# ( $\boldsymbol{H}, \boldsymbol{k}$ ) STABLE GRAPHS WITH MINIMUM SIZE 

Aneta Dudek*<br>Artur Szymański and MaŁgorzata Zwonek<br>Faculty of Applied Mathematics AGH<br>Mickiewicza 30, 30-059 Kraków, Poland


#### Abstract

Let us call a $G(H, k)$ graph vertex stable if it contains a subgraph $H$ ever after removing any of its $k$ vertices. By $Q(H, k)$ we will denote the minimum size of an $(H, k)$ vertex stable graph. In this paper, we are interested in finding $Q\left(C_{3}, k\right), Q\left(C_{4}, k\right), Q\left(K_{1, p}, k\right)$ and $Q\left(K_{s}, k\right)$.


Keywords: graph, stable graph.
2000 Mathematics Subject Classification: 05C35.

## 1. Introduction

We deal with simple graphs without loops and multiple edges. As usual $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively, $|G|, e(G)$ the order and the size of $G$ and $\operatorname{deg}_{G}(v)$ the degree of $v \in V(G)$. By $C_{n}$ we denote the cycle of order $n$ and by $K_{r}$ the complete graph on $r$ vertices and by $K_{1, p}$ the star on $1+p$ vertices. The union $G \cup H$ of graphs $G$ and $H$ is defined by $V(G \cup H):=V(G) \cup V(H), E(G \cup H):=E(G) \cup E(H)$, and we shall suppose that the components of the union are vertex disjoint.

By $G-e$ we shall denote the graph without the edge $e$ and by $G-v$ the graph obtained from $G$ by deleting the vertex $v \in V(G)$ and its incident edges.

In [1] G.Y. Katona and P. Frankl considered the following problem. What is the minimum size of a $r$-uniform hypergraph such that after removing any $k$ hyperedges there is still a hamiltonian chain. To give a lover bound of the minimum size of the mentioned $r$-uniform hypergraphs the authors

[^0]of [1] define the $\left(P_{4}, k\right)$ edge stable graph as the graph in which after removing any $k$ edges there is still $P_{4}$ and ask about the minimum size of $\left(P_{4}, k\right)$ edge stable graph. This was intended as an attempt to solve the problem of finding the minimum size of a $\left(P_{4}, k\right)$ edge stable graph. In [2] G.Y. Katona and I. Horváth considered the minimum size of $\left(P_{n}, k\right)$ edge stable graphs. It is worth pointing out that there is no other result concerning edge stable graphs.

The aim of this paper is to consider a similar problem but in a vertex version. So let us give the following definition:

Definition 1. Let us call a $(H, k)$ graph vertex stable if it contains a connected subgraph $H$ ever after removing any of its $k$ vertices. By $Q(H, k)$ we will denote the minimum size of an $(H, k)$ vertex stable graph.

In this paper we estimate $Q\left(C_{3}, k\right), Q\left(C_{4}, k\right), Q\left(K_{1, p}, k\right)$ and give lower and upper bounds for $Q\left(K_{s}, k\right)$. For simplicity we will write stable instead of vertex stable.

The proofs are based on the facts given below.
Definition 2. W say that an $(H, k)$ stable graph $G$ is $(H, k)$ strong stable if $G$ is not $(H, k+1)$ stable and $G-e$ is not $(H, k)$ stable for every $e \in E(G)$.

Proposition 1. If $G$ is an $(H, k)$ stable graph with minimum size, then $G$ is an $(H, k)$ strong stable graph. Thus $Q(H, k) \leq e(G)$ where $G$ is an $(H, k)$ strong stable graph.

Proof. Suppose $G$ is an $(H, k)$ stable graph with minimum size. Then clearly $G-e$ for any $e \in E(G)$ is not ( $H, k$ ) stable. Suppose $G$ is $(H, k+1)$ stable and $\operatorname{deg}_{G}(v)>0$, then $G-v$ is $(H, k)$ stable with smaller size than $e(G)$, a contradiction.

Lemma 1. If $G$ is an $(H, k)$ strong stable graph then every vertex as well as every edge of $G$ belongs to some subgraph of $G$ isomorphic to $H$.

Proof. Suppose there is an edge $e$ which is not in any $E(H)$. Then $G-e$ is still $(H, k)$ stable with a smaller size than $e(G)$, a contradiction. If there exists a vertex $v$ which is not in any $V(H)$, then each edge incident with $v$ is not in $E(H)$, a contradiction.

Corollary 1. If $G$ is an $(H, k)$ stable graph with a minimum size than every vertex as well as every edge of $G$ belongs to some subgraph of $G$ isomorphic to $H$.

$$
\text { 2. } Q\left(C_{n}, k\right)
$$

Theorem 2. $Q\left(C_{3}, k\right)=3 k+3$.
Proof. Let $G_{k}$ be a graph which is a vertex-disjoint union of $k+1$ triangles. Clearly, $G_{k}$ is a $\left(C_{3}, k\right)$ strong stable graph so $Q\left(C_{3}, k\right) \leq 3 k+3$.

We prove $Q\left(C_{3}, k\right) \geq 3 k+3$ by induction on $k$. It is clear that $Q\left(C_{3}, 0\right)=3$. Suppose that the statement holds for any $k<k_{0}$. We prove the validity of our claim for $k_{0}$ indirectly.

Suppose that there is a graph $G_{k_{0}}$ which is $\left(C_{3}, k\right)$ strong stable but $e\left(G_{k_{0}}\right)<3 k+3$. If the maximum degree in $G_{k_{0}}$ is at most 2 , then by Lemma 1 the graph consists only of cycle components. Since the number of edges in the graph is at most $3 k+2$, at most $k$ components can be a triangle. So removing a vertex from each of these will destroy all triangles, a contradiction.

If there is a vertex $v$ of degree greater or equal to 3 , then $G_{k_{0}}-v$ is clearly a ( $C_{3}, k-1$ ) strong stable graph with less than $3 k$ edges, a contradiction again.

Lemma 2. If $G$ is $(H, k)$ stable, then $G-v$ is $(H, k-1)$ stable for any $v \in V(G)$. Moreover, if some edges in $G-v$ cannot be contained in any $H$ subgraphs, then the graph obtained from $G-v$ by removing all these edges is still $(H, k-1)$ stable.

Proof. The first part of the proof follows from the definition of an $(H, k)$ stable graph. From Corollary 1 it follows that all edges in $(H, k-1)$ stable graphs belong to some $H$ subgraph which finishes the proof.

Theorem 3. $Q\left(C_{4}, k\right)=4 k+4$.
Proof. Let $G_{k}$ be a graph which is a vertex-disjoint union of $(k+1) C_{4}$. Clearly, $G_{k}$ is a $\left(C_{4}, k\right)$ stable graph so $Q\left(C_{4}, k\right) \leq 4 k+4$.

We prove $Q\left(C_{4}, k\right) \geq 4 k+4$ by induction on $k$. It is clear that $Q\left(C_{4}, 0\right)=4$. Suppose that the statement holds for any $k<k_{0}$. We prove the validity of our claim for $k_{0}$ indirectly.

Suppose that there is a graph $G_{k_{0}}$ which is $\left(C_{4}, k\right)$ stable with minimum size and $e\left(G_{k_{0}}\right)<4 k+4$. From Corollary 1 it follows that $\operatorname{deg}_{G_{k_{0}}}(x) \geq 2$ for every $x \in V\left(G_{k_{0}}\right)$.

We shall consider the following cases:
Case 1. $\Delta\left(G_{k_{0}}\right) \geq 4$.
Let $\operatorname{deg}_{G_{k_{0}}}(x) \geq 4$. Then $G_{k_{0}}-x$ is a ( $C_{4}, k-1$ ) stable graph with smaller size than $4 k$, a contradiction.

Case 2. $\Delta\left(G_{k_{0}}\right) \leq 3$.
Suppose first that $G_{k_{0}}$ contains a cycle as a component. Corollary 1 implies that it is $C_{4}$. If we delete one vertex of this $C_{4}$, then the remaining 2 edges of $C_{4}$ are not contained in any $C_{4}$ subgraphs. However, the graph without these 4 edges is still $\left(C_{4}, k-1\right)$ stable by Lemma 2 . This contradicts the inductive hypothesis. Next suppose that $x_{1} x_{2} \in E\left(G_{k_{0}}\right)$ and deg $_{G_{k_{0}}}\left(x_{1}\right)=3$, $\operatorname{deg}_{G_{k_{0}}}\left(x_{2}\right)=2$. By deleting $x_{1}$ using Lemma 2 we can derive a similar contradiction as before. Hence $G_{k_{0}}$ contains only cubic components.

If $K_{4}$ is a component of $G_{k_{0}}$ then it may be replaced by $C_{4}$ since both of them are $\left(C_{4}, 0\right)$ stable, and we get a graph with smaller size than $G_{k_{0}}$, a contradiction. Since the order of a $\left(C_{4}, 1\right)$ cubic graph is at least 6 , then $Q\left(C_{4}, 1\right) \geq 9>8$. Since the order of a $\left(C_{4}, 2\right)$ cubic graph is at least 10 (see [3]), then we may estimate $Q\left(C_{4}, 2\right) \geq 15>12$. Denote by ( $x_{1} x_{2} x_{3} x_{4}$ ) a cycle $C_{4}$ in a cubic graph. If $x_{1} x_{3}$ or $x_{2} x_{4}$ is in $E\left(G_{k_{0}}\right)$ it is in a contradiction with Corollary 1 or $K_{4}$ is a component of $G_{k_{0}}$. So we assume neither $x_{1} x_{3}$ nor $x_{2} x_{4}$ is in $E\left(G_{k_{0}}\right)$. In the same way as before after deleting $x_{1}$ and $x_{3}$ we may remove all edges from the cycle ( $x_{1} x_{2} x_{3} x_{4}$ ) and all edges incident with vertices of the cycle and by Lemma 2 we get a $\left(C_{4}, k-2\right)$ stable graph with smaller size than $4 k-4$, a contradiction.

For $n \geq 6$ and $k \geq 0$ it is easy to see that a $(k+1)$ disjoint union of $C_{n}$ is a $\left(C_{n}, k\right)$ strong stable graph. The following theorem is evident.

Theorem 4. $Q\left(C_{n}, k\right) \leq k n+n$.

$$
\text { 3. } \quad Q\left(K_{1, p}, k\right)
$$

Theorem 5. Let $p \geq 3$. Then $Q\left(K_{1, p}, k\right)=p k+p$.

Proof. Let $G_{k}$ be a graph which is a vertex-disjoint union of $k+1$ stars $K_{1, p}$. Clearly, $G_{k}$ is a $\left(K_{1, p}, k\right)$ strong stable graph so $Q\left(K_{1, p}, k\right) \leq p k+p$.

We prove $Q\left(K_{1, p}, k\right) \geq p k+p$ by induction on $k$. It is clear that $Q\left(K_{1, p}, 0\right)=p$. Suppose that the statement holds for any $k<k_{0}$. We prove the validity of our claim for $k_{0}$ indirectly.

Suppose that there is a graph $G_{k_{0}}$ which is $\left(K_{1, p}, k\right)$ strong stable with minimum size but $e\left(G_{k_{0}}\right)<p k+p$. From Lemma 1 it follows there is at least one vertex $v$ of degree at least $p$. So $G_{k_{0}}-v$ is clearly a ( $K_{1, p}, k-1$ ) strong stable graph with size smaller than $p k$, a contradiction.
Observe that a disjoint union of $(k+1)$ stars $K_{(1, p)}$ is a ( $K_{(1, p), k)}$ strong stable graph.

$$
\text { 4. } Q\left(K_{s}, k\right)
$$

Let $k \geq 0$ and $s \geq 0$. Let $G=(V(G) ; E(G))$ be a graph of order greater than $k+s$.

For a fixed $k, k>0$ cases for $s=0,1,2$ are trivial, the case for $s=3$ was considered as $C_{3}$, so we turn to the case $s=4$.

## 4.1. $\quad Q\left(K_{4}, k\right)$

Theorem 6.

$$
Q\left(K_{4}, k\right)= \begin{cases}6 & \text { for } k=0 \\ 5 k+5 & \text { for } k \geq 1\end{cases}
$$

Proof. It is obvious that $Q\left(K_{4}, 0\right)=6$. Let $G_{k}$ be a graph which is a vertex-disjoint union of $\frac{k+1}{2} K_{5}$ for $k$ odd, and a vertex-disjoint union of $\left(\frac{k-2}{2} K_{5}\right) \cup K_{6}$ for $k$ even. Clearly, $G_{k}$ is $\left(K_{4}, k\right)$ strong stable, so $Q\left(K_{4}, k\right) \leq$ $5 k+5$.

We prove $Q\left(K_{4}, k\right) \geq 5 k+5$ by induction on $k$. It is easy to see that $Q\left(K_{4}, 1\right)=10$. Suppose that the statement holds for any $k<k_{0}$. We prove the validity of our claim for $k_{0}$ indirectly.

Suppose that there is a $G_{k_{0}}$ graph which is ( $\left.K_{4}, k\right)$ strong stable with minimum size and $e\left(G_{k_{0}}\right)<5 k+5$.

We shall consider the following cases.
Case 1. $\Delta\left(G_{k_{0}}\right) \geq 5$.
Let $v \in G_{k_{0}}$ and $\operatorname{deg}_{G_{k_{0}}}(v) \geq 5$. Then $G_{k_{0}}-v$ is $\left(K_{4}, k-1\right)$ strong stable and $e\left(G_{k_{0}}-v\right)<5 k$, a contradiction.

Case 2. $\Delta\left(G_{k_{0}}\right)=4$ and $\delta\left(G_{k_{0}}\right)=3$.
Let $v, z \in V\left(G_{k_{0}}\right)$ and $\operatorname{deg}_{G_{k_{0}}}(v)=4, \operatorname{deg}_{G_{k_{0}}}(z)=3$.
Subcase 2a. Suppose $v z \in E\left(G_{k_{0}}\right)$. Since edges incident to $z$ in $G_{k_{0}}-v$ are not in $K_{4}$, then we may remove them. The graph obtained is ( $K_{4}, k-1$ ) strong stable and $e\left(G_{k_{0}}-v\right)<5 k-1$, a contradiction.

Subcase 2b. Suppose there is no vertex of degree 3 adjacent to vertex of degree 4 . It is easy to see that by Lemma 1 since every edge must be in $K_{4}$ it means that $G_{k_{0}}$ contains $K_{4}$ as a component ( $K_{5}$ will be considered in Case 3). Deleting one vertex from $K_{4}$ we get three edges which cannot be in any $K_{4}$ so we may delete them. We get a $\left(K_{4}, k-1\right)$ strong stable graph with smaller size than $5 k-1$, a contradiction.

Case 3. $\Delta\left(G_{k_{0}}\right)=4$ and $\delta\left(G_{k_{0}}\right)=4$.
By Lemma 1 we have that every edge must be in $K_{4}$, so it means that $G_{k_{0}}$ is a vertex disjoint union of $K_{5}$. Because $e\left(G_{k_{0}}\right)<5 k+5$, there is at most $\left(\left\lceil\frac{k+1}{2}\right\rceil-1\right) K_{5}$. If we delete $k$ vertices, two from every $K_{5}$, we will destroy all $K_{4}$, a contradiction.
Observe that the family given in the above theorem is also ( $K_{4}, k$ ) strong stable with minimum size.

### 4.2. The upper bound of $Q\left(K_{s}, k\right)$ for $s \geq 5$

The following assumption will be needed throughout this subsection

1. $k \geq 0$ and $s \geq 5$ is fixed,
2. $1 \leq r \leq k+1, j \in\{1,2, \ldots, r\}, i_{j} \geq s$ and $i_{1} \leq i_{2} \leq \ldots \leq i_{r}$.

Let $\mathcal{A}_{r}^{\left(K_{s}, k\right)}$ be a family of graphs consisting of vertex disjoint unions of $r$ complete graphs $K_{i_{j}}$ satisfying the following condition:

$$
\sum_{j=1}^{r}\left(i_{j}-s\right)+r-1=k
$$

For simplicity, we will write $\mathcal{A}_{r}^{\left(K_{s}, k\right)}$ without repetition of the above assumption.

Observe that for $r=1$ the family $\mathcal{A}_{r}^{\left(K_{s}, k\right)}$ is reduced to a complete graph $K_{s+k}$, and for $r=k+1$ it consist only of a vertex disjoint union of $k+1$ graphs $K_{s}$. Obviously, these graphs are ( $K_{s}, k$ ) strong stable.

For a fixed $k$, we will show that all graphs from $\mathcal{A}_{r}^{\left(K_{s}, k\right)}$ are ( $\left.K_{s}, k\right)$ strong stable and give the construction of a family $A\left(K_{s}, k\right)$ with the smallest size. This gives us an upper bound of $Q\left(K_{s}, k\right)$.

Lemma 3. For a fixed $k, k \geq 0$. Then $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$ is $\left(K_{s}, k\right)$ strong stable.
Proof. The proof will be divided into two steps. Let $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$.
Step 1. We show that $G$ is $\left(K_{s}, k\right)$ stable.
Deleting $\sum_{j=1}^{r}\left(i_{j}-s\right)=k-(r-1)$ vertices we obtain a union of complete graphs in which:

Case 1a. There is a complete graph of order greater than or equal to $s+r-1$. Hence after removing any $r-1$ vertices from the graph we still have $K_{s}$.

Case 1b. All complete graphs have their size less than $s+r-1$.
It means that it is a union of exactly $r$ complete graphs and each of them contains $K_{s}$. Hence after removing any $r-1$ vertices we still have $K_{s}$.

Step 2. We show that $G$ is not $\left(K_{s}, k+1\right)$ stable and $G-e$ is not $\left(K_{s}, k\right)$ stable for every $e \in E(G)$.

Deleting $k$ vertices from $G$ we obtain that the order of the remaining graph is: $i_{1}+i_{2}+\ldots+i_{r}=r(s-1)+1$. So we may create a union of $r$ graphs containing $(r-1)$ graphs $K_{s-1}$ and exactly one $K_{s}$. The proof is completed by removing one vertex or one edge from $K_{s}$.

Definition 3. For a fixed $k, k \geq 0$. We call $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$ a balanced union if $\left|i_{j}-i_{q}\right| \in\{0,1\} j, q \in\{1,2, \ldots, r\}$.

Remark 1. For a fixed $k$ and $r$ there is exactly one balanced union $B_{r}^{\left(K_{s}, k\right)} \in$ $\mathcal{A}_{r}^{\left(K_{s}, k\right)}$.

Proof. For a fixed $k$ and $r$ let $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$. Suppose $G$ consists of a vertex disjoint union of $p$ graphs $K_{s+i+1}$ and $r-p$ graphs $K_{s+i} . G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$ therefore:

$$
\begin{aligned}
& \sum_{1}^{r-p}(s+i-s)+\sum_{1}^{p}(s+i+1-s)+r-1=k, \\
& (r-p) i+p(i+1)+r-1=k \\
& r i+p+r-1=k
\end{aligned}
$$

Hence $p=k-r i-r+1$ and $i=\frac{(k-r+1)}{r}-\frac{p}{r}$. Obviously, $i$ must be an integer. Moreover, $0 \leq p<r$, so there is exactly one $p$ such that $\left\lfloor\frac{k-r+1}{r}\right\rfloor=\frac{(k-r+1)}{r}-\frac{p}{r}=i$. Therefore $G$ is a unique balanced union, hence, $G=B_{r}^{\left(K_{s}, k\right)}$.

We leave it to the reader to verify that:
Proposition 7. For a fixed $k$ and $r, B_{r}^{\left(K_{s}, k\right)}$ has the smallest possible size among all graphs $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$.

Lemma 4. Let $s \geq 5$. There exists $k_{1}(s)$ such that $e\left(B_{2}^{\left(K_{s}, k\right)}\right)<e\left(K_{s+k}\right)$ for $k \geq k_{1}(s)$.

Proof. Let $B_{2}^{\left(K_{s}, k\right)}=K_{i_{1}} \cup K_{i_{2}}$. We will consider two cases:
Case 1. $i_{1}=i_{2}$.
Then

$$
k=\sum_{j=1}^{2}\left(i_{j}-s\right)+2-1=2\left(i_{1}-s\right)+1
$$

so $i_{1}=\frac{1}{2}(k-1+2 s)$ and the inequality:

$$
2\binom{\frac{1}{2}(k-1+2 s)}{2}=e\left(K_{i_{1}}\right)+e\left(K_{i_{2}}\right)=e\left(B_{2}^{\left(K_{s}, k\right)}\right)<e\left(K_{s+k}\right)=\binom{s+k}{2}
$$

holds for $k \geq k_{1}(s)=\left\lceil\sqrt{2 s_{2}+6 s+4}\right\rceil$.
Case 2. $i_{1}+1=i_{2}$.
A similar inequality holds for $k \geq k_{1}(s)=\left\lceil\sqrt{2 s_{2}+6 s+5}\right\rceil$.

It is easily seen that:
Proposition 8. If $B_{2}^{\left(K_{s}, k\right)}=K_{i_{1}} \cup K_{i_{2}}$ and $G=K_{i_{1}+1} \cup K_{i_{2}}$, then $G=$ $B_{2}^{\left(K_{s}, k+1\right)}$.

Lemma 5. Let $k_{1}(s)$ be a value given by Lemma 4. If $K_{s+k^{\prime}}$ is a component of $B_{r}^{\left(K_{s}, k\right)}$ for $k^{\prime} \geq k_{1}(s)$, then there is a graph $B_{r^{\prime}}^{\left(K_{s}, k\right)}$ such that $e\left(B_{r^{\prime}}^{\left(K_{s}, k\right)}\right)<$ $e\left(B_{r}^{\left(K_{s}, k\right)}\right)$ and $r^{\prime}>r$.

Proof. Suppose that $K_{s+k^{\prime}}$ and $K_{s+k^{\prime}+1}$ are components of $B_{r}^{\left(K_{s}, k\right)}$ for $k^{\prime} \geq k_{1}(s)$. Note that $K_{s+k^{\prime}}$ is a ( $K_{s}, k^{\prime}$ ) strong stable graph. From Lemma 4 it follows that there are integers $i_{1}$ and $i_{2}$ such that

$$
e\left(K_{i_{1}}\right) \cup e\left(K_{i_{2}}\right)=e\left(B_{2}^{\left(K_{s}, k^{\prime}\right)}\right)<e\left(K_{s+k^{\prime}}\right) .
$$

Denote by $H^{*}$ a graph obtained by replacing all $K_{s+k^{\prime}}$ in $B_{r}^{\left(K_{s}, k\right)}$ by $K_{i_{1}} \cup K_{i_{2}}$ and replacing all $K_{s+k^{\prime}+1}$ in $B_{r}^{\left(K_{s}, k\right)}$ by $K_{i_{1}+1} \cup K_{i_{2}}$.

It is obvious that $e\left(H^{*}\right)<e\left(B_{r}^{\left(K_{s}, k\right)}\right)$. Moreover, $H^{*}$ is $\left(K_{s}, k\right)$ strong stable and it is a balanced union, therefore there is an integer $r^{\prime}$ such $H^{*}=$ $B_{r^{\prime}}^{\left(K_{s}, k\right)}$.

Lemma 5 may be used to show by similar arguments as in Lemma 4 that there exists $k_{n}(s)$ such that $e\left(B_{n+1}^{\left(K_{s}, k\right)}\right)<e\left(B_{n}^{\left(K_{s}, k\right)}\right)$ for $k \geq k_{n}(s)$.

Thus we may construct graphs $A\left(K_{s}, k\right)$ such that for $k_{n}(s) \leq k<$ $k_{n+1}(s), A\left(K_{s}, k\right)=B_{n+1}^{\left(K_{s}, k\right)}$. From the above construction the following theorem follows easily:

Theorem 9. $Q\left(K_{s}, k\right) \leq e\left(A\left(K_{s}, k\right)\right) \leq e(G)$ for every $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$ where $r \in\{1, \ldots, k+1\}$.

From the proof of Remark 1 we have the following estimation of this upper bound by sizes of ( $K_{s}, k$ ) strong stable balanced unions

$$
Q\left(K_{s}, k\right) \leq \min _{r \in\{1, \ldots, k+1\}}\left(r\binom{s+i_{r}}{2}+p_{r}\left(s+i_{r}\right)\right),
$$

where $i_{r}=\left\lfloor\frac{k-r+1}{r}\right\rfloor$ and $p_{r}=k-r+1-r i_{r}$.
For a sufficiently large $k$, we may estimate the upper bound differently.

Theorem 10. There is an integer $k(s)$ such that $Q\left(K_{s}, k\right) \leq(2 s-3)(k+1)$ for $k>k(s)$.

Proof. Let $G$ be a vertex disjoint union of $p$ graphs $K_{2 s-2}$ and $r-p$ graphs $K_{2 s-3}$ where $r \in\{1, \ldots, k+1\}$ and $p \in\{0, \ldots, r\}$. Suppose that $G \in \mathcal{A}_{r}^{\left(K_{s}, k\right)}$. Then

$$
\begin{aligned}
& \sum_{1}^{r-p}(2 s-3-s)+\sum_{1}^{p}(2 s-2-s)+r-1=k \\
& r(s-3)+p+r-1=k \\
& r(s-2)+p-1=k
\end{aligned}
$$

If $k>(s-2)(s-2)+(s-2)-1$, then $r \geq(s-2)$. Hence $p \in\{0, \ldots, s-2$, $\ldots, r\}$, and so there is a pair $r^{\prime}, p^{\prime}$ (not necessarily unique) which satisfies the equation. Therefore $G=B_{r^{\prime}}^{\left(K_{s}, k\right)}$

Now we will show by induction on $k$ that $e\left(B_{r^{\prime}}^{\left(K_{s}, k\right)}\right)=(2 s-3)(k+1)$.
For some integer $a>(s-2)$ let $k=a(s-2)-1$, then $r^{\prime}=a$ and $p^{\prime}=0$. Therefore $B_{r^{\prime}}^{\left(K_{s}, k\right)}$ is a vertex disjoint union of $a$ complete graphs $K_{(2 s-3)}$. So $e\left(B_{r^{\prime}}^{\left(K_{s}, k\right)}\right)=a\binom{2 s-3}{2}$ where $a=\frac{k+1}{s-2}$, hence $e\left(B_{r^{\prime}}^{\left(K_{s}, k\right)}\right)=\frac{k+1}{s-2}(s-2)(2 s-3)=$ $(k+1)(2 s-3)$.

For $k+1$ we shall consider two cases:

Case 1. $p^{\prime}<r^{\prime}$.
Denote by $G$ a graph obtained by replacing one $K_{2 s-3}$ in $B_{r^{\prime}}^{\left(K_{s}, k\right)}$ by $K_{2 s-2}$. Then it is easy to see that $G=B_{r^{\prime}}^{\left(K_{s}, k+1\right)}$ and $e\left(B_{r^{\prime}}^{\left(K_{s}, k+1\right)}\right)=e\left(B_{r^{\prime}}^{\left(K_{s}, k\right)}\right)+$ $(2 s-3)$ and by induction $e\left(B_{r^{\prime}}^{\left(K_{s}, k+1\right)}\right)=(k+1)(2 s-3)+(2 s-3)=$ $((k+1)+1)(2 s-3)$.

Case 2. $p^{\prime}=r^{\prime}$.
Since $B_{r^{\prime}}^{\left(K_{s}, k\right)}$ is a vertex disjoint union of $r^{\prime}$ graphs $K_{2 s-2}$ so: $r^{\prime}(2 s-3-s)+r^{\prime}+r^{\prime}-1=k$, hence $r^{\prime}=\frac{k+1}{s-1}$. Now let us consider a graph $B_{r^{\prime \prime}}^{\left(K_{s}, k+1\right)}$ which is a vertex disjoint balanced union of $p^{\prime \prime}$ graphs $K_{2 s-2}$ and $r^{\prime \prime}-p^{\prime \prime}$ graphs $K_{2 s-3}$, where $r^{\prime \prime}=r^{\prime}+1$ and $p^{\prime \prime} \in\left\{0, \ldots, r^{\prime \prime}\right\}$.

Then

$$
\begin{aligned}
& r^{\prime \prime}(2 s-3-s)+p^{\prime \prime}+r^{\prime \prime}-1=k+1, \\
& \left(r^{\prime}+1\right)(2 s-3-s)+p^{\prime \prime}+\left(r^{\prime}+1\right)-1=k+1, \\
& (2 s-3-s)+p^{\prime \prime}+r^{\prime}(2 s-3-s)+r^{\prime}+r^{\prime}+1-1=k+1+r^{\prime}, \\
& (2 s-3-s)+p^{\prime \prime}+k+1=k+1+r^{\prime}, \\
& p^{\prime \prime}=r^{\prime}-(2 s-3-s) .
\end{aligned}
$$

Observe that $B_{r^{\prime \prime}}^{\left(K_{s}, k+1\right)}$ can be constructed from $B_{r^{\prime}}^{\left(K_{s}, k\right)}$ by replacing $r^{\prime}-p^{\prime \prime}$ graphs $K_{2 s-2}$ with $K_{2 s-3}$ and adding one graph $K_{2 s-3}$. Therefore,

$$
e\left(B_{r^{\prime \prime}}^{\left(K_{s}, k+1\right)}\right)=e\left(B_{r^{\prime}}^{\left(K_{s}, k\right)}\right)-\left(r^{\prime}-p^{\prime \prime}\right)(2 s-2)+e\left(K_{2 s-3}\right),
$$

and by induction

$$
\begin{aligned}
& \left(B_{r^{\prime \prime}}^{\left(K_{s}, k+1\right)}\right)=(k+1)(2 s-3)-\left(r^{\prime}-r^{\prime}+(2 s-3-s)\right)(2 s-2)+\binom{2 s-3}{2} \\
& =(k+1)(2 s-3)-(s-3)(2 s-2)+(2 s-3)(s-2) \\
& =(k+1)(2 s-3)+(2 s-3)=((k+1)+1)(2 s-3) .
\end{aligned}
$$

Conjecture 1. There is an integer $k(s)$ such that $Q\left(K_{s}, k\right)=(2 s-3)(k+1)$ for $k>k(s)$.

## 4.3. $\quad Q\left(K_{s}, k\right)$ for $s \geq 5$ and $s \geq s(k)$

Now we assume $s \geq 5$ is fixed.
Theorem 11. For every $k \in N$ there exists $s(k)$ such that $Q\left(K_{s}, k\right)=\binom{s+k}{2}$ for every $s \geq s(k)$.

Proof. For $k=0$ the proof is evident, we may assume $k \geq 1$. The inequality $Q\left(K_{s}, k\right) \leq\binom{ s+k}{2}$ is immediate. Now we prove that $Q\left(K_{s}, k\right) \geq$ $\binom{s+k}{2}$. Let $G$ be a $\left(K_{s}, k\right)$ stable graph with $e(G)=Q\left(K_{s}, k\right)$. Let $|V(G)|=$ $s+k+\beta$ where $\beta \geq 0$. The proof falls naturally into two cases.

Case 1. $0 \leq \beta \leq k$.

Subcase 1a. There are at most $\beta$ vertices $x \in V(G)$ such that $\operatorname{deg}_{G}(x) \leq$ $s+k-2$. Therefore, there are at least $s+k$ vertices $x \in V(G)$ such that $\operatorname{deg}_{G}(x) \geq s+k-1$. Then

$$
Q\left(K_{s}, k\right) \geq \frac{(s+k)(s+k-1)}{2}=\binom{s+k}{2} .
$$

Subcase 1b. There are at least $\beta+1$ vertices $x \in V(G)$ such that $\operatorname{deg}_{G}(x) \leq s+k-2$.

Assume that $s \geq 2 k^{2}+5 k+2$. Put: $B=\left\{v_{j} \in V(G) ; j=1,2, \ldots, \beta+1\right.$ and $\operatorname{deg}_{G}\left(v_{j}\right) \leq s+k-2$ for every $\left.j=1,2, \ldots, \beta+1\right\}$ and $W=\{v \in V(G)$; such that there is $v_{j} \in B$ such that $\left.v v_{j} \notin E(G)\right\}$.

The number of elements in $W$ is bounded above by the number of elements of $V(G)$ that are not adjacent to some $v_{j}$ for $j=1, \ldots, \beta+1$. But each element $v_{j}$ is not adjacent to at most $s+k+\beta-(s-1)$ elements from $V(G)$ (there are $s+k+\beta$ elements in $V(G)$ and $v_{j}$ is adjacent to at least $s-1$ elements). Note that in this reasoning $v_{j}$ lies in $W$. Therefore, we get $|W| \leq(\beta+1)(s+k+\beta-(s-1))=(\beta+1)(k+\beta+1)$. Since $0 \leq \beta \leq k$ we estimate $|W| \leq(k+1)(2 k+1)$. Observe that $2 k^{2}+5 k+2=(k+1)(2 k+1)+2 k+1$. Therefore, we may find vertices $w_{1}, w_{2}, \ldots, w_{k} \in V(G) \backslash(W \cup B)$. Observe that $w_{i} v_{j} \in E(G)$ for every $i=1,2, \ldots, k$ and $j=1,2, \ldots, \beta+1$. Denote by $G^{\prime}$ a graph obtained from a graph $G$ by removing all the vertices $w_{i}$ for $i=1,2, \ldots, k . G$ is $\left(K_{s}, k\right)$ stable so $G^{\prime}$ contains $K_{s}$ as a subgraph. Since we removed exactly $k$ vertices and $w_{i} \neq v_{j}$ for every $i=1,2, \ldots, k$ and $j=1,2, \ldots, \beta+1$ we have $\left|V\left(G^{\prime}\right)\right|=s+\beta$ and every vertex of $B$ is a vertex of $G^{\prime}$. We deduce there is at least one vertex of $B$ which is a vertex in a complete subgraph $K_{s}$. Since $\operatorname{deg}_{G^{\prime}}\left(v_{j}\right) \leq s-2<s-1$ for every $j=1,2, \ldots, \beta+1$ we get a contradiction.

Case 2. $\beta \geq k+1$.
If $s \geq k^{2}+k+1$, then since Lemma 1 implies that the minimum degree is $\geq s-1$,

$$
Q\left(K_{s}, k\right) \geq \frac{(s+2 k+1)(s-1)}{2} \geq\binom{ s+k}{2} .
$$

Since $k^{2}+k+1<2 k^{2}+5 k+2$ for $k \geq 1$ we complete the proof with $s(k):=2 k^{2}+5 k+2$.

Remark 2. It follows from the proof that $K_{s+k}$ is the only ( $K_{s}, k$ ) stable graph with minimum size for $s \geq 2 k^{2}+5 k+2$.

## Acknowledgement

The research was partially supported by the Research Training Network COMBSTRU. The author wishes to express his gratitude to Gyula Y. Katona for suggesting the problem and for many stimulating pieces of advice.

## References

[1] P. Frankl and G.Y. Katona, Extremal k-edge-hamiltonian hypergraphs, accepted for publication in Discrete Math.
[2] I. Horváth and G.Y. Katona, Extremal stable graphs, manuscript.
[3] R. Greenlaw and R. Petreschi, Cubic Graphs, ACM Computing Surveys, No. 4, (1995).

Received 8 January 2007
Revised 16 October 2007
Accepted 26 October 2007


[^0]:    *This work was carried out while the first author was visiting UPC in Barcelona.

