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PATHS OF LOW WEIGHT IN PLANAR GRAPHS

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Abstract

The existence of paths of low degree sum of their vertices in planar graphs is investigated. The main results of the paper are:

1. Every 3-connected simple planar graph G that contains a k-path, a path on k vertices, also contains a k-path P such that for its weight (the sum of degrees of its vertices) in G it holds

$$w_G(P) := \sum_{u \in V(P)} \deg_G(u) \le \frac{3}{2} k^2 + \mathcal{O}(k) \,.$$

2. Every plane triangulation T that contains a k-path also contains a k-path P such that for its weight in T it holds

$$w_T(P) := \sum_{u \in V(P)} \deg_T(u) \le k^2 + 13k.$$

3. Let G be a 3-connected simple planar graph of circumference c(G). If $c(G) \geq \sigma |V(G)|$ for some constant $\sigma > 0$ then for any k, $1 \leq k \leq c(G)$, G contains a k-path P such that

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) \le \left(\frac{3}{\sigma} + 3\right)k.$$

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1. NOTATION

We will adapt the convention that a graph is *planar* if it can be embedded in the plane (without edges crossing), and *plane* if it is already embedded in the plane. This paper will be concerned with simple plane graphs. The sets of vertices, edges and faces of such a graph G will be denoted by V(G), E(G)and F(G), respectively, or by V, E and F if G is known from the context.

The facial walk of a face α of a connected plane graph G is the shortest closed walk induced by all edges incident with α . The degree of a face α is the length of its facial walk and is denoted by $\deg_G(\alpha)$ or $\deg(\alpha)$ if G is known from the context. The degree of a vertex x of a graph is the number of edges incident with x. Analogously the notation $\deg_G(x)$ or $\deg(x)$ is used for the degree of a vertex x. Let a *k*-vertex be a vertex of degree k. Let a *k*-face be defined similarly.

Let an (a, b)-edge be an edge f if an a-vertex and a b-vertex are endvertices of f.

A 3-face α is said to be the (a, b, c)-triangle or a triangle of type (a, b, c) if vertices incident with α have degrees a, b and c.

Let a *k*-path and a *k*-cycle be a path and a cycle on *k* vertices, respectively. Let a *k*-path be an (a_1, a_2, \ldots, a_k) -path if it passes through the vertices u_1, u_2, \ldots, u_k in order with $a_i = \deg(u_i)$ for all $i = 1, 2, \ldots, k$.

Let an (x; a, b, c)-star be a star $K_{1,3}$ which is a subgraph of a graph G with a central x-vertex and an a-vertex, a b-vertex and a c-vertex as leaves.

Let $\mathcal{P}_{\kappa}(\delta,\rho)$ be the family of all κ -connected simple plane graphs with minimum vertex degree at least δ , $\delta \geq 3$, and minimum face degree at least ρ . It is easy to see that $\mathcal{P}_{\kappa}(\delta,\rho)$ is not empty only for $(\delta,\rho) \in$ $\{(3,3), (3,4), (4,3), (3,5), (5,3)\}$. Let $\mathcal{P}(\delta,\rho) := \mathcal{P}_3(\delta,\rho)$ and $\mathcal{P}(\delta,\bar{3}) := \{G \in \mathcal{P}(\delta,3): G \text{ is a triangulation}\}.$

For a subgraph H of a planar graph G, the weight $w_G(H)$ of H is defined to be the sum of degrees of vertices of H in G; namely

$$w_G(H) = \sum_{u \in V(H)} \deg_G(u) \,.$$

2. INTRODUCTION AND RESULTS

It is well known that every planar graph contains a vertex of degree at most 5. In 1955 Kotzig [15, 16] proved that every 3-connected planar graph

contains an edge of the weight at most 13 in general and most 11 in the absence of 3-vertices, respectively. These bounds are best possible.

In 1940 Lebesgue [17] proved that every graph G of minimum degree 5 contains a star $K_{1,3}$ such that its central vertex has, in G, degree at most 5 and every its leaf has degree at most 8. Recently van den Heuvel and McGuinness [11] proved that every planar graph of minimum degree at least 3 contains a (3, a)-edge for a $3 \leq a \leq 11$, or an (a, 4, b)-path for $3 \leq a \leq 7$ and $a \leq b \leq 11$, or a (5; a, b, c)-star for $3 \leq a \leq 6$, $a \leq b \leq 7$ and $b \leq c \leq 11$.

The following recent result, that strengthens all above mentioned ones, will be applied later in this paper.

Theorem 1 [9]. Every planar graph G of minimum degree $\delta(G) \geq 3$ contains

- (i) a (3, a)-edge for $3 \le a \le 10$, or
- (ii) an (a, 4, b)-path for a = 4 and $4 \le b \le 10$,

or
$$a = 5$$
 and $5 \le b \le 9$,

or
$$6 \le a \le 7$$
 and $6 \le b \le 8$, or

(iii) a (5; a, b, c)-star for $4 \le a \le 5, 5 \le b \le 6$ and $5 \le c \le 7$, or a = b = c = 6.

Moreover, for every $S \in \{(3,10)\text{-}edge, (4,4,9)\text{-}path, (5,4,8)\text{-}path, (6,4,8)\text{-}path, (7,4,7)\text{-}path, (5;5,6,7)\text{-}star, (5;6,6,6)\text{-}star\}$ there is a 3-connected plane graph H containing S and no other subgraph from the above list.

In generalizing Kotzig's theorem there are several other natural directions. Two possibilities are as follows.

Let $k \ge 1$ be an integer.

(A) Find the smallest integer $f = f(k, \delta, \rho)$ such that whenever a graph $G \in \mathcal{P}_3(\delta, \rho)$ contains at least k vertices, there is a connected subgraph H of G of order k whose weight

$$w_G(H) = \sum_{u \in V(H)} \deg_G(u) \le f(k, \delta, \rho) \,.$$

(B) Find the smallest integer $w = w(k, \delta, \rho)$ such that whenever a graph $G \in \mathcal{P}_3(\delta, \rho)$ contains a k-path there is a k-path P_k in G with weight

$$w_G(P_k) = \sum_{u \in V(P_k)} \deg_G(u) \le w(k, \delta, \rho) \,.$$

The possibility (A) was investigated by Enomoto and Ota [4]. They proved that for $k \ge 4$

$$8k - 5 \le f(k, 3, 3) \le 8k - 1$$

and conjectured the precise value of f(k, 3, 3) to be 8k - 5.

The problem (B) was formulated in [5]. The precise values of $w(k, \delta, 3)$ are known only for small k, e.g. w(1,3,3) = 5, w(2,3,3) = f(2,3,3) = 13, w(2,4,3) = f(2,4,3) = 11 (Kotzig [15]), w(2,5,3) = f(2,5,3) = 11 (Wernicke [21]), w(3,5,3) = f(3,5,3) = 17 (Franklin [7]), and w(3,3,3) = f(3,3,3) = 21 (Ando, Iwasaki and Kaneko [1]). For greater k only estimations are known, see e.g. surveys in [10, 13, 14]. In [5, 6] it was proved that

$$k \log_2 k \le w(k, 3, 3) \le 5k^2$$

Madaras [18] improved the upper bound showing that $w(k,3,3) \leq \frac{5}{2}k(k+1)$.

Applying Theorem 1 we are able to prove the first main result of this paper

Theorem 2. Let k be an integer, $k \ge 4$. Then

- (i) every plane triangulation T, that contains a k-path, also contains a k-path P such that $w_T(P) \leq k^2 + 13k$, and
- (ii) every 3-connected planar graph, that contains a k-path, also contains a k-path P such that $w_G(P) \le w(k,3,3) \le \frac{3}{2}k^2 + \mathcal{O}(k)$.

Note. As shown in [12], no analogue of Theorem 2 can be proved for the family $\mathcal{P}_2(3,3)$. More precisely: For every pair of integers $m, k, m \ge k \ge 3$, there is a 2-connected planar graph G in which every k-path P_k has weight at least m, that is $w_G(P_k) \ge m$.

For k = 2 the situation is different. In 1972 Erdös conjectured that Kotzig's theorem holds for all planar graphs G with minimum degree $\delta(G) \geq 3$. This conjecture was proved by Barnette, see [8].

The restriction to 4-connected planar graphs brings a different behaviour. In 2000 Mohar [19] proved that every 4-connected planar graph of order at least k contains a k-path P_k of weight

$$w_G(P_k) \le 6k - 1;$$

the bound being tight. The difference is that every 4-connected plane graph contains a k-path whose weight is bounded from above by a function linear

in k while on the other side there are 3-connected plane graphs in which all k-paths have weight bounded from below by a function which is not linear in k.

Developing the ideas of Mohar's proof [19] we show that a linear in k upper bound is also true for a wider family of plane graphs. Namely, the second main result of this paper is

Theorem 3. For $G \in \mathcal{P}(\delta, \rho)$ let c(G) be the length of a longest cycle of G. Let k be an integer, $3 \leq k \leq c(G)$. If $c(G) \geq \sigma |V(G)|$ for some positive number σ then G contains a k-path P_k such that

$$w_G(P_k) < \left(\left(\frac{2\rho}{\rho - 2} - \delta \right) \frac{1}{\sigma} + \delta \right) k.$$

In fact Theorem 3 is a corollary of the following more general result

Theorem 4. Let G be a graph with n = |V(G)| vertices, e = |E(G)| edges, the length of a longest cycle c = c(G), and minimum vertex degree $\delta = \delta(G)$. Let k be a positive integer $k \leq c$. Then G contains a k-path P with

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) \le \left(\frac{2e}{c} + \delta\left(1 - \frac{n}{c}\right)\right) k.$$

Immediately we have

Corollary 5. Let G be a hamiltonian graph on n vertices, and let k be a positive integer, $k \leq n$. Then G contains a k-path P such that

$$w_G(P) \le \frac{2e}{n} k$$
.

Every 4-connected planar graph G is known to be hamiltonian [20]. Hence c(G) = |V(G)| and, because in this case $e \leq 3n - 6$, we immediately obtain the above mentioned elegant Mohar's theorem [19].

Let S be a set of three vertices of a 3-connected planar graph G such that the graph G - S obtained from G by removing S is disconnected (S is called a 3-separator in this case). It is known that G - S consists of exactly two components \mathcal{A} and \mathcal{B} . G is called to be *essentially* 4-connected if it is 3-connected and $|V(\mathcal{A})| = 1$ or $|V(\mathcal{B})| = 1$ for every 3-separator S of G. **Theorem 6.** Let G be an essentially 4-connected planar graph, and let k be an integer $1 \le k \le \frac{|V(G)|}{2}$. Then G contains a k-path P such that

$$w_G(P) \le 9k - 1$$

Moreover, there exists an essentially 4-connected planar graph H in which every k-path P has weight

$$w_H(P) \ge \frac{15}{2} k - \frac{13}{2}.$$

The rest of the paper is organized as follows. In Sections 3 and 4 we prove Theorem 2. Theorems 3, 4 and 6 are proved in Section 5. In Section 6 we add some remarks concerning the results and open problems.

3. Proof of Theorem 2(i)

First we prove the following theorem

Theorem 7. For a given positive integer $k \ge 4$, let G be a 3-connected plane graph in which every r-face, $r \ge 4$, contains at most two vertices of degree greater than k, and if it contains exactly two, then they are adjacent. Then G contains a k-path P of the weight

$$w(P) \le k^2 + 13k$$

that has at most four vertices of degree greater than k.

Proof. We give a constructive proof of Theorem 7. For convenience a vertex x of G is called *major* if $\deg_G(x) > k$, and is called *minor* otherwise.

For a vertex x of G let C_x be a cycle induced in G by all edges of all faces incident with the vertex x but not having x as an endvertex (i.e., edges not incident with x). Clearly all neighbours of x are in $V(C_x)$ and the length of C_x is at least $\deg_G(x)$.

Let M = M(G) be a subgraph of G induced on major vertices of G. The graph G and the subgraph M have the properties mentioned in the lemmas below: **Lemma 1.** Let G contain a major vertex x with $\deg_M(x) = d$ and let $\deg_G(x) \ge kd + 1$. Then G contains a k-path P of which all vertices are minor and therefore $w(P) \le k^2$.

Proof. Since $\deg_G(x) \ge kd + 1$ then for the cycle \mathcal{C}_x we have $|\mathcal{C}_x| > kd$, but on \mathcal{C}_x there are exactly d major vertices. Hence at least one path between two consecutive major vertices of \mathcal{C}_x contains at least k minor vertices. So we have a k-path P with $w(P) \le k^2$.

Lemma 2. If G contains a major vertex x with $\deg_M(x) \leq 2$, then G contains a k-path P all vertices, except possibly x, are minor and $w(P) \leq k^2 + k$.

Proof. In the case $\deg_M(x) \leq 1$ the proof is clear. Let x be a 2-vertex in M then $\deg_G(x) \geq k+1$. If $\deg_G(x) \geq 2k+1$, then, by Lemma 1, G contains a required path. Let $\deg_G(x) \leq 2k$. On \mathcal{C}_x there are two major vertices, say y and z, which divide \mathcal{C}_x into two subpaths both consisting of minor vertices that contain two subpaths P_a and P_b with $a + b \geq k - 1$ starting in minor vertices u and v, respectively, which are neighbours of x in G. These two paths together with the edges ux and xv form an l-path, $l \geq k$. This path contains a k-path P as a subgraph all vertices of which, except possibly x, are minor. So for P we have

$$w_G(P) \le k(k-1) + 2k = k^2 + k$$
.

Due to Lemma 1 and Lemma 2 we may suppose that $\deg_M(x) = d \ge 3$ and $\deg_G(x) \le kd$ for every major vertex x of G. Then we apply to M our Theorem 1. By it there is a major vertex u with $\deg_M(u) = a \le 5$. Due to hypothesis of Theorem 7 all other vertices of \mathcal{C}_u are minor. Among these major neighbours of u there are at most two vertices, say y and z, whose degrees are not known, and

- (i) if $\deg_M(u) = 3$ then there is a major neighbour v with $\deg_M(v) = b \le 10$, or
- (ii) if $\deg_M(u) = 4$ then there are two more major neighbours of u, say v and w, such that $\deg_M(v) = b$ and $\deg_M(w) = c$ with $4 \le b \le 5$ and $4 \le c \le 10$ or $6 \le b \le 7$ and $6 \le c \le 8$, respectively, or
- (iii) if $\deg_M(u) = 5$ then there are three more major neighbours v, w and x of degrees b, c, and d, respectively, where $b \leq 5, c \leq 6, d \leq 7$ or b = c = d = 6.

We may suppose that (1) and (2) below hold:

(1) For the vertex $t \in \{v, w, x\}$ with $\deg_M(t) = r \deg_G(t) \leq (r-1)k$.

Proof of (1). Suppose $\deg_G(t) \ge (r-1)k+1$. Consider the cycle C_t . Except of the vertex u of degree at most 5k there are other r-1 major vertices on C_t . These major vertices split C_t into r-1 subpaths. At least one of these subpaths contains a k-path P on k-1 minor vertices and the k-th (possibly exceptional) vertex of this path is either a minor vertex or the vertex u which has degree at most 5k. This means that the k-path P has at most one major vertex and we have $w_G(P) \le k(k-1) + 5k = k^2 + 4k$.

(2) For the vertex $u \deg_G(u) \leq 2k$ holds.

Proof of (2). Suppose the contrary. Let the cycle C_u have at least 2k + 1 vertices. There are at most five major vertices among them (all are neighbours of u), namely the vertices y and z, and at most three from the set $\{v, w, x\}$ as described above. The vertices y and z divide the cycle C_u into two subpaths P_p and P_q with $p + q \ge 2k - 1$. Hence $\max\{p,q\} \ge k$ and on C_u there is a k-path P all vertices of which except of at most three (from the set $\{v, w, x\}$) are minor. This path has in the case (i) the weight $w_G(P) \le k(k-1) + (b-1)k \le k^2 - k + 10k - k = k^2 + 8k$ because P contains at most one major neighbour of degree $\le 9k$. In the case (ii) (the case (iii)) the path P has weight

$$w_G(P) \le k(k-2) + (b-1)k + (c-1)k \le k^2 + 11k$$
$$(w_G(P) \le k(k-3) + (b-1)k + (c-1)k + (d-1)k \le k^2 + 12k)$$

and contains at most two major vertices (at most three major vertices, respectively).

Because u is a major vertex, $|\mathcal{C}_u| \geq k + 1$ and the vertices y and z divide the cycle \mathcal{C}_u into two subpaths that contain two subpaths P_p and P_q with $p+q \geq k-1$ starting in vertices u^* and v^* which are neighbours of u in G. These two subpaths together with the vertex u and edges u^*u and uv^* form an l-path, $l \geq k$, which contains as a subgraph a k-path P passing through at least (k-4) minor vertices and at most four major vertices all from the set $\{u, v, w, x\}$.

Applying Properties (1) and (2), and distinguishing three cases (i), (ii) and (iii) according to the degree of u in M we obtain, similarly as in the

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proof of (2), the following upper bound on $w_G(P)$, the weight of P,

$$w_G(P) \le k^2 + 13k$$

From Theorem 7 we immediately have the upper bound for Theorem 2(i).■

4. Proof of Theorem 2(ii)

To prove Theorem 2(ii) consider first the following construction and then modify the proof of Theorem 2(i). Suppose $H \in \mathcal{P}(3,3)$.

Let $G_0 = H, G_1, \ldots, G_p = G$ be a sequence of plane graphs defined as follows: If G_i , $i = 0, 1, \ldots, p-1$, is a 3-connected plane graph having an *r*-face α , $r \ge 4$, incident with two non-adjacent major vertices u and v we insert a diagonal d = uv into α joining the vertices u and v. The result is a 3-connected plane graph $G_{i+1} = G_i + d$. If G_i does not contain any face α having the above-mentioned property we put i = p and $G = G_p$.

It is easy to see that the graph $G_p = G$ satisfies the hypothesis of Theorem 7. If G contains a major vertex x of degree $\deg_M(x) \leq 2$ then, analogously as in the proof of Lemma 2, one can prove that G contains a k-path P of the weight $w(P) \leq k^2 + k$ with at most one major vertex. Clearly this path is also present in the graph H. If for each major vertex x in $G \deg_M(x) \ge 3$ then, by Theorem 1, there is a major vertex u with $3 \leq \deg_M(u) \leq 5$ such that on \mathcal{C}_u there are at most two major vertices y and z of unknown degree and at most three major vertices with known degree bounds. Because of our above construction there is, on $\mathcal{C}_u - \{y, z\}$, at most one edge or one pair of adjacent edges, incident neither with y nor z that is not present in H. These edges together with the vertices y and z divide the cycle \mathcal{C}_{μ} into at most three subpaths consisting of at least k-2 vertices with at least k - 4 minor ones among them. All these subpaths are clearly present in H. Two longest ones of them joined through the vertex u form in H a path Q on at least $\frac{2}{3}(k-2) + 1 = \frac{2k-1}{3}$ vertices. On the path Q there are the vertex u, at most two vertices from the set $\{v, w, x\}$ and all remaining vertices are minor. Hence, by Theorem 1, for the weight of Q we have $w(Q) \leq (\lceil \frac{2k-1}{3} \rceil - 3)k + 5k + 7k + 8k = \lceil \frac{2k-1}{3} \rceil k + 17k.$

Hence we have

Theorem 8. Every graph $G \in \mathcal{P}(3,3)$ contains a $\lceil \frac{2k-1}{3} \rceil$ -path Q of the weight $w(Q) \leq \lceil \frac{2k-1}{3} \rceil k + 17k$ with at most four major vertices.

From this theorem we immediately have our Theorem 2(ii).

5. LIGHT PATHS WITH LINEAR WEIGHTS

In this section we are going to prove Theorems 3, 4 and 6. We start with

Proof of Theorem 4. Let C be a longest cycle of G. For the number e of edges of G,

$$2e = \sum_{x \in V(G)} \deg_G(x) = \sum_{x \in V(\mathcal{C})} \deg_G(x) + \sum_{x \in V(G) \setminus V(\mathcal{C})} \deg_G(x).$$

Thus,

$$\sum_{x \in V(\mathcal{C})} \deg_G(x) = 2e - \sum_{x \in V(G) \setminus V(\mathcal{C})} \deg_G(x) \le 2e - \delta \cdot |V(G) \setminus V(\mathcal{C})| = 2e - \delta(n-c).$$

For $x \in V(\mathcal{C})$, let P(x) denote the path on \mathcal{C} starting in x and following a fixed orientation of \mathcal{C} such that |V(P(x))| = k (because $k \leq c$ this is possible). Then every vertex of \mathcal{C} is contained in exactly k of these c paths. Hence,

$$\sum_{x \in V(\mathcal{C})} \Big(\sum_{y \in V(P(x))} \deg_G(y) \Big) = k \sum_{x \in V(\mathcal{C})} \deg_G(x) \le (2e + \delta(c - n))k \,.$$

Among these c paths there is one, say P, with

$$\sum_{x \in V(P)} \deg_G(x) \le \left(\frac{2e}{c} + \delta\left(1 - \frac{n}{c}\right)\right) k \,,$$

and Theorem is proved.

Proof of Theorem 3. Theorem 3 is a simple consequence of Theorem 4. For a connected plane graph G with e edges, f faces and minimum face degree ρ we have

$$2e = \sum_{\alpha \in F(G)} \deg_G(\alpha) \ge \rho f \,.$$

This immediately yields $f \leq \frac{2e}{\rho}$. Using this inequality and Euler's polyhedral formula we obtain $e \leq \frac{\rho(n-2)}{\rho-2}$. Applying this fact together with the

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inequality $c \geq \sigma n$ on parameters c and n in Theorem 4 we obtain

$$w_G(P) \le \left(\frac{2e}{c} + \delta\left(1 - \frac{n}{c}\right)\right)k \le \left(\frac{2\rho(n-2)}{c(\rho-2)} + \delta - \frac{\delta n}{c}\right)k$$
$$= \left(\left(\frac{2\rho}{\rho-2} - \delta\right)\frac{n}{c} + \delta - \frac{4\rho}{c(\rho-2)}\right)k < \left(\left(\frac{2\rho}{\rho-2} - \delta\right)\frac{1}{\sigma} + \delta\right)k$$

which is the statement of Theorem 3. Note that $(\frac{2\rho}{\rho-2} - \delta)$ is positive for all five admissible pairs (δ, ρ) .

Proof of Theorem 6. Theorem 4 and next theorem give the upper bound in Theorem 6.

Theorem 9. For every 3-connected essentially 4-connected plane graph G on n vertices there is

 $n \leq 2c-4$

where c = c(G) is the length of a longest cycle of G.

Proof. Consider G to be embedded into the plane π . For a cycle \mathcal{C} of G, the bounded and the unbounded region of $\pi \setminus \mathcal{C}$ are denoted by $\operatorname{int}(\mathcal{C})$ and $\operatorname{out}(\mathcal{C})$, respectively. A cycle \mathcal{C} of G is called to be int -feasible if, for every $x \in V(G) \cap \operatorname{int}(\mathcal{C})$, $\operatorname{deg}(x) = 3$, $\operatorname{N}(x) \subseteq \operatorname{V}(\mathcal{C})$, and any two $y, z \in \operatorname{N}(x)$ are not adjacent on \mathcal{C} . A cycle to be out -feasible is defined similarly. Recall that N(x) denotes the neighbourhood of x.

Lemma 3. Given an int-feasible cycle C of G on at least 4 vertices, $|V(G) \cap int(C)| \leq \frac{|V(C)|}{2} - 2$.

Proof. By induction on $c = |V(\mathcal{C})|$. If c = 4 then \mathcal{C} is int-feasible only if $|V(G) \cap \operatorname{int}(\mathcal{C})| = 0$.

Let c > 4, $d = |V(G) \cap \operatorname{int}(\mathcal{C})| > 0$, and ϕ be an orientation of \mathcal{C} . Consider a fixed $x \in V(G) \cap \operatorname{int}(\mathcal{C})$ and let x_1, x_2, x_3 be the neighbours of x on \mathcal{C} met in this sequence following ϕ . For i = 1, 2, 3, let \mathcal{C}_i be the cycle obtained by the union of the path on \mathcal{C} from x_i to x_{i+1} following ϕ and the two edges xx_i and $xx_{i+1}(x_4 = x_1), c_i = |V(\mathcal{C}_i)|$, and $d_i = |V(G) \cap \operatorname{int}(\mathcal{C}_i)|$. Obviously, \mathcal{C}_i is int-feasible and $c_i \geq 4$, i = 1, 2, 3. We have $c_1 + c_2 + c_3 = c + 6$, $d_1 + d_2 + d_3 = d - 1$, and, by induction hypothesis, $d_i \leq \frac{c_i}{2} - 2(i = 1, 2, 3)$. This implies $d \leq \frac{c}{2} - 2$.

A consequence of a result of Tutte [20] is the following

Lemma 4. G contains a cycle \mathcal{T} (a so-called Tutte-cycle) such that deg(x) = 3 and $N(x) \subseteq V(\mathcal{T})$ for every $x \in V(G) \setminus V(\mathcal{T})$.

If a Tutte-cycle \mathcal{T} of G is not int-feasible (assume the edge yz belongs to \mathcal{T} for certain $y, z \in N(x), x \in V(G) \cap \operatorname{int}(\mathcal{T})$) then the cycle obtained from \mathcal{T} by removing the edge yz and adding the edges xy and yz is also a (longer!) Tutte-cycle of G. Hence, we may assume that there is a Tutte-cycle \mathcal{C} being both int-feasible and out-feasible. Since $c = |V(\mathcal{C})| \geq 6$ if $s = |V(G) \setminus V(\mathcal{C})| > 0$ and $n = c \geq 4$ if s = 0, we may apply Lemma 3. Because \mathcal{C} is also out-feasible, by symmetry, we have $|V(G) \cap \operatorname{out}(\mathcal{C})| \leq \frac{|V(\mathcal{C})|}{2} - 2$. Hence, $s \leq c - 4$, $n \leq 2c - 4$.

To prove the second part of Theorem 6, take a 4-connected plane triangulation H containing only 5- and 6-vertices such that the distance between arbitrary two 5-vertices is at least k. Such triangulations are well known, see e.g. [3]. They are the duals to the famous fullerene graphs. Let K(H) be the graph obtained from H by inserting a new vertex into each face α and joining it to all vertices incident with α . Then K(H) contains 12 vertices of degree 10, |V(H)| - 12 vertices of degree 12, the remaining vertices of K(H)are independent 3-vertices, and every 3-separator forms the neighbourhood of a 3-vertex. Since every k-path P contains at most one vertex of degree 10, the proof is complete.

6. Remarks

6.1. Comparison of the upper bounds

Theorem 4 provides the upper bound $(\frac{3n}{c} + 3 - \frac{12}{c})k$ on $w_G(P)$ for a k-path P of graphs G from the family $\mathcal{P}(3,3)$ if G has circumference c = c(G) and $k \leq c$. Here n = |V(G)|.

Theorem 2, on the other side, gives for $G \in \mathcal{P}(3, \overline{3})$, with G containing a k-path, the upper bound k(k+13) on $w_G(P)$.

It is easy to see that for plane triangulations the first mentioned upper bound is better than the second one if and only if

$$\frac{3n-12}{c} - 10 \le k \le c \text{ and } c \ge -5 + \sqrt{3n+13}.$$

Note that the best known lower bound for circumference of a 3-connected planar graph G is $\Omega(n^{\log_3 2})$ recently proved by Chen and Yu [2].

7. MATCHINGS AND KOTZIG'S TYPE THEOREMS

The idea of the proof of Theorem 4 can be used e.g. in proving the following Kotzig's type theorem.

Theorem 10. For a graph G let n, e, δ and m be the number of its vertices, edges, the minimum degree, and the edge independence number, respectively. Let M be a matching of G of the cardinality m. Then G contains an edge $h \in M$ of weight

$$w_G(h) \le \frac{2e - \delta(n - 2m)}{m}$$
.

Proof. Let $M = \{h_1, h_2, \dots, h_m\}$ be a maximum matching of G. Put $V_1 = V(M)$ and $V_2 = V \setminus V_1$. Then

$$\sum_{h \in M} w_G(h) = \sum_{u \in V_1} \deg_G(u) = 2e - \sum_{v \in V_2} \deg_G(v) \le 2e - \delta(n - 2m),$$

which immediately yields the required inequality.

For a graph G having perfect matching there is $m = \frac{n}{2}$, so we get

Corollary 11. If a graph G has a perfect matching M then G contains an edge $h \in M$ of weight

$$w_G(h) \le \frac{4e}{n}$$
.

The number e edges of planar graph G is bounded by $e \leq 3n - 6$ or $e \leq 2n - 4$ in general and in the absence of 3-faces, respectively. Using these inequalities in Corollary 11 we obtain

Corollary 12. If a planar graph G has a perfect matching M then it contains an edge h of weight

$$w_G(h) \leq 11$$
 and $w_G(h) \leq 7$

in general and in the absence of 3-faces, respectively.

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References

- K. Ando, S. Iwasaki and A. Kaneko, Every 3-connected planar graph has a connected subgraph with small degree sum, Annual Meeting of Mathematical Society of Japan, 1993, Japanese.
- [2] G. Chen and X. Yu, Long cycles in 3-connected graphs, J. Combin. Theory (B) 86 (2002) 80–99.
- [3] E. Etourneau, Existence and connectivity of planar having 12 vertices of degree 5 and, n - 12 vertices of degree 6, Colloq. Math. Soc. János Bolyai 10 (1975) 645-655.
- [4] H. Enomoto and K. Ota, Connected subgraphs with small degree sum in 3connected planar graphs, J. Graph Theory 30 (1999) 191–203.
- [5] I. Fabrici and S. Jendrol', Subgraphs with restricted degrees of their vertices in planar 3-connected graphs, Graphs Combin. 13 (1997) 245-250.
- [6] I. Fabrici and S. Jendrol', Subgraphs with restricted degrees of their vertices in planar graphs. Discrete Math. 191 (1998) 83–90.
- [7] P. Franklin, The four color problem, Amer. J. Math. 44 (1922) 225–236.
- [8] B. Grünbaum, New views on some old questions of combinatorial geometry, Int. Teorie Combinatorie, Rome 1 (1976) 451–468.
- [9] J. Harant and S. Jendrol', On the existence of specific stars in planar graph, Graphs and Combinatorics 23 (2007) 529–543.
- [10] J. Harant, S. Jendrol' and M. Tkáč, On 3-connected plane graphs without triangular faces, J. Combin. Theory (B) 77 (1999) 150–161.
- [11] J. van den Heuvel and S. McGuinness, Coloring the square of a planar graph, J. Graph Theory 42 (2003) 110–124.
- [12] S. Jendrol', Paths with restricted degrees of their vertices in planar graphs, Czechoslovak Math. J. 49 (1999) 481–490.
- [13] S. Jendrol', T. Madaras, R. Soták and Z. Tuza, On light cycles in plane triangulations, Discrete Math. 197/198 (1999) 453–467.

- [14] S. Jendrol' and H.-J. Voss, Light subgraphs of graphs embedded in the plane and in the projective plane — a survey, P.J. Šafárik University Košice, IM Preprint series (A) No. 1 (2004).
- [15] A. Kotzig, Contribution to the theory of Eulerian polyhedra, Mat. Čas. SAV (Math. Slovaca) 5 (1955) 101–113.
- [16] A. Kotzig, Extremal polyhedral graphs, Ann. New York Acad. Sci. 319 (1979) 565–570.
- [17] H. Lebesgue, Quelques conséquences simples de la formule d'Euler, J. Math. Pures Appl. 19 (1940) 27–43.
- [18] T. Madaras, Note on weights of paths in polyhedral graphs, Discrete Math. 203 (1999) 267–269.
- B. Mohar, Light paths in 4-connected graphs in the plane and other surfaces, J. Graph Theory 34 (2000) 170–179.
- [20] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99–116.
- [21] P. Wernicke, Über den kartographischen Vierfarbensatz, Math. Ann. 58 (1904) 413–426.

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