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MAXIMAL k-INDEPENDENT SETS IN GRAPHS

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Abstract

A subset of vertices of a graph G is *k-independent* if it induces in G a subgraph of maximum degree less than k. The minimum and maximum cardinalities of a maximal k-independent set are respectively denoted $i_k(G)$ and $\beta_k(G)$. We give some relations between $\beta_k(G)$ and $\beta_j(G)$ and between $i_k(G)$ and $i_j(G)$ for $j \neq k$. We study two families of extremal graphs for the inequality $i_2(G) \leq i(G) + \beta(G)$. Finally we give an upper bound on $i_2(G)$ and a lower bound when G is a cactus.

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1. Introduction

For notation and graph theory terminology, we in general follow [6, 7]. In a graph $G = (V, E)$ of order $n(G) = n$, the *neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$. If X is a subset of vertices, then $N_G(X) = \cup_{v \in X} N_G(v)$. The *closed neighbohoods* of v and X are respectively $N_G[v] = N(v) \cup \{v\}$ and $N[X] = N(X) \cup X$. The *degree* of a vertex v of G, denoted by $d_G(v)$, is the order of its neighborhood. For a subset A of V, let us denote by $G[A]$ the subgraph induced in G by A. If x is a vertex of V, then $d_A(x) = |N(x) \cap A|$ and $\Delta(A) = \max\{d_A(x) \mid x \in A\}$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. We denote the set of leaves of a graph G by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell(G)$, $|S(G)| = s(G)$. If $T = P_2$, then $\ell(P_2) = s(P_2) = 2$. A *double star* $S_{p,q}$ is obtained by attaching p leaves at an endvertex of a path P_2 and q leaves at the second one. A *cactus* is a graph in which every edge is contained in at most one cycle. A graph is called *trivial* if its order is $n = 1$.

An *independent set* is a set of vertices whose induced subgraph has no edge. The *independence number* $\beta(G)$ is the maximum cardinality of an independent set in G. The *independence domination number* $i(G)$ is the minimum cardinality of a maximal independent set in G.

In [5] Fink and Jacobson generalized the concepts of independent and dominating sets. A subset X of V is *k-independent* if the maximum degree of the subgraph induced by the vertices of X is less or equal to $k-1$. The subset X is *k-dominating* if every vertex of $V - X$ is adjacent to at least k vertices in X. The *lower k-independence number* $i_k(G)$ is the minimum cardinality of a maximal k-independent set in G, the *k-independence number* $\beta_k(G)$ is the maximum cardinality of a maximal k-independent set, and the *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. A k-independent set with maximum cardinality of a graph G is called a $\beta_k(G)$-set. Similarly we define a $i_k(G)$-set and a $\gamma(G)$-set. For $k = 1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets and so $i_1(G) = i\ (G)$, $\beta_1(G) = \beta(G)$, and $\gamma_1(G) = \gamma(G)$.

Note that Borowiecki and Michalak [2] gave a generalization of the concept of k-independence by considering other hereditary-induced properties than the property for a subgraph to have maximum degree at most $k-1$.

On the same way that the minmax parameter i is more difficult to study than β, very few results are known on i_k while the literature on β_k, and even more on γ_k, is rather copious. The irregularity of the behaviour of i_k is shown for instance by the followings two facts. The well-known inequalities $\gamma(G) \le i(G) \le \beta(G)$ only extend to $\gamma_k(G) \le \beta_k(G)$ [3] but $i_k(G)$ may be smaller than $\gamma_k(G)$. The sequence $(\beta_k(G))$ is always non-decreasing while the sequence $(i_k(G))$ is not necessarily monotone. In this paper we show some properties related to β_k and i_k.

A *matching* in a graph G is a collection of pairwise non-adjacent edges.

The matching is called *induced* if no two edges of the matching are joined by an edge in G.

2. Bounds on β_k and i_k.

Theorem 1. *For every graph G and integers j, k with $1 \leq j \leq k$, $\beta_{k+1}(G) \leq \beta_j(G) + \beta_{k-j+1}(G)$.*

Proof. Let T be a maximum $(k+1)$-independent set of G and X both a j-independent and j-dominating set of $G[T]$. Such a set X exists by [3]. Thus $\beta_j(G) \geq |X|$. Let $Y = T - X$. Since X is j-dominating in $G[T]$, $\Delta(G[Y]) \leq k - j$. Hence Y is a $(k-j+1)$-independent set and therefore $\beta_{k-j+1}(G) \geq |Y| = |T| - |X| \geq \beta_{k+1}(G) - \beta_j(G)$. ■

Corollary 2. *For every graph G and every integer $k \geq 1$,*

(a) $\beta_{k+1}(G) \leq \beta_k(G) + \beta(G)$,

(b) $\beta_{k+1}(G) \leq 2\beta_{\lceil k+1/2 \rceil}(G)$,

(c) $\beta_{k+1}(G) \leq (k+1)\beta(G)$.

The next theorem gives a structural property of the graphs satisfying (c).

Theorem 3. *Let $k \geq 2$ be an integer and G a graph such that $\beta_k(G) = k\beta(G)$. Then every $\beta_k(G)$-set T is the disjoint union of $\beta(G)$ cliques U^j, $1 \leq j \leq \beta$, of order k and every vertex $v \in V \setminus T$ has at least one clique U^j entirely contained in its neighborhood.*

Proof. Since T is a k-independent set, $\Delta(T) \leq k - 1$. Let X_1 be a maximal independent set of $G[T]$. Every vertex of $T \setminus X_1$ has at least one neighbor in X_1 and thus, $\Delta(T \setminus X_1) \leq k - 2$. Let X_2 be a maximal independent set of $G[T \setminus X_1]$. Every vertex of $T \setminus (X_1 \cup X_2)$ has at least one neighbor in X_1 and one in X_2, and thus $\Delta(T \setminus (X_1 \cup X_2)) \leq k - 3$. We continue the process until the choice of a maximal independent set X_{k-1} of $G[T \setminus (X_1 \cup \cdots \cup X_{k-2})]$. Then $\Delta(T \setminus (X_1 \cup \cdots \cup X_{k-1}) \leq 0$ and thus the set $X_k = T \setminus (X_1 \cup \cdots \cup X_{k-1})$ is independent. Therefore every set X_i is independent in G and $|X_i| \leq \beta(G)$ for $1 \leq i \leq k$. Hence $|T| = \sum_{i=1}^{k} |X_i| \leq k\beta(G)$ and since $T = \beta_k(G) = k\beta(G)$, $|X_i| = \beta(G)$ ($= \beta$ for short) for $1 \leq i \leq k$. Let $X_1 = \{u_1^1, u_1^2, \ldots, u_1^\beta\}$. Then $k\beta = |T| = |N_T[u_1^1] \cup N_T[u_1^2] \cup \cdots \cup N_T[u_1^\beta]| \leq \sum_{j=1}^{\beta} |N_T[u_1^j]| \leq k\beta$ since $d_T(u_1^j) \leq k - 1$ for $1 \leq j \leq \beta$.

Therefore the sets $N_T[u_1^j]$ are disjoint and $|N[u_1^j]| = k$ for $1 \le j \le \beta$. If one of the sets $N_T[u_1^j]$, say $N_T[u_1^1]$, does not induce a clique, let a and b two non-adjacent vertices of $N_T(u_1^1)$. Then $\{a, b, u_1^2, \ldots, u_1^\beta\}$ is an independent set of $\beta + 1$ elements of G, a contradiction. Hence each $U^j = N[u_1^j]$ is a clique and G is the disjoint unions of β cliques of order k. Let now v be any vertex in $V \setminus T$. If every clique U^j contains a vertex which is not adjacent to v, say $u_1^j v \notin E(G)$ for $1 \le j \le \beta$, then $\{v, u_1^1, u_1^2, \ldots, u_1^\beta\}$ is an independent set of G of $\beta + 1$ elements, a contradiction which completes the proof. ■

Corollary 4. *Every connected graph with order n and clique number $\omega < n$ satisfies $\beta_\omega(G) < \omega\beta(G)$.*

Proof. If $\beta_\omega(G) = \omega\beta(G)$, then every $\beta_\omega(G)$-set T consists of disjoint cliques $K_{\omega(G)}$. Since G is connected and different from K_ω, $V \setminus T$ is not empty and every vertex $v \in V \setminus T$ forms with one of these cliques a clique of order $\omega + 1$, a contradiction. ■

Theorem 5. *For every graph G and integers j, k with $1 \le j \le k$, $i_{k+1}(G) \le (k - j + 2)i_j(G)$. Equality can occur only when $j = 1$ or $j = k$.*

Proof. Let S be a $i_j(G)$-set, $X = \{x \in S | d_S(x) = j - 1\}$ and $Y = S \setminus X = \{y \in S | d_S(y) < j - 1\}$. Since $j < k + 1$, the set S is a $(k + 1)$-independent set of G. Let I be a maximal $(k + 1)$-independent set of G containing S, $A = N_{I \setminus S}(X)$ and $B = I \setminus (A \cup S)$. Since I is $(k + 1)$-independent, $d_A(x) \le k - j + 1$ for every $x \in X$, which implies $|A| \le (k - j + 1)|X|$, and $d_I(y) \le k$ for every $y \in Y$. Since the j-independent set S is maximal in G, $d_Y(v) \ge j$ for every $v \in B$. Hence the number $m(Y, B)$ of edges of G between Y and B satisfies $j|B| \le m(Y, B) \le k|Y|$, which implies $|B| \le k|Y|/j \le (k - j + 1)|Y|$. Therefore

$$i_{k+1}(G) \le |I| = |A| + |B| + |S| \le (k - j + 1)(|X| + |Y|) + |S| = (k - j + 2)i_j(G).$$

If $i_{k+1}(G) = (k - j + 2)i_j(G)$, then $i_{k+1}(G) = |I|$, $|A| = (k - j + 1)|X|$, and $|B| = k|Y|/j = (k - j + 1)|Y|$. Equality $|A| = (k - j + 1)|X|$ implies $d_A(x) = k - j + 1$ for every $x \in X$. Equality $|B| = k|Y|/j$ implies $d_B(y) = k$ for every $y \in Y$ and $d_Y(v) = j$ for every $v \in B$. Finally, $k/j = k - j + 1$ if and only if $j = 1$ or $j = k$.

Case $j = 1$. If $i_{k+1}(G) = (k+1)i(G)$ for some $k \geq 1$, then $Y = \emptyset$, $X = S$ is an independent set, $A = I \setminus S$, $|A| = k|X|$, the neighborhoods in A of the $i(G)$ vertices of S are disjoint and each of order k, and $G[A]$ has maximum degree at most $k-1$.

Case $j = k$. If $i_{k+1}(G) = 2i_k(G)$ for some $k \geq 1$, then $|A| = |X|$, $|B| = |Y|$, the edges of G between A and X form a perfect matching M and the edges of G between B and Y form a k-regular bipartite graph. ■

Corollary 6. *For every graph G of order n and maximum degree Δ, $i_\Delta(G) \geq n/2$, and this bound is sharp.*

Proof. Obvious consequence of $i_{k+1}(G) \leq 2i_k(G)$ obtained from Theorem 5 when $j = k$ and $i_{\Delta+1}(G) = n$.

Let G be obtained by attaching one pendant vertex at each vertex of a clique K_k. Then $n = 2k$, $\Delta = k$ and $i_k(G) = k$. Hence $i_\Delta = n/2$. ■

Corollary 7. *If a graph G of order $n \geq 2$ satisfies $i_2(G) = 2i(G)$, then G contains an induced matching of size $i(G)$.*

Proof. In the equality case $k = j = 1$ in Theorem 5, $S = X$, G contains a perfect matching between A and X, and this matching is induced of size $S = i(G)$ since $G[A]$ has maximum degree 0. ■

The converse of Corollary 7 is not true. For instance the cycle C_6 admits an induced mathing M of size $i(C_6) = 2$ but $i_2(C_6) = 3 < 2i(G)$.

The inequality $i_2(G) \leq 2i(G)$ cannot be improved to $i_2(G) \leq 2\gamma(G)$, even for trees, as shown by the caterpillar obtained by adding $k \geq 5$ pendant vertices at each vertex of a path P_3. However the next theorem improves it to $i_2(G) \leq \gamma(G) + i(G)$ in the class of trees and unicyclic graphs.

Theorem 8. *If the graph G contains at most one cycle, then $i_2(G) \leq \gamma(G) + i(G)$.*

Proof. Let S be a $i(G)$-set and I a maximal 2-independent set of G containing S. With the notation of Theorem 5, $X = S$ is independent, $A = N_I(S) = I \setminus S$, and the edges of $G[I]$ form an induced matching M between A and a subset A' of S. Let Z be a $\gamma(G)$-set, M_1 the edges of M with no endvertex in Z, and A_1 (A'_1 respectively) the set of the endvertices of the edges of M_1 in A (A' respectively). If $\gamma(G) < |M|$, then $M_1 \neq \emptyset$

and since M is induced, the vertices of $A_1 \cup A'_1$ cannot be dominated by vertices in $Z \cap (A \cup A')$. Hence the set $W = Z \setminus (A \cup A')$ is not empty and dominates $A_1 \cup A'_1$. Therefore the induced subgraph $G[W \cup A_1 \cup A'_1]$ of order $|W| + 2|M_1|$ contains at least $3|M_1|$ edges. Moreover, since Z contains at least one endvertex of each edge in $M \setminus M_1$, $|W| \leq |Z| - |M \setminus M_1| = (\gamma(G) - |M|) + |M_1| < |M_1|$. Thus $3|M_1| > |W| + 2|M_1|$, which contradicts the assumption that G contains at most one cycle. Therefore $\gamma(G) \geq |M| = |A|$ and $i_2(G) \leq |S| + |A| \leq i(G) + \gamma(G)$. ■

The result of Theorem 8 is not valid for all graphs as shown by the following example. We consider eight disjoint triangles $x_i y_i z_i$ and identify the vertex x_i with x_{i+1} for $i = 1, 3, 5, 7$. Let w_1, w_2, w_3, w_4 denote the resulting new vertices. To complete G, we add the edges w_1w_2, w_1w_3 and w_1w_4. Then $\{w_2, y_3, w_3, y_5, w_4, y_7, y_1, z_1, y_2, z_2\}$ is a $i_2(G)$-set and thus $i_2(G) = 10, \gamma(G) = 4$, and $i(G) = 5$. By attaching q triangles at each vertex w_i instead of 2, $\gamma(G)$ does not change while now $i(G) = 3 + q$ and $i_2(G) = 6 + 2q$. Therefore the difference $i_2(G) - (i(G) + \gamma(G))$ can be done arbitrarily large and the ratio $i_2(G)/(i(G) + \gamma(G))$ arbitrarily close to 2.

The next corollary is another consequence of Theorem 5. A graph G is *well-covered* if $i(G) = \beta(G)$ and *well-k-covered* if $i_k(G) = \beta_k(G)$.

Corollary 9. *For any $k \geq 1$, $i_{k+1}(G) \leq (k+1)i(G) \leq ki(G) + \beta(G)$ and if $i_{k+1}(G) = ki(G) + \beta(G)$, then G is well-covered and well-$(k+1)$-covered.*

Proof. The inequality comes from Theorem 5 with $j = 1$. If $i_{k+1}(G) = ki(G) + \beta(G)$ then $i(G) = \beta(G)$, that is G is well-covered, and thus $i_{k+1}(G) = (k+1)\beta(G)$. Therefore $\beta_{k+1}(G) \leq i_{k+1}(G)$ from Corollary 2, which implies $\beta_{k+1}(G) = i_{k+1}(G)$ and proves that G is well-$(k+1)$-covered. ■

3. Graphs with $i_2 = i + \beta$

In this section we are interested in graphs G satisfying the equality in Corollary 9 when $k = 1$. We describe two particular classes of them defined by forbidden subgraphs.

Definition 10

- The graphs F of the family $\mathcal{F}$ are formed by five disjoint cliques X_i of cardinality at least 2 together with all the edges between X_i and X_{i+1} for $1 \leq i \leq 5$ (mod 5).

- The graphs C_4 and g are shown in Figure 1.

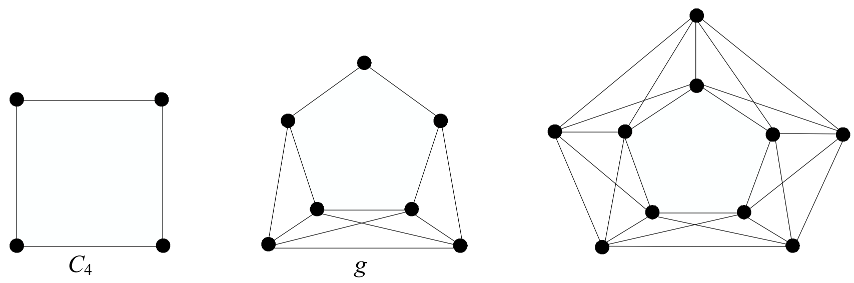

Figure 1

Clearly every non-trivial clique and every graph of $\mathcal{F}$ satisfies $i_2(G) = i(G) + \beta(G)$ with $i(G) = \beta(G) = 1$ for a clique, $i(G) = \beta(G) = 2$ for a graph of $\mathcal{F}$.

Theorem 11. *Let G be a graph such that $i_2(G) = i(G) + \beta(G)$. Then*

1. *If G is g-free, the component of G are λ_1 non-trivial cliques with $\lambda_1 = i(G)$.*
2. *If G is C_4-free, the components of G are $\lambda_1 \geq 0$ non-trivial cliques and $\lambda_2 \geq 0$ graphs of $\mathcal{F}$ with $\lambda_1 + 2\lambda_2 = i(G)$.*

Proof. From Corollary 9, $i(G) = \beta(G)$ and $i_2(G) = \beta_2(G)$. The maximal independent sets of G have all the same cardinality and each maximal 2-independent set induces a matching M of size $i(G)$. In particular, G has no isolated vertex. We make an induction on the common value λ of $i(G)$ and $\beta(G)$. If $\lambda = 1$, then G is a clique of cardinality at least 2. For $\lambda > 1$, suppose the property true when $i_2(G) = i(G) + \beta(G) < 2\lambda$ and let G be a g-free or C_4-free graph such that $i_2(G) = i(G) + \beta(G) = \beta_2(G) = 2\lambda$. Let a be a vertex of G such that $\beta(N[a])$ is maximum and let $G' = G - N[a]$. Every maximal independent set of G' can be completed to a maximal independent set of G by adding a. Hence $i(G') = \beta(G') = \lambda - 1$. If S' is a $i_2(G')$-set then, by Theorem 5, $|S'| = i_2(G') \leq 2i(G') = 2\lambda - 2$. The 2-independent set $S' \cup \{a\}$ of G can be completed to a maximal 2-independent set of G by adding at most one vertex of $N(a)$. Hence $i_2(G) \leq |S'| + 2 \leq 2\lambda$. Since $i_2(G) = 2\lambda$, $i_2(G') = 2\lambda - 2 = i(G') + \beta(G')$. By the induction hypothesis applied to G', which is g-free or C_4-free as G, the components of G' are μ_1 non-trivial cliques if G is g-free, μ_1 non-trivial cliques and μ_2 graphs of $\mathcal{F}$ if G is C_4-free, with $\mu_1 + 2\mu_2 = \lambda - 1$. To continue, we distinguish two cases.

Case 1. $N[a]$ is a clique of G. By the choice of a, $N[x]$ is a clique for every vertex x of G. Therefore the components of G are λ non-trivial cliques.

Case 2. $N[a]$ is not a clique. Let b and c be two non-adjacent vertices of $N(a)$. If for each component H of G', $V(H) \setminus (N(b) \cup N(c)) \neq \emptyset$ if H is a clique and $\beta(V(H) \setminus N(b) \cup N(c)) = 2$ if $H \in \mathcal{F}$, then $\beta(G) \geq \beta(G')+2 = \lambda+1$ which is impossible. Therefore there exists a component H of G' such that either H is a non-trivial clique and $V(H) \subseteq N(b) \cup N(c)$ or $H \in \mathcal{F}$ and $\beta(V(H) \setminus N(b) \cup N(c)) < 2$.

Subcase 2.1. Suppose first that $H \in \mathcal{F}$ and $\beta(V(H) \setminus (N(b) \cup N(c)) \leq 1$. Then G is not g-free and thus is C_4-free. We will prove that this subcase is impossible. Let X_i, $1 \leq i \leq 5$ be the five cliques of H as described in the definition of $\mathcal{F}$. Then $V(H) \setminus (N(b) \cup N(c))$ is a (possibly empty) clique U and since G is C_4-free, $N(b) \cap N(c) \cap V(H) = \emptyset$. If $V(H) \subseteq N(b)$, then a maximal 2-independent set I of G containing $\{a, b\}$ contains no other vertex in $N[a] \cup V(H)$ and at most $2\mu_1 + 4(\mu_2 - 1) = 2\lambda - 6$ vertices in $G' - V(H)$, that is $|I| \leq 2\lambda - 4$ which is impossible. Hence $V(H)$ is not contained in $N(b)$, neither in $N(c)$ by symmetry. If $V(H) \cap N(b)$ and $V(H) \cap N(c)$ are cliques, then $U \neq \emptyset$ and H contains an edge uv with $u \in U$ and $v \in N(c)$. The set $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. If $N(b) \cap V(H)$ is not a clique, say b is adjacent to $x_1 \in X_1$ and to $x_3 \in X_3$, then by C_4-freeness, b is adjacent to every vertex x_2 of X_2 and to no vertex of $X_4 \cup X_5$ for otherwise $V(H) \subseteq N(b)$. Similarly, if $V(H) \cap N(c)$ is neither a clique then $N(c)$ entirely contains X_4 or X_5, say $N(c)$ contains vertices in X_3 and in X_5 and entirely contains X_4. The set U is contained in $X_1 \cup X_5$. If $U \neq \emptyset$, then U contains a vertex u adjacent to some vertex v in X_2 or in X_4, say $v \in X_4$. The set $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. If $U = \emptyset$, then b (respectively c) entirely dominates X_1 (respectively X_5). Let u and v be vertices in X_4. Again $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. Finally, if $V(H) \cap N(b)$ is not a clique and $V(H) \cap N(c)$ is a clique C, we can find two adjacent vertices u and v with u in C or in U, depending on whether U is or not equal to $\emptyset$, and v in C. The set $S = \{u, v, a, b\}$ dominates $V(H) \cup N[a]$. In any case, the set S is 2-independent and a maximal 2-independent set of G containing S contains no other vertex in $V(H) \cup N[a]$ and at most $2\mu_1 + 4(\mu_2 - 1) = 2\lambda - 6$ vertices in $G' - V(H)$. Hence $i_2(G) \leq 2\lambda - 2$, a contradiction. Therefore Subcase 2.1 is impossible.

Subcase 2.2. H is a non-trivial clique contained in $N(b) \cup N(c)$. Since every maximal 2-independent set S of G contains at most two vertices in each clique-component and four vertices in each $\mathcal{F}$-component of G', $2\lambda = i_2(G) \leq 2(\mu_1 - 1) + 4\mu_2 + i_2(V(H) \cup N[a]) = 2\lambda - 4 + i_2(V(H) \cup N[a])$, which gives $i_2(V(H) \cup N[a]) \geq 4$. Hence if S is a maximal 2-independent set of $G[V(H) \cup N[a]]$ containing $\{a, c\}$, then $|S \cap (V(H) \setminus N(c)| \geq 2$. Let x and x' be two vertices in $V(H) \setminus N(c) \subseteq V(H) \cap N(b)$. Similarly, $V(H) \setminus N(b)$ contains at least two vertices y and y' which are adjacent to c. The induced subgraph $G[\{a, b, c, x, x', y, y'\}]$ is equal to g. Hence G is C_4-free and $V(H)$ is partitioned into $V(H) \cap N(b)$ and $V(H) \cap N(c)$.

If $\beta(N[a]) > 2$, let $\{a_1, a_2, \ldots, a_\ell\}$ be a $\beta(N[a])$-set with $\ell \geq 3$. For each pair $\{a_i, a_j\}$ of nonadjacent vertices of $N(a)$, there exists, by Subcase 2.1, a non-trivial clique-component H_{ij} of G' contained in $N(a_i) \cap N(a_j)$. Since G is C_4-free, $V(H_{ij}) \cap N(a_i)$ and $V(H_{ij}) \cap N(a_j)$ partition $V(H_{ij})$ and the $\ell(\ell-1)/2$ cliques H_{ij} are different. Then any maximal independent set I of G containing $\{a_1, a_2, \ldots, a_\ell\}$ satisfies $\beta(G) - |I| \geq \ell(\ell-1)/2 + 1 - \ell > 0$, contradicting $i(G) = \beta(G)$. Therefore $\beta(N[a]) = 2$.

Let u be a vertex of $N(a)$ adjacent to c but not to b, if any. As above, let H' be the clique-component of G' contained in $N(b) \cup N(u)$. If $H' \neq H$, then $N(b)$ contains at least one vertex in $V(H)$, one vertex in $V(H')$ and a, that is $\beta(N[b]) \geq 3$, in contradiction to the choice of a. Therefore $H' = H$ and $N(u) \cap V(H) = N(c) \cap V(H)$. Similarly, $N(v) \cap V(H) = N(b) \cap V(H)$ for every vertex v of $N(a)$ adjacent to b but not to c. Hence every vertex z in $N[a] \cup H$ satisfies $\beta(N[z] \cap (V(H) \cup N[a])) = 2$ and by the choice of a, $N[z] \subseteq V(H) \cup N[a]$ and $N[a] \cup V(H)$ forms a component L of G. For each vertex z of L, the vertices of L which are not adjacent to z form a clique. The clique $V(L) \setminus N[a]$ is H. Let $\mathcal{B}, \mathcal{C}, \mathcal{X}, \mathcal{Y}$ be respectively the cliques $V(L) \setminus N[b]$, $V(L) \setminus N[c]$, $V(L) \setminus N[x]$, $V(L) \setminus N[y]$. Let $B = \mathcal{C} \cap \mathcal{Y}$, $A = \mathcal{Y} \cap \mathcal{X}$, $C = \mathcal{X} \cap \mathcal{B}$, $Y = \mathcal{B} \cap\ H$, $X = H \cap \mathcal{C}$. Then $a \in A$, $b \in B$, $c \in C$, $x \in X$ and $y \in Y$. Since G is C_4-free, $A \cap B = \emptyset$ and (A, B) form a partition of $\mathcal{Y}$. Similarly (A, C), (C, Y), (Y, X) and (X, B) respectively form a partition of $\mathcal{X}$, $\mathcal{B}$, $V(H)$ and $\mathcal{C}$. Finally if, say, $|A| = 1$, then every maximal 2-independent set of G containing $\{x, y, a\}$ contains no other vertex in L and thus at most $2(\mu_1 - 1) + 4\mu_2 + 3 = 2(\lambda - 1) + 1$ vertices, in contradiction to $i_2(G) = 2\lambda$. Therefore each of the five cliques A, B, X, Y, C has at least two vertices and L is a graph of $\mathcal{F}$. The components of G' are μ_1 non-trivial cliques and μ_2 graphs of $\mathcal{F}$ with $\mu_1 + 2\mu_2 = \lambda - 1$. Hence the components of G are $\lambda_1 = \mu_1 - 1$ non-trivial cliques and $\lambda_2 = \mu_2 + 1$ graphs of $\mathcal{F}$ with $\lambda_1 + 2\lambda_2 = \mu_1 + 2\mu_2 + 1 = \lambda$. This completes the proof. ■

4. Bounds on i_2

In [4], it is proved that every graph G of maximum degree $\Delta \geq 1$ satisfies $i_k(G) \geq (n + k - 1)/(\Delta + 1)$ for $1 \leq k \leq n - 1$ and examples of extremal graphs are given for $k \geq 3$. Here we slightly improve the bound when $k = 2$ and characterize the extremal graphs.

Theorem 12. *Let G be a connected graph of order $n \geq 2$ and maximum degree Δ. Then $i_2(G) \geq (n+2)/(\Delta+1)$, with equality if and only if $G = P_2$ or G is obtained from a double star $S_{\Delta-1,\Delta-1}$ by adding zero or more edges between its leaves without creating a vertex of degree larger than Δ.*

Proof. If $n = 2$, then $i_2(P_2) = 2 = (n + 2)/(\Delta + 1)$. If $n = 3$ then $G = P_3$ or C_3 and $i_2(G) = 2 > (n + 2)/(\Delta + 1)$. So assume that $n \geq 4$ and $\Delta \geq 2$ since G is connected. Let S be a $i_2(G)$-set, p be the number of edges in $G[S]$ and t the number of edges joining the vertices in S and $V - S$. Assume first that $p \geq 1$. Then since the p edges are independent, $t \leq 2p(\Delta - 1) + (|S| - 2p)\Delta$. Also since S dominates $V - S$, $t \geq |V - S|$. It follows that $|V - S| \leq t \leq 2p(\Delta - 1) + (|S| - 2p)\Delta$. Thus

$$i_2(G) = |S| \geq (n + 2p)/(\Delta + 1) \geq (n + 2)/(\Delta + 1).$$

If further $i_2(G) = (n + 2)/(\Delta + 1)$, then we must have equality throughout the above inequality chain, in particular we have $p = 1$, every vertex of $\langle S\rangle$ has degree Δ and every vertex of $V - S$ is adjacent to exactly one vertex of S. If $\langle S\rangle$ contains an isolated vertex say u, then $S \cup \{v\}$ is a 2-independent set of G, where $v \in V - S$ is any neighbor of u, contradicting the maximality of S. Therefore S contains only two adjacent vertices, each of degree Δ, and G has the structure described in the theorem. The converse is easy to show.

Now assume that $p = 0$. Then $t \leq \Delta|S|$. If $V - S$ contains any vertex, say w, that has only one neighbor in S then $S \cup \{w\}$ is a 2-independent set of G, a contradiction with the maximality of S. Thus each vertex of $V - S$ has at least two neighbors in S and hence $t \geq 2|V - S|$. It follows that $\Delta|S| \geq t \geq 2|V - S|$ and so $i_2(G) \geq 2n/(\Delta + 2)$. Notice that $2n/(\Delta + 2) \geq (n + 2)/(\Delta + 1)$ for $n \geq 4$ with equality if and only if $n = 4$ and $\Delta = 2$.

If further $i_2(G) = (n+2)/(\Delta+1)$ then $n = 4$, $\Delta = 2$ and every vertex of $V - S$ has exactly two neighbors in S. Thus G is a cycle C_4 which is obtained from a double star $S_{1,1}$ by adding an edge joining the two leaves. ■

In [1], Blidia *et al.* have given an upper bound on $i_2(G)$ for every nontrivial connected bipartite graph.

Theorem 13. *If G is a connected nontrivial bipartite graph with $s(G)$ support vertices, then $i_2(G) \leq (n + s(G))/2$.*

When G is a cactus, this upper bound can be extended to non-bipartite graphs. First we give a lemma related to matchings in cactus.

Lemma 14. *In every cactus G with k odd cycles, there exists a matching of size k containing exactly one edge in each odd cycle of G.*

Proof. We proceed by induction on the number of odd cycles. Clearly the property is true for $k = 0$ and $k = 1$. Let $k \geq 2$. Assume the property true for cactus with less than k odd cycles and let G be a cactus with k odd cycles. Let $C = x_1x_2 \cdots x_{2p+1}$ with $p \geq 1$ be an odd cycle of G. For each $x_i \in V(C)$, let $A_i = N(x_i) \setminus V(C)$. By the definition of cactus, all the sets A_i are disjoint. Let G' be the graph obtained from G by contracting the cycle C into one vertex c. More precisely, $V(G') = (V(G) \setminus V(C)) \cup \{c\}$ and for $1 \leq i \leq 2p+1$, the edges between x_i and A_i are replaced by the edges between c and A_i. Every cycle $\mathcal{C} \neq C$ of G is unchanged in G' if $V(\mathcal{C}) \cap V(C) = \emptyset$ or is replaced by a cycle $\mathcal{C}'$ of same length and containing c if $|V(\mathcal{C}) \cap V(C)| = 1$. Hence G' is a cactus with $k-1$ odd cycles and by the inductive hypothesis, contains a matching M' of size $k-1$ with exactly one edge in each of its odd cycles. All the edges of M' are edges of G except possibly one, say cy_1 with $y_1 \in A_1$. In this case, the edge cy_1 belongs to an odd cycle $\mathcal{C}_1'$ of G' corresponding to an odd cycle $\mathcal{C}_1$ of G containing x_1. The set $M = M' \cup \{x_2x_3\}$ if $M' \subseteq E(G)$, $M = (M' \setminus \{cy_1\}) \cup \{x_1y_1, x_2x_3\}$ if $cy_1 \in M'$, is a matching of G containing exactly one edge in each of its odd cycles. ■

Theorem 15. *If G is a connected nontrivial cactus graph with k odd cycles and $s(G)$ support vertices, then $i_2(G) \leq (n + s(G) + k)/2$ and this bound is sharp.*

Proof. Let G be a connected nontrivial cactus graph with k odd cycles and $s(G)$ support vertices. If $k = 0$, then G is a bipartite graph and hence by Theorem 13 the result is valid. So assume that G contains at least one odd cycle. By Lemma 14, there exists in G a matching M of size k containing one edge in each odd cycle of G. We subdivide each edge of M by exactly one vertex. Let D be the set of such vertices and $G' = (V', E')$ the resulting graph. Then every vertex of D has degree two and G' is a connected bipartite graph of order $n + k$ with $s(G') = s(G)$ and different from a tree. Let C be a set of leaves of G' so that every support vertex has exactly one leaf in C. Clearly $|C| = s(G)$. Let A and B be the two classes of the bipartition of $G'[V' \setminus C]$ with $|A| \leq |B|$. Then $|B| \geq (n + k - s(G'))/2 \geq |A| > 0$. Let S_A, C_A denote the set of support vertices and leaves of G' belonging to A, respectively, and let $A' = A \setminus (S_A \cup C_A)$. Likewise, we define S_B, C_B and B'. The 2-independent set $S' = A \cup C$ is maximal in G' since every leaf of B is adjacent to a support vertex of A, which has degree one in $G'[S']$, and the other vertices of B have at least two neighbors in A. Its order satisfies

$$\left|S'\right| = |A \cup C| \leq (n + k - s(G'))/2 + |C| = (n + s(G) + k)/2.$$

We shall construct a maximal 2-independent set S of G with $|S| \leq (n + s(G) + k)/2$. Let $D_A = D \cap A$, $D_B = D \cap B$, $F_B = N(D_A) \cap B$ and $F_A = N(D_B) \cap A$. Note that each of $G[F_A]$ and $G[F_B]$ consists of disjoint copies of P_2. Then $F_B \subset B \setminus (C_B \cup D_B)$, $F_A \subset A \setminus (C_A \cup D_A)$. Thus each component in $G[A \setminus D_A]$ is either an isolated vertex or a path P_2. So $A \setminus D_A$ is a 2-independent set but $(A \setminus D_A) \cup C$ may be not 2-independent. This occurs if $F_A \cap S_A \neq \emptyset$. In that case delete from C all leaves adjacent to $F_A \cap S_A$ and let $C' \subseteq C$ be the resulting set. Thus $(A \setminus D_A) \cup C'$ is a 2-independent set. To extend it to a maximal 2-independent set of G, we can only add vertices of $B \setminus D_B$ having only one neighbor in $A \setminus D_A$ and this neighbor must be isolated in $G[(A \setminus D_A) \cup C']$. Hence we add at most one endvertex of each edge of $G[F_B]$, that is at most $|D_A|$ vertices. Thus $i_2(G) \leq |(A \setminus D_A) \cup C'| + |D_A| \leq |A \cup C| = |S'|$. This completes the proof. Odd cycles are examples of graphs attaining the bounds. ■

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