

PATHS OF LOW WEIGHT IN PLANAR GRAPHS

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Abstract

The existence of paths of low degree sum of their vertices in planar graphs is investigated. The main results of the paper are:

1. Every 3-connected simple planar graph G that contains a k -path, a path on k vertices, also contains a k -path P such that for its weight (the sum of degrees of its vertices) in G it holds

$$w_G(P) := \sum_{u \in V(P)} \deg_G(u) \leq \frac{3}{2}k^2 + \mathcal{O}(k).$$

2. Every plane triangulation T that contains a k -path also contains a k -path P such that for its weight in T it holds

$$w_T(P) := \sum_{u \in V(P)} \deg_T(u) \leq k^2 + 13k.$$

3. Let G be a 3-connected simple planar graph of circumference $c(G)$. If $c(G) \geq \sigma|V(G)|$ for some constant $\sigma > 0$ then for any k , $1 \leq k \leq c(G)$, G contains a k -path P such that

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) \leq \left(\frac{3}{\sigma} + 3\right)k.$$

Keywords: planar graphs, polytopal graphs, paths, weight of an edge, weight of a path.

2000 Mathematics Subject Classification: 05C10, 05C38, 52B10.

1. NOTATION

We will adapt the convention that a graph is *planar* if it can be embedded in the plane (without edges crossing), and *plane* if it is already embedded in the plane. This paper will be concerned with simple plane graphs. The sets of vertices, edges and faces of such a graph G will be denoted by $V(G)$, $E(G)$ and $F(G)$, respectively, or by V , E and F if G is known from the context.

The *facial walk* of a face α of a connected plane graph G is the shortest closed walk induced by all edges incident with α . The *degree* of a face α is the length of its facial walk and is denoted by $\deg_G(\alpha)$ or $\deg(\alpha)$ if G is known from the context. The degree of a vertex x of a graph is the number of edges incident with x . Analogously the notation $\deg_G(x)$ or $\deg(x)$ is used for the degree of a vertex x . Let a *k-vertex* be a vertex of degree k . Let a *k-face* be defined similarly.

Let an (a, b) -*edge* be an edge f if an a -vertex and a b -vertex are endvertices of f .

A 3-face α is said to be the (a, b, c) -*triangle* or a *triangle of type* (a, b, c) if vertices incident with α have degrees a , b and c .

Let a *k-path* and a *k-cycle* be a path and a cycle on k vertices, respectively. Let a *k-path* be an $(a_1, a_2, \dots, a_k)$ -*path* if it passes through the vertices $u_1, u_2, \dots, u_k$ in order with $a_i = \deg(u_i)$ for all $i = 1, 2, \dots, k$.

Let an $(x; a, b, c)$ -*star* be a star $K_{1,3}$ which is a subgraph of a graph G with a central x -vertex and an a -vertex, a b -vertex and a c -vertex as leaves.

Let $\mathcal{P}_\kappa(\delta, \rho)$ be the family of all κ -connected simple plane graphs with minimum vertex degree at least δ , $\delta \geq 3$, and minimum face degree at least ρ . It is easy to see that $\mathcal{P}_\kappa(\delta, \rho)$ is not empty only for $(\delta, \rho) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$. Let $\mathcal{P}(\delta, \rho) := \mathcal{P}_3(\delta, \rho)$ and $\mathcal{P}(\delta, \bar{3}) := \{G \in \mathcal{P}(\delta, 3) : G \text{ is a triangulation}\}$.

For a subgraph H of a planar graph G , the *weight* $w_G(H)$ of H is defined to be the sum of degrees of vertices of H in G ; namely

$$w_G(H) = \sum_{u \in V(H)} \deg_G(u).$$

2. INTRODUCTION AND RESULTS

It is well known that every planar graph contains a vertex of degree at most 5. In 1955 Kotzig [15, 16] proved that every 3-connected planar graph

contains an edge of the weight at most 13 in general and most 11 in the absence of 3-vertices, respectively. These bounds are best possible.

In 1940 Lebesgue [17] proved that every graph G of minimum degree 5 contains a star $K_{1,3}$ such that its central vertex has, in G , degree at most 5 and every its leaf has degree at most 8. Recently van den Heuvel and McGuinness [11] proved that every planar graph of minimum degree at least 3 contains a $(3, a)$ -edge for a $3 \leq a \leq 11$, or an $(a, 4, b)$ -path for $3 \leq a \leq 7$ and $a \leq b \leq 11$, or a $(5; a, b, c)$ -star for $3 \leq a \leq 6$, $a \leq b \leq 7$ and $b \leq c \leq 11$.

The following recent result, that strengthens all above mentioned ones, will be applied later in this paper.

Theorem 1 [9]. *Every planar graph G of minimum degree $\delta(G) \geq 3$ contains*

- (i) *a $(3, a)$ -edge for $3 \leq a \leq 10$, or*
- (ii) *an $(a, 4, b)$ -path for $a = 4$ and $4 \leq b \leq 10$,
or $a = 5$ and $5 \leq b \leq 9$,
or $6 \leq a \leq 7$ and $6 \leq b \leq 8$, or*
- (iii) *a $(5; a, b, c)$ -star for $4 \leq a \leq 5$, $5 \leq b \leq 6$ and $5 \leq c \leq 7$,
or $a = b = c = 6$.*

Moreover, for every $\mathcal{S} \in \{(3, 10)$ -edge, $(4, 4, 9)$ -path, $(5, 4, 8)$ -path, $(6, 4, 8)$ -path, $(7, 4, 7)$ -path, $(5; 5, 6, 7)$ -star, $(5; 6, 6, 6)$ -star $\}$ there is a 3-connected plane graph H containing $\mathcal{S}$ and no other subgraph from the above list.

In generalizing Kotzig's theorem there are several other natural directions. Two possibilities are as follows.

Let $k \geq 1$ be an integer.

- (A) Find the smallest integer $f = f(k, \delta, \rho)$ such that whenever a graph $G \in \mathcal{P}_3(\delta, \rho)$ contains at least k vertices, there is a connected subgraph H of G of order k whose weight

$$w_G(H) = \sum_{u \in V(H)} \deg_G(u) \leq f(k, \delta, \rho).$$

- (B) Find the smallest integer $w = w(k, \delta, \rho)$ such that whenever a graph $G \in \mathcal{P}_3(\delta, \rho)$ contains a k -path there is a k -path P_k in G with weight

$$w_G(P_k) = \sum_{u \in V(P_k)} \deg_G(u) \leq w(k, \delta, \rho).$$

The possibility (A) was investigated by Enomoto and Ota [4]. They proved that for $k \geq 4$

$$8k - 5 \leq f(k, 3, 3) \leq 8k - 1$$

and conjectured the precise value of $f(k, 3, 3)$ to be $8k - 5$.

The problem (B) was formulated in [5]. The precise values of $w(k, \delta, 3)$ are known only for small k , e.g. $w(1, 3, 3) = 5$, $w(2, 3, 3) = f(2, 3, 3) = 13$, $w(2, 4, 3) = f(2, 4, 3) = 11$ (Kotzig [15]), $w(2, 5, 3) = f(2, 5, 3) = 11$ (Wernicke [21]), $w(3, 5, 3) = f(3, 5, 3) = 17$ (Franklin [7]), and $w(3, 3, 3) = f(3, 3, 3) = 21$ (Ando, Iwasaki and Kaneko [1]). For greater k only estimations are known, see e.g. surveys in [10, 13, 14]. In [5, 6] it was proved that

$$k \log_2 k \leq w(k, 3, 3) \leq 5k^2.$$

Madaras [18] improved the upper bound showing that $w(k, 3, 3) \leq \frac{5}{2}k(k+1)$.

Applying Theorem 1 we are able to prove the first main result of this paper

Theorem 2. *Let k be an integer, $k \geq 4$. Then*

- (i) *every plane triangulation T , that contains a k -path, also contains a k -path P such that $w_T(P) \leq k^2 + 13k$, and*
- (ii) *every 3-connected planar graph, that contains a k -path, also contains a k -path P such that $w_G(P) \leq w(k, 3, 3) \leq \frac{3}{2}k^2 + \mathcal{O}(k)$.*

Note. As shown in [12], no analogue of Theorem 2 can be proved for the family $\mathcal{P}_2(3, 3)$. More precisely: For every pair of integers m, k , $m \geq k \geq 3$, there is a 2-connected planar graph G in which every k -path P_k has weight at least m , that is $w_G(P_k) \geq m$.

For $k = 2$ the situation is different. In 1972 Erdős conjectured that Kotzig's theorem holds for all planar graphs G with minimum degree $\delta(G) \geq 3$. This conjecture was proved by Barnette, see [8].

The restriction to 4-connected planar graphs brings a different behaviour. In 2000 Mohar [19] proved that every 4-connected planar graph of order at least k contains a k -path P_k of weight

$$w_G(P_k) \leq 6k - 1;$$

the bound being tight. The difference is that every 4-connected plane graph contains a k -path whose weight is bounded from above by a function linear

in k while on the other side there are 3-connected plane graphs in which all k -paths have weight bounded from below by a function which is not linear in k .

Developing the ideas of Mohar's proof [19] we show that a linear in k upper bound is also true for a wider family of plane graphs. Namely, the second main result of this paper is

Theorem 3. *For $G \in \mathcal{P}(\delta, \rho)$ let $c(G)$ be the length of a longest cycle of G . Let k be an integer, $3 \leq k \leq c(G)$. If $c(G) \geq \sigma|V(G)|$ for some positive number σ then G contains a k -path P_k such that*

$$w_G(P_k) < \left(\left(\frac{2\rho}{\rho-2} - \delta \right) \frac{1}{\sigma} + \delta \right) k.$$

In fact Theorem 3 is a corollary of the following more general result

Theorem 4. *Let G be a graph with $n = |V(G)|$ vertices, $e = |E(G)|$ edges, the length of a longest cycle $c = c(G)$, and minimum vertex degree $\delta = \delta(G)$. Let k be a positive integer $k \leq c$. Then G contains a k -path P with*

$$w_G(P) = \sum_{u \in V(P)} \deg_G(u) \leq \left(\frac{2e}{c} + \delta \left(1 - \frac{n}{c} \right) \right) k.$$

Immediately we have

Corollary 5. *Let G be a hamiltonian graph on n vertices, and let k be a positive integer, $k \leq n$. Then G contains a k -path P such that*

$$w_G(P) \leq \frac{2e}{n} k.$$

Every 4-connected planar graph G is known to be hamiltonian [20]. Hence $c(G) = |V(G)|$ and, because in this case $e \leq 3n - 6$, we immediately obtain the above mentioned elegant Mohar's theorem [19].

Let S be a set of three vertices of a 3-connected planar graph G such that the graph $G - S$ obtained from G by removing S is disconnected (S is called a *3-separator* in this case). It is known that $G - S$ consists of exactly two components $\mathcal{A}$ and $\mathcal{B}$. G is called to be *essentially 4-connected* if it is 3-connected and $|V(\mathcal{A})| = 1$ or $|V(\mathcal{B})| = 1$ for every 3-separator S of G .

Theorem 6. *Let G be an essentially 4-connected planar graph, and let k be an integer $1 \leq k \leq \frac{|V(G)|}{2}$. Then G contains a k -path P such that*

$$w_G(P) \leq 9k - 1.$$

Moreover, there exists an essentially 4-connected planar graph H in which every k -path P has weight

$$w_H(P) \geq \frac{15}{2}k - \frac{13}{2}.$$

The rest of the paper is organized as follows. In Sections 3 and 4 we prove Theorem 2. Theorems 3, 4 and 6 are proved in Section 5. In Section 6 we add some remarks concerning the results and open problems.

3. PROOF OF THEOREM 2(i)

First we prove the following theorem

Theorem 7. *For a given positive integer $k \geq 4$, let G be a 3-connected plane graph in which every r -face, $r \geq 4$, contains at most two vertices of degree greater than k , and if it contains exactly two, then they are adjacent. Then G contains a k -path P of the weight*

$$w(P) \leq k^2 + 13k$$

that has at most four vertices of degree greater than k .

Proof. We give a constructive proof of Theorem 7. For convenience a vertex x of G is called *major* if $\deg_G(x) > k$, and is called *minor* otherwise.

For a vertex x of G let $\mathcal{C}_x$ be a cycle induced in G by all edges of all faces incident with the vertex x but not having x as an endvertex (i.e., edges not incident with x). Clearly all neighbours of x are in $V(\mathcal{C}_x)$ and the length of $\mathcal{C}_x$ is at least $\deg_G(x)$.

Let $M = M(G)$ be a subgraph of G induced on major vertices of G . The graph G and the subgraph M have the properties mentioned in the lemmas below:

Lemma 1. *Let G contain a major vertex x with $\deg_M(x) = d$ and let $\deg_G(x) \geq kd + 1$. Then G contains a k -path P of which all vertices are minor and therefore $w(P) \leq k^2$.*

Proof. Since $\deg_G(x) \geq kd + 1$ then for the cycle $\mathcal{C}_x$ we have $|\mathcal{C}_x| > kd$, but on $\mathcal{C}_x$ there are exactly d major vertices. Hence at least one path between two consecutive major vertices of $\mathcal{C}_x$ contains at least k minor vertices. So we have a k -path P with $w(P) \leq k^2$. ■

Lemma 2. *If G contains a major vertex x with $\deg_M(x) \leq 2$, then G contains a k -path P all vertices, except possibly x , are minor and $w(P) \leq k^2 + k$.*

Proof. In the case $\deg_M(x) \leq 1$ the proof is clear. Let x be a 2-vertex in M then $\deg_G(x) \geq k + 1$. If $\deg_G(x) \geq 2k + 1$, then, by Lemma 1, G contains a required path. Let $\deg_G(x) \leq 2k$. On $\mathcal{C}_x$ there are two major vertices, say y and z , which divide $\mathcal{C}_x$ into two subpaths both consisting of minor vertices that contain two subpaths P_a and P_b with $a + b \geq k - 1$ starting in minor vertices u and v , respectively, which are neighbours of x in G . These two paths together with the edges ux and xv form an l -path, $l \geq k$. This path contains a k -path P as a subgraph all vertices of which, except possibly x , are minor. So for P we have

$$w_G(P) \leq k(k - 1) + 2k = k^2 + k. \quad \blacksquare$$

Due to Lemma 1 and Lemma 2 we may suppose that $\deg_M(x) = d \geq 3$ and $\deg_G(x) \leq kd$ for every major vertex x of G . Then we apply to M our Theorem 1. By it there is a major vertex u with $\deg_M(u) = a \leq 5$. Due to hypothesis of Theorem 7 all other vertices of $\mathcal{C}_u$ are minor. Among these major neighbours of u there are at most two vertices, say y and z , whose degrees are not known, and

- (i) if $\deg_M(u) = 3$ then there is a major neighbour v with $\deg_M(v) = b \leq 10$, or
- (ii) if $\deg_M(u) = 4$ then there are two more major neighbours of u , say v and w , such that $\deg_M(v) = b$ and $\deg_M(w) = c$ with $4 \leq b \leq 5$ and $4 \leq c \leq 10$ or $6 \leq b \leq 7$ and $6 \leq c \leq 8$, respectively, or
- (iii) if $\deg_M(u) = 5$ then there are three more major neighbours v , w and x of degrees b, c , and d , respectively, where $b \leq 5, c \leq 6, d \leq 7$ or $b = c = d = 6$.

We may suppose that (1) and (2) below hold:

- (1) *For the vertex $t \in \{v, w, x\}$ with $\deg_M(t) = r \deg_G(t) \leq (r - 1)k$.*

Proof of (1). Suppose $\deg_G(t) \geq (r - 1)k + 1$. Consider the cycle $\mathcal{C}_t$. Except of the vertex u of degree at most $5k$ there are other $r - 1$ major vertices on $\mathcal{C}_t$. These major vertices split $\mathcal{C}_t$ into $r - 1$ subpaths. At least one of these subpaths contains a k -path P on $k - 1$ minor vertices and the k -th (possibly exceptional) vertex of this path is either a minor vertex or the vertex u which has degree at most $5k$. This means that the k -path P has at most one major vertex and we have $w_G(P) \leq k(k - 1) + 5k = k^2 + 4k$. ■

- (2) *For the vertex u $\deg_G(u) \leq 2k$ holds.*

Proof of (2). Suppose the contrary. Let the cycle $\mathcal{C}_u$ have at least $2k + 1$ vertices. There are at most five major vertices among them (all are neighbours of u), namely the vertices y and z , and at most three from the set $\{v, w, x\}$ as described above. The vertices y and z divide the cycle $\mathcal{C}_u$ into two subpaths P_p and P_q with $p + q \geq 2k - 1$. Hence $\max\{p, q\} \geq k$ and on $\mathcal{C}_u$ there is a k -path P all vertices of which except of at most three (from the set $\{v, w, x\}$) are minor. This path has in the case (i) the weight $w_G(P) \leq k(k - 1) + (b - 1)k \leq k^2 - k + 10k - k = k^2 + 8k$ because P contains at most one major neighbour of degree $\leq 9k$. In the case (ii) (the case (iii)) the path P has weight

$$w_G(P) \leq k(k - 2) + (b - 1)k + (c - 1)k \leq k^2 + 11k$$

$$(w_G(P) \leq k(k - 3) + (b - 1)k + (c - 1)k + (d - 1)k \leq k^2 + 12k)$$

and contains at most two major vertices (at most three major vertices, respectively). ■

Because u is a major vertex, $|\mathcal{C}_u| \geq k + 1$ and the vertices y and z divide the cycle $\mathcal{C}_u$ into two subpaths that contain two subpaths P_p and P_q with $p + q \geq k - 1$ starting in vertices u^* and v^* which are neighbours of u in G . These two subpaths together with the vertex u and edges u^*u and uv^* form an l -path, $l \geq k$, which contains as a subgraph a k -path P passing through at least $(k - 4)$ minor vertices and at most four major vertices all from the set $\{u, v, w, x\}$.

Applying Properties (1) and (2), and distinguishing three cases (i), (ii) and (iii) according to the degree of u in M we obtain, similarly as in the

proof of (2), the following upper bound on $w_G(P)$, the weight of P ,

$$w_G(P) \leq k^2 + 13k.$$

From Theorem 7 we immediately have the upper bound for Theorem 2(i). ■

4. PROOF OF THEOREM 2(ii)

To prove Theorem 2(ii) consider first the following construction and then modify the proof of Theorem 2(i). Suppose $H \in \mathcal{P}(3, 3)$.

Let $G_0 = H, G_1, \dots, G_p = G$ be a sequence of plane graphs defined as follows: If G_i , $i = 0, 1, \dots, p-1$, is a 3-connected plane graph having an r -face α , $r \geq 4$, incident with two non-adjacent major vertices u and v we insert a diagonal $d = uv$ into α joining the vertices u and v . The result is a 3-connected plane graph $G_{i+1} = G_i + d$. If G_i does not contain any face α having the above-mentioned property we put $i = p$ and $G = G_p$.

It is easy to see that the graph $G_p = G$ satisfies the hypothesis of Theorem 7. If G contains a major vertex x of degree $\deg_M(x) \leq 2$ then, analogously as in the proof of Lemma 2, one can prove that G contains a k -path P of the weight $w(P) \leq k^2 + k$ with at most one major vertex. Clearly this path is also present in the graph H . If for each major vertex x in G $\deg_M(x) \geq 3$ then, by Theorem 1, there is a major vertex u with $3 \leq \deg_M(u) \leq 5$ such that on $\mathcal{C}_u$ there are at most two major vertices y and z of unknown degree and at most three major vertices with known degree bounds. Because of our above construction there is, on $\mathcal{C}_u - \{y, z\}$, at most one edge or one pair of adjacent edges, incident neither with y nor z that is not present in H . These edges together with the vertices y and z divide the cycle $\mathcal{C}_u$ into at most three subpaths consisting of at least $k-2$ vertices with at least $k-4$ minor ones among them. All these subpaths are clearly present in H . Two longest ones of them joined through the vertex u form in H a path Q on at least $\frac{2}{3}(k-2) + 1 = \frac{2k-1}{3}$ vertices. On the path Q there are the vertex u , at most two vertices from the set $\{v, w, x\}$ and all remaining vertices are minor. Hence, by Theorem 1, for the weight of Q we have $w(Q) \leq (\lceil \frac{2k-1}{3} \rceil - 3)k + 5k + 7k + 8k = \lceil \frac{2k-1}{3} \rceil k + 17k$.

Hence we have

Theorem 8. *Every graph $G \in \mathcal{P}(3, 3)$ contains a $\lceil \frac{2k-1}{3} \rceil$ -path Q of the weight $w(Q) \leq \lceil \frac{2k-1}{3} \rceil k + 17k$ with at most four major vertices.*

From this theorem we immediately have our Theorem 2(ii). ■

5. LIGHT PATHS WITH LINEAR WEIGHTS

In this section we are going to prove Theorems 3, 4 and 6. We start with

Proof of Theorem 4. Let $\mathcal{C}$ be a longest cycle of G . For the number e of edges of G ,

$$2e = \sum_{x \in V(G)} \deg_G(x) = \sum_{x \in V(\mathcal{C})} \deg_G(x) + \sum_{x \in V(G) \setminus V(\mathcal{C})} \deg_G(x).$$

Thus,

$$\sum_{x \in V(\mathcal{C})} \deg_G(x) = 2e - \sum_{x \in V(G) \setminus V(\mathcal{C})} \deg_G(x) \leq 2e - \delta \cdot |V(G) \setminus V(\mathcal{C})| = 2e - \delta(n - c).$$

For $x \in V(\mathcal{C})$, let $P(x)$ denote the path on $\mathcal{C}$ starting in x and following a fixed orientation of $\mathcal{C}$ such that $|V(P(x))| = k$ (because $k \leq c$ this is possible). Then every vertex of $\mathcal{C}$ is contained in exactly k of these c paths. Hence,

$$\sum_{x \in V(\mathcal{C})} \left(\sum_{y \in V(P(x))} \deg_G(y) \right) = k \sum_{x \in V(\mathcal{C})} \deg_G(x) \leq (2e + \delta(c - n))k.$$

Among these c paths there is one, say P , with

$$\sum_{x \in V(P)} \deg_G(x) \leq \left(\frac{2e}{c} + \delta \left(1 - \frac{n}{c} \right) \right) k,$$

and Theorem is proved. ■

Proof of Theorem 3. Theorem 3 is a simple consequence of Theorem 4. For a connected plane graph G with e edges, f faces and minimum face degree ρ we have

$$2e = \sum_{\alpha \in F(G)} \deg_G(\alpha) \geq \rho f.$$

This immediately yields $f \leq \frac{2e}{\rho}$. Using this inequality and Euler's polyhedral formula we obtain $e \leq \frac{\rho(n-2)}{\rho-2}$. Applying this fact together with the

inequality $c \geq \sigma n$ on parameters c and n in Theorem 4 we obtain

$$\begin{aligned} w_G(P) &\leq \left(\frac{2e}{c} + \delta \left(1 - \frac{n}{c} \right) \right) k \leq \left(\frac{2\rho(n-2)}{c(\rho-2)} + \delta - \frac{\delta n}{c} \right) k \\ &= \left(\left(\frac{2\rho}{\rho-2} - \delta \right) \frac{n}{c} + \delta - \frac{4\rho}{c(\rho-2)} \right) k < \left(\left(\frac{2\rho}{\rho-2} - \delta \right) \frac{1}{\sigma} + \delta \right) k \end{aligned}$$

which is the statement of Theorem 3. Note that $(\frac{2\rho}{\rho-2} - \delta)$ is positive for all five admissible pairs (δ, ρ) . $\blacksquare$

Proof of Theorem 6. Theorem 4 and next theorem give the upper bound in Theorem 6.

Theorem 9. *For every 3-connected essentially 4-connected plane graph G on n vertices there is*

$$n \leq 2c - 4$$

where $c = c(G)$ is the length of a longest cycle of G .

Proof. Consider G to be embedded into the plane π . For a cycle $\mathcal{C}$ of G , the bounded and the unbounded region of $\pi \setminus \mathcal{C}$ are denoted by $\text{int}(\mathcal{C})$ and $\text{out}(\mathcal{C})$, respectively. A cycle $\mathcal{C}$ of G is called to be *int-feasible* if, for every $x \in V(G) \cap \text{int}(\mathcal{C})$, $\deg(x) = 3$, $N(x) \subseteq V(\mathcal{C})$, and any two $y, z \in N(x)$ are not adjacent on $\mathcal{C}$. A cycle to be *out-feasible* is defined similarly. Recall that $N(x)$ denotes the neighbourhood of x .

Lemma 3. *Given an int-feasible cycle $\mathcal{C}$ of G on at least 4 vertices, $|V(G) \cap \text{int}(\mathcal{C})| \leq \frac{|V(\mathcal{C})|}{2} - 2$.*

Proof. By induction on $c = |V(\mathcal{C})|$. If $c = 4$ then $\mathcal{C}$ is int-feasible only if $|V(G) \cap \text{int}(\mathcal{C})| = 0$.

Let $c > 4$, $d = |V(G) \cap \text{int}(\mathcal{C})| > 0$, and ϕ be an orientation of $\mathcal{C}$. Consider a fixed $x \in V(G) \cap \text{int}(\mathcal{C})$ and let x_1, x_2, x_3 be the neighbours of x on $\mathcal{C}$ met in this sequence following ϕ . For $i = 1, 2, 3$, let $\mathcal{C}_i$ be the cycle obtained by the union of the path on $\mathcal{C}$ from x_i to x_{i+1} following ϕ and the two edges xx_i and xx_{i+1} ($x_4 = x_1$), $c_i = |V(\mathcal{C}_i)|$, and $d_i = |V(G) \cap \text{int}(\mathcal{C}_i)|$. Obviously, $\mathcal{C}_i$ is int-feasible and $c_i \geq 4$, $i = 1, 2, 3$.

We have $c_1 + c_2 + c_3 = c + 6$, $d_1 + d_2 + d_3 = d - 1$, and, by induction hypothesis, $d_i \leq \frac{c_i}{2} - 2$ ($i = 1, 2, 3$). This implies $d \leq \frac{c}{2} - 2$. ■

A consequence of a result of Tutte [20] is the following

Lemma 4. *G contains a cycle $\mathcal{T}$ (a so-called Tutte-cycle) such that $\deg(x) = 3$ and $N(x) \subseteq V(\mathcal{T})$ for every $x \in V(G) \setminus V(\mathcal{T})$.*

If a Tutte-cycle $\mathcal{T}$ of G is not int-feasible (assume the edge yz belongs to $\mathcal{T}$ for certain $y, z \in N(x), x \in V(G) \cap \text{int}(\mathcal{T})$) then the cycle obtained from $\mathcal{T}$ by removing the edge yz and adding the edges xy and yz is also a (longer!) Tutte-cycle of G . Hence, we may assume that there is a Tutte-cycle $\mathcal{C}$ being both int-feasible and out-feasible. Since $c = |V(\mathcal{C})| \geq 6$ if $s = |V(G) \setminus V(\mathcal{C})| > 0$ and $n = c \geq 4$ if $s = 0$, we may apply Lemma 3. Because $\mathcal{C}$ is also out-feasible, by symmetry, we have $|V(G) \cap \text{out}(\mathcal{C})| \leq \frac{|V(\mathcal{C})|}{2} - 2$. Hence, $s \leq c - 4$, $n \leq 2c - 4$. ■

To prove the second part of Theorem 6, take a 4-connected plane triangulation H containing only 5- and 6-vertices such that the distance between arbitrary two 5-vertices is at least k . Such triangulations are well known, see e.g. [3]. They are the duals to the famous fullerene graphs. Let $K(H)$ be the graph obtained from H by inserting a new vertex into each face α and joining it to all vertices incident with α . Then $K(H)$ contains 12 vertices of degree 10, $|V(H)| - 12$ vertices of degree 12, the remaining vertices of $K(H)$ are independent 3-vertices, and every 3-separator forms the neighbourhood of a 3-vertex. Since every k -path P contains at most one vertex of degree 10, the proof is complete. ■

6. REMARKS

6.1. Comparison of the upper bounds

Theorem 4 provides the upper bound $(\frac{3n}{c} + 3 - \frac{12}{c})k$ on $w_G(P)$ for a k -path P of graphs G from the family $\mathcal{P}(3, 3)$ if G has circumference $c = c(G)$ and $k \leq c$. Here $n = |V(G)|$.

Theorem 2, on the other side, gives for $G \in \mathcal{P}(3, \bar{3})$, with G containing a k -path, the upper bound $k(k + 13)$ on $w_G(P)$.

It is easy to see that for plane triangulations the first mentioned upper bound is better than the second one if and only if

$$\frac{3n - 12}{c} - 10 \leq k \leq c \text{ and } c \geq -5 + \sqrt{3n + 13}.$$

Note that the best known lower bound for circumference of a 3-connected planar graph G is $\Omega(n^{\log_3 2})$ recently proved by Chen and Yu [2].

7. MATCHINGS AND KOTZIG'S TYPE THEOREMS

The idea of the proof of Theorem 4 can be used e.g. in proving the following Kotzig's type theorem.

Theorem 10. *For a graph G let n, e, δ and m be the number of its vertices, edges, the minimum degree, and the edge independence number, respectively. Let M be a matching of G of the cardinality m . Then G contains an edge $h \in M$ of weight*

$$w_G(h) \leq \frac{2e - \delta(n - 2m)}{m}.$$

Proof. Let $M = \{h_1, h_2, \dots, h_m\}$ be a maximum matching of G . Put $V_1 = V(M)$ and $V_2 = V \setminus V_1$. Then

$$\sum_{h \in M} w_G(h) = \sum_{u \in V_1} \deg_G(u) = 2e - \sum_{v \in V_2} \deg_G(v) \leq 2e - \delta(n - 2m),$$

which immediately yields the required inequality. ■

For a graph G having perfect matching there is $m = \frac{n}{2}$, so we get

Corollary 11. *If a graph G has a perfect matching M then G contains an edge $h \in M$ of weight*

$$w_G(h) \leq \frac{4e}{n}.$$

The number e edges of planar graph G is bounded by $e \leq 3n - 6$ or $e \leq 2n - 4$ in general and in the absence of 3-faces, respectively. Using these inequalities in Corollary 11 we obtain

Corollary 12. *If a planar graph G has a perfect matching M then it contains an edge h of weight*

$$w_G(h) \leq 11 \text{ and } w_G(h) \leq 7$$

in general and in the absence of 3-faces, respectively.

Acknowledgement

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104. Support of Slovak VEGA Grant 1/3004/06 is acknowledged as well. The authors wish to thank the referees for their thoughtful suggestions.

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Received 22 November 2006

Revised 4 May 2007

Accepted 4 May 2007