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(H, k) STABLE GRAPHS WITH MINIMUM SIZE

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Abstract

Let us call a G (H, k) graph vertex stable if it contains a subgraph H ever after removing any of its k vertices. By $Q(H, k)$ we will denote the minimum size of an (H, k) vertex stable graph. In this paper, we are interested in finding $Q(C_3, k)$, $Q(C_4, k)$, $Q(K_{1,p}, k)$ and $Q(K_s, k)$.

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1. Introduction

We deal with simple graphs without loops and multiple edges. As usual $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively, $|G|$, $e(G)$ the order and the size of G and $deg_G(v)$ the degree of $v \in V(G)$. By C_n we denote the cycle of order n and by K_r the complete graph on r vertices and by $K_{1,p}$ the star on $1+p$ vertices. The union $G \cup H$ of graphs G and H is defined *by* $V(G \cup H) := V(G) \cup V(H)$, $E(G \cup H) := E(G) \cup E(H)$, and we shall suppose that the components of the union are vertex disjoint.

By $G - e$ we shall denote the graph without the edge e and by $G - v$ the graph obtained from G by deleting the vertex $v \in V(G)$ and its incident edges.

In [1] G.Y. Katona and P. Frankl considered the following problem. What is the minimum size of a r-uniform hypergraph such that after removing any k hyperedges there is still a hamiltonian chain. To give a lover bound of the minimum size of the mentioned r-uniform hypergraphs the authors

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of [1] define the (P_4, k) edge stable graph as the graph in which after removing any k edges there is still P_4 and ask about the minimum size of (P_4, k) edge stable graph. This was intended as an attempt to solve the problem of finding the minimum size of a (P_4, k) edge stable graph. In [2] G.Y. Katona and I. Horváth considered the minimum size of (P_n, k) edge stable graphs. It is worth pointing out that there is no other result concerning edge stable graphs.

The aim of this paper is to consider a similar problem but in a vertex version. So let us give the following definition:

Definition 1. Let us call a (H, k) graph *vertex stable* if it contains a connected subgraph H ever after removing any of its k vertices. By $Q(H, k)$ we will denote the minimum size of an (H, k) vertex stable graph.

In this paper we estimate $Q(C_3, k)$, $Q(C_4, k)$, $Q(K_{1,p}, k)$ and give lower and upper bounds for $Q(K_s, k)$. For simplicity we will write stable instead of vertex stable.

The proofs are based on the facts given below.

Definition 2. W say that an (H, k) stable graph G is (H, k) *strong stable* if G is not $(H, k+1)$ stable and $G - e$ is not (H, k) stable for every $e \in E(G)$.

Proposition 1. *If G is an (H, k) stable graph with minimum size, then G is an (H, k) strong stable graph. Thus $Q(H, k) \leq e(G)$ where G is an (H, k) strong stable graph.*

Proof. Suppose G is an (H, k) stable graph with minimum size. Then clearly $G - e$ for any $e \in E(G)$ is not (H, k) stable. Suppose G is $(H, k+1)$ stable and $deg_G(v) > 0$, then $G - v$ is (H, k) stable with smaller size than $e(G)$, a contradiction. ■

Lemma 1. *If G is an (H, k) strong stable graph then every vertex as well as every edge of G belongs to some subgraph of G isomorphic to H.*

Proof. Suppose there is an edge e which is not in any $E(H)$. Then $G - e$ is still (H, k) stable with a smaller size than $e(G)$, a contradiction. If there exists a vertex v which is not in any $V(H)$, then each edge incident with v is not in $E(H)$, a contradiction. ■

Corollary 1. *If G is an (H, k) stable graph with a minimum size than every vertex as well as every edge of G belongs to some subgraph of G isomorphic to H.*

2. $Q(C_n, k)$

Theorem 2. $Q(C_3, k) = 3k + 3$.

Proof. Let G_k be a graph which is a vertex-disjoint union of $k+1$ triangles. Clearly, G_k is a (C_3, k) strong stable graph so $Q(C_3, k) \leq 3k + 3$.

We prove $Q(C_3, k) \geq 3k + 3$ by induction on k. It is clear that $Q(C_3, 0) = 3$. Suppose that the statement holds for any $k < k_0$. We prove the validity of our claim for k_0 indirectly.

Suppose that there is a graph G_{k_0} which is (C_3, k) strong stable but $e(G_{k_0}) < 3k + 3$. If the maximum degree in G_{k_0} is at most 2, then by Lemma 1 the graph consists only of cycle components. Since the number of edges in the graph is at most $3k + 2$, at most k components can be a triangle. So removing a vertex from each of these will destroy all triangles, a contradiction.

If there is a vertex v of degree greater or equal to 3, then $G_{k_0} - v$ is clearly a $(C_3, k - 1)$ strong stable graph with less than $3k$ edges, a contradiction again. ■

Lemma 2. *If G is (H, k) stable, then $G - v$ is $(H, k - 1)$ stable for any $v \in V(G)$. Moreover, if some edges in $G - v$ cannot be contained in any H subgraphs, then the graph obtained from $G - v$ by removing all these edges is still $(H, k - 1)$ stable.*

Proof. The first part of the proof follows from the definition of an (H, k) stable graph. From Corollary 1 it follows that all edges in $(H, k - 1)$ stable graphs belong to some H subgraph which finishes the proof. ■

Theorem 3. $Q(C_4, k) = 4k + 4$.

Proof. Let G_k be a graph which is a vertex-disjoint union of $(k + 1)$ C_4. Clearly, G_k is a (C_4, k) stable graph so $Q(C_4, k) \leq 4k + 4$.

We prove $Q(C_4, k) \geq 4k + 4$ by induction on k. It is clear that $Q(C_4, 0) = 4$. Suppose that the statement holds for any $k < k_0$. We prove the validity of our claim for k_0 indirectly.

Suppose that there is a graph G_{k_0} which is (C_4, k) stable with minimum size and $e(G_{k_0}) < 4k + 4$. From Corollary 1 it follows that $deg_{G_{k_0}}(x) \geq 2$ for every $x \in V(G_{k_0})$.

We shall consider the following cases:

Case 1. $\Delta(G_{k_0}) \geq 4$.
Let $deg_{G_{k_0}}(x) \geq 4$. Then $G_{k_0} - x$ is a $(C_4, k-1)$ stable graph with smaller size than $4k$, a contradiction.

Case 2. $\Delta(G_{k_0}) \leq 3$.
Suppose first that G_{k_0} contains a cycle as a component. Corollary 1 implies that it is C_4. If we delete one vertex of this C_4, then the remaining 2 edges of C_4 are not contained in any C_4 subgraphs. However, the graph without these 4 edges is still $(C_4, k-1)$ stable by Lemma 2. This contradicts the inductive hypothesis. Next suppose that $x_1x_2 \in E(G_{k_0})$ and $deg_{G_{k_0}}(x_1) = 3$, $deg_{G_{k_0}}(x_2) = 2$. By deleting x_1 using Lemma 2 we can derive a similar contradiction as before. Hence G_{k_0} contains only cubic components.

If K_4 is a component of G_{k_0} then it may be replaced by C_4 since both of them are $(C_4, 0)$ stable, and we get a graph with smaller size than G_{k_0}, a contradiction. Since the order of a $(C_4, 1)$ cubic graph is at least 6, then $Q(C_4, 1) \geq 9 > 8$. Since the order of a $(C_4, 2)$ cubic graph is at least 10 (see [3]), then we may estimate $Q(C_4, 2) \geq 15 > 12$. Denote by $(x_1x_2x_3x_4)$ a cycle C_4 in a cubic graph. If x_1x_3 or x_2x_4 is in $E(G_{k_0})$ it is in a contradiction with Corollary 1 or K_4 is a component of G_{k_0}. So we assume neither x_1x_3 nor x_2x_4 is in $E(G_{k_0})$. In the same way as before after deleting x_1 and x_3 we may remove all edges from the cycle $(x_1x_2x_3x_4)$ and all edges incident with vertices of the cycle and by Lemma 2 we get a $(C_4, k-2)$ stable graph with smaller size than $4k - 4$, a contradiction. ■

For $n \geq 6$ and $k \geq 0$ it is easy to see that a $(k+1)$ disjoint union of C_n is a (C_n, k) strong stable graph. The following theorem is evident.

Theorem 4. *$Q(C_n, k) \leq kn + n$.*

3. $Q(K_{1,p}, k)$

Theorem 5. *Let $p \geq 3$. Then $Q(K_{1,p}, k) = pk + p$.*

Proof. Let G_k be a graph which is a vertex-disjoint union of $k+1$ stars $K_{1,p}$. Clearly, G_k is a $(K_{1,p}, k)$ strong stable graph so $Q(K_{1,p}, k) \leq pk + p$.

We prove $Q(K_{1,p}, k) \geq pk + p$ by induction on k. It is clear that $Q(K_{1,p}, 0) = p$. Suppose that the statement holds for any $k < k_0$. We prove the validity of our claim for k_0 indirectly.

Suppose that there is a graph G_{k_0} which is $(K_{1,p}, k)$ strong stable with minimum size but $e(G_{k_0}) < pk + p$. From Lemma 1 it follows there is at least one vertex v of degree at least p. So $G_{k_0} - v$ is clearly a $(K_{1,p}, k-1)$ strong stable graph with size smaller than pk, a contradiction. ∎

Observe that a disjoint union of $(k+1)$ stars $K_{(1,p)}$ is a $(K_{(1,p),}k)$ strong stable graph.

4. $Q(K_s, k)$

Let $k \geq 0$ and $s \geq 0$. Let $G = (V(G); E(G))$ be a graph of order greater than $k + s$.

For a fixed k, $k > 0$ cases for $s = 0, 1, 2$ are trivial, the case for $s = 3$ was considered as C_3, so we turn to the case $s = 4$.

4.1. $Q(K_4, k)$

Theorem 6.

$$Q(K_4, k) = \begin{cases} 6 & \textit{for} \;\; k = 0, \\ 5k + 5 & \textit{for} \;\; k \geq 1. \end{cases}$$

Proof. It is obvious that $Q(K_4, 0) = 6$. Let G_k be a graph which is a vertex-disjoint union of $\frac{k+1}{2}$ K_5 for k odd, and a vertex-disjoint union of $(\frac{k-2}{2}$ $K_5) \cup K_6$ for k even. Clearly, G_k is (K_4, k) strong stable, so $Q(K_4, k) \leq 5k + 5$.

We prove $Q(K_4, k) \geq 5k + 5$ by induction on k. It is easy to see that $Q(K_4, 1) = 10$. Suppose that the statement holds for any $k < k_0$. We prove the validity of our claim for k_0 indirectly.

Suppose that there is a G_{k_0} graph which is (K_4, k) strong stable with minimum size and $e(G_{k_0}) < 5k + 5$.

We shall consider the following cases.

Case 1. $\Delta(G_{k_0}) \geq 5$.
Let $v \in G_{k_0}$ and $deg_{G_{k_0}}(v) \geq 5$. Then $G_{k_0} - v$ is $(K_4, k-1)$ strong stable and $e(G_{k_0} - v) < 5k$, a contradiction.

Case 2. $\Delta(G_{k_0}) = 4$ and $\delta(G_{k_0}) = 3$.
Let $v, z \in V(G_{k_0})$ and $deg_{G_{k_0}}(v) = 4$, $deg_{G_{k_0}}(z) = 3$.

Subcase 2a. Suppose $vz \in E(G_{k_0})$. Since edges incident to z in $G_{k_0} - v$ are not in K_4, then we may remove them. The graph obtained is $(K_4, k-1)$ strong stable and $e(G_{k_0} - v) < 5k - 1$, a contradiction.

Subcase 2b. Suppose there is no vertex of degree 3 adjacent to vertex of degree 4. It is easy to see that by Lemma 1 since every edge must be in K_4 it means that G_{k_0} contains K_4 as a component (K_5 will be considered in Case 3). Deleting one vertex from K_4 we get three edges which cannot be in any K_4 so we may delete them. We get a $(K_4, k-1)$ strong stable graph with smaller size than $5k - 1$, a contradiction.

Case 3. $\Delta(G_{k_0}) = 4$ and $\delta(G_{k_0}) = 4$.
By Lemma 1 we have that every edge must be in K_4, so it means that G_{k_0} is a vertex disjoint union of K_5. Because $e(G_{k_0}) < 5k + 5$, there is at most $(\lceil \frac{k+1}{2} \rceil - 1)$ K_5. If we delete k vertices, two from every K_5, we will destroy all K_4, a contradiction.

Observe that the family given in the above theorem is also (K_4, k) strong stable with minimum size. ■

4.2. The upper bound of $Q(K_s, k)$ for $s \geq 5$

The following assumption will be needed throughout this subsection

1. $k \geq 0$ and $s \geq 5$ is fixed,
2. $1 \leq r \leq k+1, j \in \{1, 2, \ldots, r\}, i_j \geq s$ and $i_1 \leq i_2 \leq \ldots \leq i_r$.

Let $\mathcal{A}_r^{(K_s,k)}$ be a family of graphs consisting of vertex disjoint unions of r complete graphs K_{i_j} satisfying the following condition:

$$\sum_{j=1}^{r} (i_j - s) + r - 1 = k\,.$$

For simplicity, we will write $\mathcal{A}_r^{(K_s,k)}$ without repetition of the above assumption.

Observe that for $r = 1$ the family $\mathcal{A}_r^{(K_s,k)}$ is reduced to a complete graph K_{s+k}, and for $r = k+1$ it consist only of a vertex disjoint union of $k+1$ graphs K_s. Obviously, these graphs are (K_s, k) strong stable.

For a fixed k, we will show that all graphs from $\mathcal{A}_{r}^{(K_s,k)}$ are (K_s,k) strong stable and give the construction of a family $A(K_s,k)$ with the smallest size. This gives us an upper bound of $Q(K_s,k)$.

Lemma 3. *For a fixed k, $k \geq 0$. Then $G \in \mathcal{A}_r^{(K_s,k)}$ is (K_s,k) strong stable.*

Proof. The proof will be divided into two steps. Let $G \in \mathcal{A}_r^{(K_s,k)}$.

Step 1. We show that G is (K_s,k) stable.
Deleting $\sum_{j=1}^{r}(i_j - s) = k - (r-1)$ vertices we obtain a union of complete graphs in which:

Case 1a. There is a complete graph of order greater than or equal to $s + r - 1$. Hence after removing any $r-1$ vertices from the graph we still have K_s.

Case 1b. All complete graphs have their size less than $s + r - 1$.
It means that it is a union of exactly r complete graphs and each of them contains K_s. Hence after removing any $r-1$ vertices we still have K_s.

Step 2. We show that G is not $(K_s, k+1)$ stable and $G - e$ is not (K_s,k) stable for every $e \in E(G)$.

Deleting k vertices from G we obtain that the order of the remaining graph is: $i_1 + i_2 + \ldots + i_r = r(s-1) + 1$. So we may create a union of r graphs containing $(r-1)$ graphs K_{s-1} and exactly one K_s. The proof is completed by removing one vertex or one edge from K_s. ■

Definition 3. For a fixed k, $k \geq 0$. We call $G \in \mathcal{A}_r^{(K_s,k)}$ a *balanced union* if $|i_j - i_q| \in \{0,1\}$ $j,q \in \{1,2,\ldots,r\}$.

Remark 1. For a fixed k and r there is exactly one balanced union $B_r^{(K_s,k)} \in \mathcal{A}_r^{(K_s,k)}$.

Proof. For a fixed k and r let $G \in \mathcal{A}_r^{(K_s,k)}$. Suppose G consists of a vertex disjoint union of p graphs K_{s+i+1} and $r-p$ graphs K_{s+i}. $G \in \mathcal{A}_r^{(K_s,k)}$ therefore:

$$\sum_{1}^{r-p}(s+i-s)+\sum_{1}^{p}(s+i+1-s)+r-1=k,$$
$$(r-p)i+p(i+1)+r-1=k,$$
$$ri+p+r-1=k.$$

Hence $p = k - ri - r + 1$ and $i = \frac{(k-r+1)}{r} - \frac{p}{r}$. Obviously, i must be an integer. Moreover, $0 \leq p < r$, so there is exactly one p such that $\left\lfloor \frac{k-r+1}{r} \right\rfloor = \frac{(k-r+1)}{r} - \frac{p}{r} = i$. Therefore G is a unique balanced union, hence, $G = B_r^{(K_s,k)}$. ■

We leave it to the reader to verify that:

Proposition 7. *For a fixed k and r, $B_r^{(K_s,k)}$ has the smallest possible size among all graphs $G \in \mathcal{A}_r^{(K_s,k)}$.*

Lemma 4. *Let $s \geq 5$. There exists $k_1(s)$ such that $e(B_2^{(K_s,k)}) < e(K_{s+k})$ for $k \geq k_1(s)$.*

Proof. Let $B_2^{(K_s,k)} = K_{i_1} \cup K_{i_2}$. We will consider two cases:

Case 1. $i_1 = i_2$.
Then

$$k = \sum_{j=1}^{2}(i_j - s) + 2 - 1 = 2(i_1 - s) + 1$$

so $i_1 = \frac{1}{2}(k - 1 + 2s)$ and the inequality:

$$2\binom{\frac{1}{2}(k-1+2s)}{2} = e(K_{i_1}) + e(K_{i_2}) = e(B_2^{(K_s,k)}) < e(K_{s+k}) = \binom{s+k}{2}$$

holds for $k \geq k_1(s) = \lceil \sqrt{2s_2 + 6s + 4} \rceil$.

Case 2. $i_1 + 1 = i_2$.
A similar inequality holds for $k \geq k_1(s) = \lceil \sqrt{2s_2 + 6s + 5}\ \rceil$. ■

It is easily seen that:

Proposition 8. *If* $B_2^{(K_s,k)} = K_{i_1} \cup K_{i_2}$ *and* $G = K_{i_1+1} \cup K_{i_2}$, *then* $G = B_2^{(K_s,k+1)}$.

Lemma 5. *Let* $k_1(s)$ *be a value given by Lemma* 4. *If* $K_{s+k'}$ *is a component of* $B_r^{(K_s,k)}$ *for* $k' \geq k_1(s)$, *then there is a graph* $B_{r'}^{(K_s,k)}$ *such that* $e(B_{r'}^{(K_s,k)}) < e(B_r^{(K_s,k)})$ *and* $r' > r$.

Proof. Suppose that $K_{s+k'}$ and $K_{s+k'+1}$ are components of $B_r^{(K_s,k)}$ for $k' \geq k_1(s)$. Note that $K_{s+k'}$ is a (K_s, k') strong stable graph. From Lemma 4 it follows that there are integers i_1 and i_2 such that

$$e(K_{i_1}) \cup e(K_{i_2}) = e\left(B_2^{(K_s,k')}\right) < e(K_{s+k'}).$$

Denote by H^* a graph obtained by replacing all $K_{s+k'}$ in $B_r^{(K_s,k)}$ by $K_{i_1} \cup K_{i_2}$ and replacing all $K_{s+k'+1}$ in $B_r^{(K_s,k)}$ by $K_{i_1+1} \cup K_{i_2}$.

It is obvious that $e(H^*) < e(B_r^{(K_s,k)})$. Moreover, H^* is (K_s, k) strong stable and it is a balanced union, therefore there is an integer r' such $H^* = B_{r'}^{(K_s,k)}$. ■

Lemma 5 may be used to show by similar arguments as in Lemma 4 that there exists $k_n(s)$ such that $e(B_{n+1}^{(K_s,k)}) < e(B_n^{(K_s,k)})$ for $k \geq k_n(s)$.

Thus we may construct graphs $A(K_s, k)$ such that for $k_n(s) \leq k < k_{n+1}(s)$, $A(K_s, k) = B_{n+1}^{(K_s,k)}$. From the above construction the following theorem follows easily:

Theorem 9. $Q(K_s, k) \leq e(A(K_s, k)) \leq e(G)$ *for every* $G \in \mathcal{A}_r^{(K_s,k)}$ *where* $r \in \{1, \dots, k+1\}$.

From the proof of Remark 1 we have the following estimation of this upper bound by sizes of (K_s, k) strong stable balanced unions

$$Q(K_s, k) \leq \min_{r \in \{1,\dots,k+1\}} \left(r \binom{s + i_r}{2} + p_r(s + i_r) \right),$$

where $i_r = \left\lfloor \frac{k-r+1}{r} \right\rfloor$ and $p_r = k - r + 1 - r i_r$.

For a sufficiently large k, we may estimate the upper bound differently.

Theorem 10. *There is an integer $k(s)$ such that $Q(K_s,k) \leq (2s-3)(k+1)$ for $k > k(s)$.*

Proof. Let G be a vertex disjoint union of p graphs K_{2s-2} and $r-p$ graphs K_{2s-3} where $r \in \{1,\ldots,k+1\}$ and $p \in \{0,\ldots,r\}$. Suppose that $G \in \mathcal{A}_r^{(K_s,k)}$. Then

$$\sum_1^{r-p}(2s-3-s) + \sum_1^{p}(2s-2-s) + r - 1 = k,$$
$$r(s-3) + p + r - 1 = k,$$
$$r(s-2) + p - 1 = k.$$

If $k > (s-2)(s-2) + (s-2) - 1$, then $r \geq (s-2)$. Hence $p \in \{0,\ldots,s-2,\ldots,r\}$, and so there is a pair r', p' (not necessarily unique) which satisfies the equation. Therefore $G = B_{r'}^{(K_s,k)}$

Now we will show by induction on k that $e(B_{r'}^{(K_s,k)}) = (2s-3)(k+1)$.

For some integer $a > (s-2)$ let $k = a(s-2)-1$, then $r' = a$ and $p' = 0$. Therefore $B_{r'}^{(K_s,k)}$ is a vertex disjoint union of a complete graphs $K_{(2s-3)}$. So $e(B_{r'}^{(K_s,k)}) = a\binom{2s-3}{2}$ where $a = \frac{k+1}{s-2}$, hence $e(B_{r'}^{(K_s,k)}) = \frac{k+1}{s-2}(s-2)(2s-3) = (k+1)(2s-3)$.

For $k+1$ we shall consider two cases:

Case 1. $p' < r'$.
Denote by G a graph obtained by replacing one K_{2s-3} in $B_{r'}^{(K_s,k)}$ by K_{2s-2}. Then it is easy to see that $G = B_{r'}^{(K_s,k+1)}$ and $e(B_{r'}^{(K_s,k+1)}) = e(B_{r'}^{(K_s,k)}) + (2s-3)$ and by induction $e(B_{r'}^{(K_s,k+1)}) = (k+1)(2s-3) + (2s-3) = ((k+1)+1)(2s-3)$.

Case 2. $p' = r'$.
Since $B_{r'}^{(K_s,k)}$ is a vertex disjoint union of r' graphs K_{2s-2} so: $r'(2s-3-s) + r' + r' - 1 = k$, hence $r' = \frac{k+1}{s-1}$. Now let us consider a graph $B_{r''}^{(K_s,k+1)}$ which is a vertex disjoint balanced union of p'' graphs K_{2s-2} and $r''-p''$ graphs K_{2s-3}, where $r'' = r'+1$ and $p'' \in \{0,\ldots,r''\}$.

Then

$$\begin{aligned}
&r''(2s-3-s)+p''+r''-1=k+1,\\
&(r'+1)(2s-3-s)+p''+(r'+1)-1=k+1,\\
&(2s-3-s)+p''+r'(2s-3-s)+r'+r'+1-1=k+1+r',\\
&(2s-3-s)+p''+k+1=k+1+r',\\
&p''=r'-(2s-3-s).
\end{aligned}$$

Observe that $B_{r''}^{(K_s,k+1)}$ can be constructed from $B_{r'}^{(K_s,k)}$ by replacing $r'-p''$ graphs K_{2s-2} with K_{2s-3} and adding one graph K_{2s-3}. Therefore,

$$e\left(B_{r''}^{(K_s,k+1)}\right)=e\left(B_{r'}^{(K_s,k)}\right)-(r'-p'')(2s-2)+e(K_{2s-3}),$$

and by induction

$$\begin{aligned}
&\left(B_{r''}^{(K_s,k+1)}\right)=(k+1)(2s-3)-(r'-r'+(2s-3-s))(2s-2)+\binom{2s-3}{2}\\
&=(k+1)(2s-3)-(s-3)(2s-2)+(2s-3)(s-2)\\
&=(k+1)(2s-3)+(2s-3)=((k+1)+1)(2s-3). \qquad \blacksquare
\end{aligned}$$

Conjecture 1. There is an integer $k(s)$ such that $Q(K_s,k)=(2s-3)(k+1)$ for $k>k(s)$.

4.3. $Q(K_s,k)$ for $s\geq 5$ and $s\geq s(k)$

Now we assume $s\geq 5$ is fixed.

Theorem 11. *For every $k\in N$ there exists $s(k)$ such that $Q(K_s,k)=\binom{s+k}{2}$ for every $s\geq s(k)$.*

Proof. For $k=0$ the proof is evident, we may assume $k\geq 1$. The inequality $Q(K_s,k)\leq\binom{s+k}{2}$ is immediate. Now we prove that $Q(K_s,k)\geq\binom{s+k}{2}$. Let G be a (K_s,k) stable graph with $e(G)=Q(K_s,k)$. Let $|V(G)|=s+k+\beta$ where $\beta\geq 0$. The proof falls naturally into two cases.

Case 1. $0\leq\beta\leq k$.

Subcase 1a. There are at most β vertices $x \in V(G)$ such that $deg_G(x) \leq s+k-2$. Therefore, there are at least $s+k$ vertices $x \in V(G)$ such that $deg_G(x) \geq s+k-1$. Then

$$Q(K_s,k) \geq \frac{(s+k)(s+k-1)}{2} = \binom{s+k}{2}.$$

Subcase 1b. There are at least $\beta+1$ vertices $x \in V(G)$ such that $deg_G(x) \leq s+k-2$.

Assume that $s \geq 2k^2+5k+2$. Put: $B = \{v_j \in V(G); j = 1,2,\ldots,\beta+1$ and $deg_G(v_j) \leq s+k-2$ for every $j = 1,2,\ldots,\beta+1\}$ and $W = \{v \in V(G);$ such that there is $v_j \in B$ such that $vv_j \notin E(G)\}$.

The number of elements in W is bounded above by the number of elements of $V(G)$ that are not adjacent to some v_j for $j = 1,\ldots,\beta+1$. But each element v_j is not adjacent to at most $s+k+\beta-(s-1)$ elements from $V(G)$ (there are $s+k+\beta$ elements in $V(G)$ and v_j is adjacent to at least $s-1$ elements). Note that in this reasoning v_j lies in W. Therefore, we get $|W| \leq (\beta+1)(s+k+\beta-(s-1)) = (\beta+1)(k+\beta+1)$. Since $0 \leq \beta \leq k$ we estimate $|W| \leq (k+1)(2k+1)$. Observe that $2k^2+5k+2 = (k+1)(2k+1)+2k+1$. Therefore, we may find vertices $w_1, w_2, \ldots, w_k \in V(G)\backslash(W \cup B)$. Observe that $w_iv_j \in E(G)$ for every $i = 1,2,\ldots,k$ and $j = 1,2,\ldots,\beta+1$. Denote by G' a graph obtained from a graph G by removing all the vertices w_i for $i = 1,2,\ldots,k$. G is (K_s,k) stable so G' contains K_s as a subgraph. Since we removed exactly k vertices and $w_i \neq v_j$ for every $i = 1,2,\ldots,k$ and $j = 1,2,\ldots,\beta+1$ we have $|V(G')| = s+\beta$ and every vertex of B is a vertex of G'. We deduce there is at least one vertex of B which is a vertex in a complete subgraph K_s. Since $deg_{G'}(v_j) \leq s-2 < s-1$ for every $j = 1,2,\ldots,\beta+1$ we get a contradiction.

Case 2. $\beta \geq k+1$.
If $s \geq k^2+k+1$, then since Lemma 1 implies that the minimum degree is $\geq s-1$,

$$Q(K_s,k) \geq \frac{(s+2k+1)(s-1)}{2} \geq \binom{s+k}{2}.$$

Since $k^2+k+1 < 2k^2+5k+2$ for $k \geq 1$ we complete the proof with $s(k) := 2k^2+5k+2$. ■

Remark 2. It follows from the proof that K_{s+k} is the only (K_s,k) stable graph with minimum size for $s \geq 2k^2+5k+2$.

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