way that the minmax parameter i is more difficult to study than β , very few results are known on i_k while the literature on β_k , and even more on γ_k , is rather copious. The irregularity of the behaviour of i_k is shown for instance by the followings two facts. The well-known inequalities $\gamma(G) \leq i(G) \leq \beta(G)$ only extend to $\gamma_k(G) \leq \beta_k(G)$ [3] but $i_k(G)$ may be smaller than $\gamma_k(G)$. The sequence $(\beta_k(G))$ is always non-decreasing while the sequence $(i_k(G))$ is not necessarily monotone. In this paper we show some properties related to β_k and i_k .

A matching in a graph G is a collection of pairwise non-adjacent edges.

The matching is called *induced* if no two edges of the matching are joined by an edge in G.

2. Bounds on
$$\beta_k$$
 and i_k .

Theorem 1. For every graph G and integers j, k with $1 \le j \le k$, $\beta_{k+1}(G) \le \beta_j(G) + \beta_{k-j+1}(G)$.

Proof. Let T be a maximum (k + 1)-independent set of G and X both a j-independent and j-dominating set of G[T]. Such a set X exists by [3]. Thus $\beta_j(G) \ge |X|$. Let Y = T - X. Since X is j-dominating in G[T], $\Delta(G[Y]) \le k - j$. Hence Y is a (k - j + 1)-independent set and therefore $\beta_{k-j+1}(G) \ge |Y| = |T| - |X| \ge \beta_{k+1}(G) - \beta_j(G)$.

Corollary 2. For every graph G and every integer $k \ge 1$,

- (a) $\beta_{k+1}(G) \leq \beta_k(G) + \beta(G),$
- (b) $\beta_{k+1}(G) \le 2\beta_{\lceil k+1/2 \rceil}(G),$
- (c) $\beta_{k+1}(G) \le (k+1)\beta(G)$.

The next theorem gives a structural property of the graphs satisfying (c).

Theorem 3. Let $k \geq 2$ be an integer and G a graph such that $\beta_k(G) = k\beta(G)$. Then every $\beta_k(G)$ -set T is the disjoint union of $\beta(G)$ cliques U^j , $1 \leq j \leq \beta$, of order k and every vertex $v \in V \setminus T$ has at least one clique U^j entirely contained in its neighborhood.

Proof. Since T is a k-independent set, $\Delta(T) \leq k - 1$. Let X_1 be a maximal independent set of G[T]. Every vertex of $T \setminus X_1$ has at least one neighbor in X_1 and thus, $\Delta(T \setminus X_1) \leq k - 2$. Let X_2 be a maximal independent set of $G[T \setminus X_1]$. Every vertex of $T \setminus (X_1 \cup X_2)$ has at least one neighbor in X_1 and one in X_2 , and thus $\Delta(T \setminus (X_1 \cup X_2)) \leq k - 3$. We continue the process until the choice of a maximal independent set X_{k-1} of $G[T \setminus (X_1 \cup \cdots \cup X_{k-2})]$. Then $\Delta(T \setminus (X_1 \cup \cdots \cup X_{k-1}) \leq 0$ and thus the set $X_k = T \setminus (X_1 \cup \cdots \cup X_{k-1})$ is independent. Therefore every set X_i is independent in G and $|X_i| \leq \beta(G)$ for $1 \leq i \leq k$. Hence $|T| = \sum_{i=1}^k |X_i| \leq k\beta(G)$ and since $T = \beta_k(G) = k\beta(G), |X_i| = \beta(G) (= \beta$ for short) for $1 \leq i \leq k$. Let $X_1 = \{u_1^1, u_1^2, \ldots, u_1^\beta\}$. Then $k\beta = |T| = |N_T[u_1^1] \cup N_T[u_1^2] \cup \cdots \cup N_T[u_1^\beta]| \leq \sum_{j=1}^\beta |N_T[u_1^j]| \leq k\beta$ since $d_T(u_1^j) \leq k - 1$ for $1 \leq j \leq \beta$.

Therefore the sets $N_T[u_1^j]$ are disjoint and $|N[u_1^j]| = k$ for $1 \leq j \leq \beta$. If one of the sets $N_T[u_1^j]$, say $N_T[u_1^1]$, does not induce a clique, let a and b two non-adjacent vertices of $N_T(u_1^1)$. Then $\{a, b, u_1^2, \ldots, u_1^\beta\}$ is an independent set of $\beta + 1$ elements of G, a contradiction. Hence each $U^j = N[u_1^j]$ is a clique and G is the disjoint unions of β cliques of order k. Let now vbe any vertex in $V \setminus T$. If every clique U^j contains a vertex which is not adjacent to v, say $u_1^j v \notin E(G)$ for $1 \leq j \leq \beta$, then $\{v, u_1^1, u_1^2, \ldots, u_1^\beta\}$ is an independent set of G of $\beta + 1$ elements, a contradiction which completes the proof.

Corollary 4. Every connected graph with order n and clique number $\omega < n$ satisfies $\beta_{\omega}(G) < \omega\beta(G)$.

Proof. If $\beta_{\omega}(G) = \omega\beta(G)$, then every $\beta_{\omega}(G)$ -set T consists of disjoint cliques $K_{\omega(G)}$. Since G is connected and different from K_{ω} , $V \setminus T$ is not empty and every vertex $v \in V \setminus T$ forms with one of these cliques a clique of order $\omega + 1$, a contradiction.

Theorem 5. For every graph G and integers j, k with $1 \le j \le k$, $i_{k+1}(G) \le (k - j + 2)i_j(G)$. Equality can occur only when j = 1 or j = k.

Proof. Let S be a $i_j(G)$ -set, $X = \{x \in S | d_S(x) = j-1\}$ and $Y = S \setminus X = \{y \in S | d_S(y) < j-1\}$. Since j < k+1, the set S is a (k+1)-independent set of G. Let I be a maximal (k+1)-independent set of G containing S, $A = N_{I \setminus S}(X)$ and $B = I \setminus (A \cup S)$. Since I is (k+1)-independent, $d_A(x) \le k - j + 1$ for every $x \in X$, which implies $|A| \le (k - j + 1)|X|$, and $d_I(y) \le k$ for every $y \in Y$. Since the j-independent set S is maximal in G, $d_Y(v) \ge j$ for every $v \in B$. Hence the number m(Y,B) of edges of G between Y and B satisfies $j|B| \le m(Y,B) \le k|Y|$, which implies $|B| \le k|Y|/j \le (k - j + 1)|Y|$. Therefore

$$i_{k+1}(G) \le |I| = |A| + |B| + |S| \le (k-j+1)(|X|+|Y|) + |S| = (k-j+2)i_j(G).$$

If $i_{k+1}(G) = (k - j + 2)i_j(G)$, then $i_{k+1}(G) = |I|$, |A| = (k - j + 1)|X|, and |B| = k|Y|/j = (k - j + 1)|Y|. Equality |A| = (k - j + 1)|X| implies $d_A(x) = k - j + 1$ for every $x \in X$. Equality |B| = k|Y|/j implies $d_B(y) = k$ for every $y \in Y$ and $d_Y(v) = j$ for every $v \in B$. Finally, k/j = k - j + 1 if and only if j = 1 or j = k. Case j = 1. If $i_{k+1}(G) = (k+1)i(G)$ for some $k \ge 1$, then $Y = \emptyset$, X = S is an independent set, $A = I \setminus S$, |A| = k|X|, the neighborhoods in A of the i(G) vertices of S are disjoint and each of order k, and G[A] has maximum degree at most k - 1.

Case j = k. If $i_{k+1}(G) = 2i_k(G)$ for some $k \ge 1$, then |A| = |X|, |B| = |Y|, the edges of G between A and X form a perfect matching M and the edges of G between B and Y form a k-regular bipartite graph.

Corollary 6. For every graph G of order n and maximum degree Δ , $i_{\Delta}(G) \ge n/2$, and this bound is sharp.

Proof. Obvious consequence of $i_{k+1}(G) \leq 2i_k(G)$ obtained from Theorem 5 when j = k and $i_{\Delta+1}(G) = n$.

Let G be obtained by attaching one pendant vertex at each vertex of a clique K_k . Then n = 2k, $\Delta = k$ and $i_k(G) = k$. Hence $i_{\Delta} = n/2$.

Corollary 7. If a graph G of order $n \ge 2$ satisfies $i_2(G) = 2i(G)$, then G contains an induced matching of size i(G).

Proof. In the equality case k = j = 1 in Theorem 5, S = X, G contains a perfect matching between A and X, and this matching is induced of size S = i(G) since G[A] has maximum degree 0.

The converse of Corollary 7 is not true. For instance the cycle C_6 admits an induced mathing M of size $i(C_6) = 2$ but $i_2(C_6) = 3 < 2i(G)$.

The inequality $i_2(G) \leq 2i(G)$ cannot be improved to $i_2(G) \leq 2\gamma(G)$, even for trees, as shown by the caterpillar obtained by adding $k \geq 5$ pendant vertices at each vertex of a path P_3 . However the next theorem improves it to $i_2(G) \leq \gamma(G) + i(G)$ in the class of trees and unicyclic graphs.

Theorem 8. If the graph G contains at most one cycle, then $i_2(G) \leq \gamma(G) + i(G)$.

Proof. Let S be a i(G)-set and I a maximal 2-independent set of G containing S. With the notation of Theorem 5, X = S is independent, $A = N_I(S) = I \setminus S$, and the edges of G[I] form an induced matching M between A and a subset A' of S. Let Z be a $\gamma(G)$ -set, M_1 the edges of M with no endvertex in Z, and A_1 (A'_1 respectively) the set of the endvertices of the edges of M_1 in A (A' respectively). If $\gamma(G) < |M|$, then $M_1 \neq \emptyset$

and since M is induced, the vertices of $A_1 \cup A'_1$ cannot be dominated by vertices in $Z \cap (A \cup A')$. Hence the set $W = Z \setminus (A \cup A')$ is not empty and dominates $A_1 \cup A'_1$. Therefore the induced subgraph $G[W \cup A_1 \cup A'_1]$ of order $|W| + 2|M_1|$ contains at least $3|M_1|$ edges. Moreover, since Z contains at least one endvertex of each edge in $M \setminus M_1$, $|W| \leq |Z| - |M \setminus M_1| = (\gamma(G) - |M|) + |M_1| < |M_1|$. Thus $3|M_1| > |W| + 2|M_1|$, which contradicts the assumption that G contains at most one cycle. Therefore $\gamma(G) \geq |M| = |A|$ and $i_2(G) \leq |S| + |A| \leq i(G) + \gamma(G)$.

The result of Theorem 8 is not valid for all graphs as shown by the following example. We consider eight disjoint triangles $x_iy_iz_i$ and identify the vertex x_i with x_{i+1} for i = 1, 3, 5, 7. Let w_1, w_2, w_3, w_4 denote the resulting new vertices. To complete G, we add the edges w_1w_2, w_1w_3 and w_1w_4 . Then $\{w_2, y_3, w_3, y_5, w_4, y_7, y_1, z_1, y_2, z_2\}$ is a $i_2(G)$ -set and thus $i_2(G) =$ $10, \gamma(G) = 4$, and i(G) = 5. By attaching q triangles at each vertex w_i instead of 2, $\gamma(G)$ does not change while now i(G) = 3 + q and $i_2(G) = 6 + 2q$. Therefore the difference $i_2(G) - (i(G) + \gamma(G))$ can be done arbitrarily large and the ratio $i_2(G)/(i(G) + \gamma(G))$ arbitrarily close to 2.

The next corollary is another consequence of Theorem 5. A graph G is well-covered if $i(G) = \beta(G)$ and well-k-covered if $i_k(G) = \beta_k(G)$.

Corollary 9. For any $k \ge 1$, $i_{k+1}(G) \le (k+1)i(G) \le ki(G) + \beta(G)$ and if $i_{k+1}(G) = ki(G) + \beta(G)$, then G is well-covered and well-(k+1)-covered.

Proof. The inequality comes from Theorem 5 with j = 1. If $i_{k+1}(G) = ki(G) + \beta(G)$ then $i(G) = \beta(G)$, that is G is well-covered, and thus $i_{k+1}(G) = (k+1)\beta(G)$. Therefore $\beta_{k+1}(G) \leq i_{k+1}(G)$ from Corollary 2, which implies $\beta_{k+1}(G) = i_{k+1}(G)$ and proves that G is well-(k+1)-covered.

3. Graphs with
$$i_2 = i + \beta$$

In this section we are interested in graphs G satisfying the equality in Corollary 9 when k = 1. We describe two particular classes of them defined by forbidden subgraphs.

Definition 10

• The graphs F of the family \mathcal{F} are formed by five disjoint cliques X_i of cardinality at least 2 together with all the edges between X_i and X_{i+1} for $1 \leq i \leq 5 \pmod{5}$.

• The graphs C_4 and g are shown in Figure 1.



Clearly every non-trivial clique and every graph of \mathcal{F} satisfies $i_2(G) = i(G) + \beta(G)$ with $i(G) = \beta(G) = 1$ for a clique, $i(G) = \beta(G) = 2$ for a graph of \mathcal{F} .

Theorem 11. Let G be a graph such that $i_2(G) = i(G) + \beta(G)$. Then

- 1. If G is g-free, the component of G are λ_1 non-trivial cliques with $\lambda_1 = i(G)$.
- 2. If G is C₄-free, the components of G are $\lambda_1 \ge 0$ non-trivial cliques and $\lambda_2 \ge 0$ graphs of \mathcal{F} with $\lambda_1 + 2\lambda_2 = i(G)$.

Proof. From Corollary 9, $i(G) = \beta(G)$ and $i_2(G) = \beta_2(G)$. The maximal independent sets of G have all the same cardinality and each maximal 2independent set induces a matching M of size i(G). In particular, G has no isolated vertex. We make an induction on the common value λ of i(G)and $\beta(G)$. If $\lambda = 1$, then G is a clique of cardinality at least 2. For $\lambda > 1$, suppose the property true when $i_2(G) = i(G) + \beta(G) < 2\lambda$ and let G be a gfree or C_4 -free graph such that $i_2(G) = i(G) + \beta(G) = \beta_2(G) = 2\lambda$. Let a be a vertex of G such that $\beta(N[a])$ is maximum and let G' = G - N[a]. Every maximal independent set of G' can be completed to a maximal independent set of G by adding a. Hence $i(G') = \beta(G') = \lambda - 1$. If S' is a $i_2(G')$ -set then, by Theorem 5, $|S'| = i_2(G') \le 2i(G') = 2\lambda - 2$. The 2-independent set $S' \cup \{a\}$ of G can be completed to a maximal 2-independent set of G by adding at most one vertex of N(a). Hence $i_2(G) \leq |S'| + 2 \leq 2\lambda$. Since $i_2(G) = 2\lambda, i_2(G') = 2\lambda - 2 = i(G') + \beta(G')$. By the induction hypothesis applied to G', which is g-free or C_4 -free as G, the components of G' are μ_1 non-trivial cliques if G is g-free, μ_1 non-trivial cliques and μ_2 graphs of \mathcal{F} if G is C_4 -free, with $\mu_1 + 2\mu_2 = \lambda - 1$. To continue, we distinguish two cases.

Case 1. N[a] is a clique of G. By the choice of a, N[x] is a clique for every vertex x of G. Therefore the components of G are λ non-trivial cliques.

Case 2. N[a] is not a clique. Let b and c be two non-adjacent vertices of N(a). If for each component H of G', $V(H) \setminus (N(b) \cup N(c)) \neq \emptyset$ if H is a clique and $\beta(V(H) \setminus N(b) \cup N(c)) = 2$ if $H \in \mathcal{F}$, then $\beta(G) \geq \beta(G') + 2 = \lambda + 1$ which is impossible. Therefore there exists a component H of G' such that either H is a non-trivial clique and $V(H) \subseteq N(b) \cup N(c)$ or $H \in \mathcal{F}$ and $\beta(V(H) \setminus N(b) \cup N(c)) < 2$.

Subcase 2.1. Suppose first that $H \in \mathcal{F}$ and $\beta(V(H) \setminus (N(b) \cup N(c))) \leq 1$. Then G is not g-free and thus is C_4 -free. We will prove that this subcase is impossible. Let X_i , $1 \le i \le 5$ be the five cliques of H as described in the definition of \mathcal{F} . Then $V(H) \setminus (N(b) \cup N(c))$ is a (possibly empty) clique U and since G is C_4 -free, $N(b) \cap N(c) \cap V(H) = \emptyset$. If $V(H) \subseteq N(b)$, then a maximal 2-independent set I of G containing $\{a, b\}$ contains no other vertex in $N[a] \cup V(H)$ and at most $2\mu_1 + 4(\mu_2 - 1) = 2\lambda - 6$ vertices in G' - V(H), that is $|I| \leq 2\lambda - 4$ which is impossible. Hence V(H) is not contained in N(b), neither in N(c) by symmetry. If $V(H) \cap N(b)$ and $V(H) \cap N(c)$ are cliques, then $U \neq \emptyset$ and H contains an edge uv with $u \in U$ and $v \in N(c)$. The set $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. If $N(b) \cap V(H)$ is not a clique, say b is adjacent to $x_1 \in X_1$ and to $x_3 \in X_3$, then by C_4 -freeness, b is adjacent to every vertex x_2 of X_2 and to no vertex of $X_4 \cup X_5$ for otherwise $V(H) \subseteq N(b)$. Similarly, if $V(H) \cap N(c)$ is neither a clique then N(c) entirely contains X_4 or X_5 , say N(c) contains vertices in X_3 and in X_5 and entirely contains X_4 . The set U is contained in $X_1 \cup X_5$. If $U \neq \emptyset$, then U contains a vertex u adjacent to some vertex v in X_2 or in X_4 , say $v \in X_4$. The set $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. If $U = \emptyset$, then b (respectively c) entirely dominates X_1 (respectively X_5). Let u and v be vertices in X_4 . Again $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. Finally, if $V(H) \cap N(b)$ is not a clique and $V(H) \cap N(c)$ is a clique C, we can find two adjacent vertices u and v with u in C or in U, depending on whether U is or not equal to \emptyset , and v in C. The set $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. In any case, the set S is 2-independent and a maximal 2-independent set of G containing S contains no other vertex in $V(H) \cup N[a]$ and at most $2\mu_1 + 4(\mu_2 - 1) = 2\lambda - 6$ vertices in G' - V(H). Hence $i_2(G) \leq 2\lambda - 2$, a contradiction. Therefore Subcase 2.1 is impossible.

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Subcase 2.2. H is a non-trivial clique contained in $N(b) \cup N(c)$. Since every maximal 2-independent set S of G contains at most two vertices in each clique-component and four vertices in each \mathcal{F} -component of G', $2\lambda = i_2(G) \leq 2(\mu_1 - 1) + 4\mu_2 + i_2(V(H) \cup N[a]) = 2\lambda - 4 + i_2(V(H) \cup N[a])$, which gives $i_2(V(H) \cup N[a]) \geq 4$. Hence if S is a maximal 2-independent set of $G[V(H) \cup N[a]]$ containing $\{a, c\}$, then $|S \cap (V(H) \setminus N(c)| \geq 2$. Let x and x' be two vertices in $V(H) \setminus N(c) \subseteq V(H) \cap N(b)$. Similarly, $V(H) \setminus N(b)$ contains at least two vertices y and y' which are adjacent to c. The induced subgraph $G[\{a, b, c, x, x', y, y'\}]$ is equal to g. Hence G is C_4 -free and V(H) is partitioned into $V(H) \cap N(b)$ and $V(H) \cap N(c)$.

If $\beta(N[a]) > 2$, let $\{a_1, a_2, \ldots, a_\ell\}$ be a $\beta(N[a])$ -set with $\ell \ge 3$. For each pair $\{a_i, a_j\}$ of nonadjacent vertices of N(a), there exists, by Subcase 2.1, a non-trivial clique-component H_{ij} of G' contained in $N(a_i) \cap N(a_j)$. Since G is C_4 -free, $V(H_{ij}) \cap N(a_i)$ and $V(H_{ij}) \cap N(a_j)$ partition $V(H_{ij})$ and the $\ell(\ell-1)/2$ cliques H_{ij} are different. Then any maximal independent set I of G containing $\{a_1, a_2, \ldots, a_\ell\}$ satisfies $\beta(G) - |I| \ge \ell(\ell-1)/2 + 1 - \ell > 0$, contradicting $i(G) = \beta(G)$. Therefore $\beta(N[a]) = 2$.

Let u be a vertex of N(a) adjacent to c but not to b, if any. As above, let H' be the clique-component of G' contained in $N(b) \cup N(u)$. If $H' \neq H$, then N(b) contains at least one vertex in V(H), one vertex in V(H') and a, that is $\beta(N[b]) \geq 3$, in contradiction to the choice of a. Therefore H' = Hand $N(u) \cap V(H) = N(c) \cap V(H)$. Similarly, $N(v) \cap V(H) = N(b) \cap V(H)$ for every vertex v of N(a) adjacent to b but not to c. Hence every vertex z in $N[a] \cup H$ satisfies $\beta(N[z] \cap (V(H) \cup N[a])) = 2$ and by the choice of $a, N[z] \subseteq V(H) \cup N[a]$ and $N[a] \cup V(H)$ forms a component L of G. For each vertex z of L, the vertices of L which are not adjacent to z form a clique. The clique $V(L) \setminus N[a]$ is H. Let $\mathcal{B}, \mathcal{C}, \mathcal{X}, \mathcal{Y}$ be respectively the cliques $V(L) \setminus N[b], V(L) \setminus N[c], V(L) \setminus N[x], V(L) \setminus N[y]$. Let $B = \mathcal{C} \cap \mathcal{Y}$, $A = \mathcal{Y} \cap \mathcal{X}, \ C = \mathcal{X} \cap \mathcal{B}, \ Y = \mathcal{B} \cap H, \ X = H \cap \mathcal{C}.$ Then $a \in A, \ b \in B,$ $c \in C, x \in X$ and $y \in Y$. Since G is C_4 -free, $A \cap B = \emptyset$ and (A, B) form a partition of \mathcal{Y} . Similarly (A, C), (C, Y), (Y, X) and (X, B) respectively form a partition of $\mathcal{X}, \mathcal{B}, V(H)$ and \mathcal{C} . Finally if, say, |A| = 1, then every maximal 2-independent set of G containing $\{x, y, a\}$ contains no other vertex in L and thus at most $2(\mu_1 - 1) + 4\mu_2 + 3 = 2(\lambda - 1) + 1$ vertices, in contradiction to $i_2(G) = 2\lambda$. Therefore each of the five cliques A, B, X, Y, C has at least two vertices and L is a graph of \mathcal{F} . The components of G' are μ_1 non-trivial cliques and μ_2 graphs of \mathcal{F} with $\mu_1 + 2\mu_2 = \lambda - 1$. Hence the components

of G are $\lambda_1 = \mu_1 - 1$ non-trivial cliques and $\lambda_2 = \mu_2 + 1$ graphs of \mathcal{F} with $\lambda_1 + 2\lambda_2 = \mu_1 + 2\mu_2 + 1 = \lambda$. This completes the proof.

4. Bounds on i_2

In [4], it is proved that every graph G of maximum degree $\Delta \geq 1$ satisfies $i_k(G) \geq (n+k-1)/(\Delta+1)$ for $1 \leq k \leq n-1$ and examples of extremal graphs are given for $k \geq 3$. Here we slightly improve the bound when k = 2 and characterize the extremal graphs.

Theorem 12. Let G be a connected graph of order $n \ge 2$ and maximum degree Δ . Then $i_2(G) \ge (n+2)/(\Delta+1)$, with equality if and only if $G = P_2$ or G is obtained from a double star $S_{\Delta-1,\Delta-1}$ by adding zero or more edges between its leaves without creating a vertex of degree larger than Δ .

Proof. If n = 2, then $i_2(P_2) = 2 = (n+2)/(\Delta+1)$. If n = 3 then $G = P_3$ or C_3 and $i_2(G) = 2 > (n+2)/(\Delta+1)$. So assume that $n \ge 4$ and $\Delta \ge 2$ since G is connected. Let S be a $i_2(G)$ -set, p be the number of edges in G[S] and t the number of edges joining the vertices in S and V - S. Assume first that $p \ge 1$. Then since the p edges are independent, $t \le 2p(\Delta-1) + (|S|-2p)\Delta$. Also since S dominates V - S, $t \ge |V - S|$. It follows that $|V - S| \le t \le 2p(\Delta - 1) + (|S| - 2p)\Delta$. Thus

$$i_2(G) = |S| \ge (n+2p)/(\Delta+1) \ge (n+2)/(\Delta+1).$$

If further $i_2(G) = (n+2)/(\Delta+1)$, then we must have equality throughout the above inequality chain, in particular we have p = 1, every vertex of $\langle S \rangle$ has degree Δ and every vertex of V - S is adjacent to exactly one vertex of S. If $\langle S \rangle$ contains an isolated vertex say u, then $S \cup \{v\}$ is a 2independent set of G, where $v \in V - S$ is any neighbor of u, contradicting the maximality of S. Therefore S contains only two adjacent vertices, each of degree Δ , and G has the structure described in the theorem. The converse is easy to show.

Now assume that p = 0. Then $t \leq \Delta |S|$. If V - S contains any vertex, say w, that has only one neighbor in S then $S \cup \{w\}$ is a 2-independent set of G, a contradiction with the maximality of S. Thus each vertex of V - Shas at least two neighbors in S and hence $t \geq 2|V - S|$. It follows that $\Delta |S| \geq t \geq 2|V - S|$ and so $i_2(G) \geq 2n/(\Delta + 2)$. Notice that $2n/(\Delta + 2) \geq (n+2)/(\Delta + 1)$ for $n \geq 4$ with equality if and only if n = 4 and $\Delta = 2$. If further $i_2(G) = (n+2)/(\Delta+1)$ then n = 4, $\Delta = 2$ and every vertex of V-S has exactly two neighbors in S. Thus G is a cycle C_4 which is obtained from a double star $S_{1,1}$ by adding an edge joining the two leaves.

In [1], Blidia *et al.* have given an upper bound on $i_2(G)$ for every nontrivial connected bipartite graph.

Theorem 13. If G is a connected nontrivial bipartite graph with s(G) support vertices, then $i_2(G) \leq (n + s(G))/2$.

When G is a cactus, this upper bound can be extended to non-bipartite graphs. First we give a lemma related to matchings in cactus.

Lemma 14. In every cactus G with k odd cycles, there exists a matching of size k containing exactly one edge in each odd cycle of G.

Proof. We proceed by induction on the number of odd cycles. Clearly the property is true for k = 0 and k = 1. Let $k \ge 2$. Assume the property true for cactus with less than k odd cycles and let G be a cactus with k odd cycles. Let $C = x_1 x_2 \cdots x_{2p+1}$ with $p \ge 1$ be an odd cycle of G. For each $x_i \in V(C)$, let $A_i = N(x_i) \setminus V(C)$. By the definition of cactus, all the sets A_i are disjoint. Let G' be the graph obtained from G by contracting the cycle C into one vertex c. More precisely, $V(G') = (V(G) \setminus V(C)) \cup \{c\}$ and for $1 \leq i \leq 2p + 1$, the edges between x_i and A_i are replaced by the edges between c and A_i . Every cycle $\mathcal{C} \neq C$ of G is unchanged in G' if $V(\mathcal{C}) \cap V(\mathcal{C}) = \emptyset$ or is replaced by a cycle \mathcal{C}' of same length and containing c if $|V(\mathcal{C}) \cap V(C)| = 1$. Hence G' is a cactus with k-1 odd cycles and by the inductive hypothesis, contains a matching M' of size k-1 with exactly one edge in each of its odd cycles. All the edges of M' are edges of G except possibly one, say cy_1 with $y_1 \in A_1$. In this case, the edge cy_1 belongs to an odd cycle \mathcal{C}'_1 of G' corresponding to an odd cycle \mathcal{C}_1 of G containing x_1 . The set $M = M' \cup \{x_2x_3\}$ if $M' \subseteq E(G), M = (M' \setminus \{cy_1\}) \cup \{x_1y_1, x_2x_3\}$ if $cy_1 \in M'$, is a matching of G containing exactly one edge in each of its odd cycles.

Theorem 15. If G is a connected nontrivial cactus graph with k odd cycles and s(G) support vertices, then $i_2(G) \leq (n + s(G) + k)/2$ and this bound is sharp.

Proof. Let G be a connected nontrivial cactus graph with k odd cycles and s(G) support vertices. If k = 0, then G is a bipartite graph and hence by Theorem 13 the result is valid. So assume that G contains at least one odd cycle. By Lemma 14, there exists in G a matching M of size k containing one edge in each odd cycle of G. We subdivide each edge of M by exactly one vertex. Let D be the set of such vertices and G' = (V', E') the resulting graph. Then every vertex of D has degree two and G' is a connected bipartite graph of order n + k with s(G') = s(G) and different from a tree. Let C be a set of leaves of G' so that every support vertex has exactly one leaf in C. Clearly |C| = s(G). Let A and B be the two classes of the bipartition of $G'[V' \setminus C]$ with $|A| \leq |B|$. Then $|B| \geq (n + k - s(G'))/2 \geq |A| > 0$. Let S_A, C_A denote the set of support vertices and leaves of G' belonging to A, respectively, and let $A' = A \setminus (S_A \cup C_A)$. Likewise, we define S_B , C_B and B'. The 2-independent set $S' = A \cup C$ is maximal in G' since every leaf of B is adjacent to a support vertex of A, which has degree one in G'[S'], and the other vertices of B have at least two neighbors in A. Its order satisfies

 $|S'| = |A \cup C| \le (n + k - s(G'))/2 + |C| = (n + s(G) + k)/2.$

We shall construct a maximal 2-independent set S of G with $|S| \leq (n + s(G) + k)/2$. Let $D_A = D \cap A$, $D_B = D \cap B$, $F_B = N(D_A) \cap B$ and $F_A = N(D_B) \cap A$. Note that each of $G[F_A]$ and $G[F_B]$ consists of disjoint copies of P_2 . Then $F_B \subset B \setminus (C_B \cup D_B)$, $F_A \subset A \setminus (C_A \cup D_A)$. Thus each component in $G[A \setminus D_A]$ is either an isolated vertex or a path P_2 . So $A \setminus D_A$ is a 2-independent set but $(A \setminus D_A) \cup C$ may be not 2-independent. This occurs if $F_A \cap S_A \neq \emptyset$. In that case delete from C all leaves adjacent to $F_A \cap S_A$ and let $C' \subseteq C$ be the resulting set. Thus $(A \setminus D_A) \cup C'$ is a 2-independent set. To extend it to a maximal 2-independent set of G, we can only add vertices of $B \setminus D_B$ having only one neighbor in $A \setminus D_A$ and this neighbor must be isolated in $G[(A \setminus D_A) \cup C']$. Hence we add at most one endvertex of each edge of $G[F_B]$, that is at most $|D_A|$ vertices. Thus $i_2(G) \leq |(A \setminus D_A) \cup C'| + |D_A| \leq |A \cup C| = |S'|$. This completes the proof. Odd cycles are examples of graphs attaining the bounds.

References

 M. Blidia, M. Chellali, O. Favaron and N. Meddah, On k-independence in graphs with emphasis on trees, Discrete Math. 307 (2007) 2209–2216.

- [2] M. Borowiecki and D. Michalak, Generalized independence and domination in graphs, Discrete Math. 191 (1998) 51–56.
- [3] O. Favaron, On a conjecture of Fink and Jacobson concerning k-domination and k-dependence, J. Combin. Theory (B) 39 (1985) 101–102.
- [4] O. Favaron, k-domination and k-independence in graphs, Ars Combin. 25 C (1988) 159–167.
- [5] J.F. Fink and M.S. Jacobson, n-domination, n-dependence and forbidden subgraphs, Graph Theory with Applications to Algorithms and Computer (John Wiley and sons, New York, 1985) 301–311.
- [6] G. Chartrand and L. Lesniak, Graphs & Digraphs: Third Edition (Chapman & Hall, London, 1996).
- [7] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

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