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WEAKLY CONNECTED DOMINATION SUBDIVISION NUMBERS

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Abstract

A set D of vertices in a graph $G = (V, E)$ is a *weakly connected dominating set of* G if D is dominating in G and the subgraph weakly induced by D is connected. The *weakly connected domination number* of G is the minimum cardinality of a weakly connected dominating set of G. The *weakly connected domination subdivision number* of a connected graph G is the minimum number of edges that must be subdivided (where each egde can be subdivided at most once) in order to increase the weakly connected domination number. We study the weakly connected domination subdivision numbers of some families of graphs.

Keywords: weakly connected domination number, weakly connected domination subdivision number.

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1. Introduction

Let $G = (V, E)$ be a simple graph with $|V(G)| = n(G)$. The *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v in G and the *closed neighbourhood* $N_G[v]$ is the set $N_G(v) \cup \{v\}$. The *degree* $d_G(v)$ of v is the number of edges incident to v in G, $d_G(v) = |N_G(v)|$. If $X \subseteq V(G)$, then $N_G[X] = \bigcup_{v \in X} N_G[v]$ is the *closed neighbouhood* of X. Let $L(G)$ be the set of all leaves of G, that is the set of vertices degree 1, and let $n_1(G)$

be the cardinality of $L(G)$. A vertex v is called a *support vertex* if v is a neighbour of a leaf. Denote by $S(G)$ the set of all supports in G and let $n_S(G)$ be the cardinality of $S(G)$.

A set $D \subseteq V(G)$ is a *dominating set of* G if for every vertex $v \in V(G) - D$ there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the *domination number of* G denoted $\gamma(G)$. A minimum dominating set of a graph G is called a $\gamma(G)$-set.

From now on, G will be assumed to be connected. The *subgraph weakly induced by a set* $D \subseteq V(G)$ is the graph $\langle D \rangle_w = (N_G[D], E_w)$, where E_w consists of the set of all edges of G having at least one vertex in D. A set $D \subseteq V(G)$ is a *weakly connected dominating set* (WCDS) of G if D is dominating and $\langle D \rangle_w$ is connected. The *weakly connected domination number* of G, denoted $\gamma_w(G)$, is the minimum cardinality of a WCDS of G. A minimum WCDS of a graph G is called a $\gamma_w(G)$-set.

In this paper we define and study the *weakly connected domination subdivision number* $sd_{\gamma_w}(G)$ of a connected graph G to be the minimum number of edges that must be subdivided (where each egde can be subdivided at most once) in order to increase the weakly connected domination number. We assume that the graph G has at least three vertices, since the weakly connected domination number of the graph K_2 does not increase when its only edge is subdivided. We show that for every connected graph of order at least 3 the weakly connected domination subdivision number is well defined. For any unexplained terms and symbols see [3].

Some results, namely Proposition 4, Theorems 10 and 16, Corollaries 18 and 19, as well as a part of Corollary 15 are independently obtained by Hattingh, Jonck and Marcus [5].

2. Preliminary Results

In this section we study basic properties of weakly connected domination subdivision numbers of graphs.

Proposition 1. *If G is a connected graph of order at least* 3 *and e is an edge of G, then for the graph G' obtained from G by subdividing e,*

$$\gamma_w(G) \leq \gamma_w(G') \leq \gamma_w(G) + 1.$$

Proof. If we subdivide any edge of G, then obviously the resulting graph cannot have smaller weakly connected domination number. Hence $\gamma_w(G) \le \gamma_w(G')$.

Let x be the vertex obtained by subdividing e. Then any $\gamma_w(G)$-set D can be extended to a WCDS of G' by adding to it x. In this way $\gamma_w(G') \le |D| + 1 = \gamma_w(G) + 1$. ■

Observation 2. *Let $G = K_{m_1,m_2,\ldots,m_k}$ be the complete k partite graph, $k \ge 2$ with $m_1 \le m_2 \le \cdots \le m_k$.*

- *If $m_1 = 1$, then $sd_{\gamma_w}(G) = 1$;*
- *If $m_1 \ge 2$, then $sd_{\gamma_w}(G) = 2$.*

The weakly connected domination number of a path or a cycle is easy to compute.

Proposition 3. *For a path P_n and a cycle C_n on $n \ge 3$ vertices,*

$$\gamma_w(P_n) = \gamma_w(C_n) = \left\lceil \frac{n-1}{2} \right\rceil.$$

An immediate consequence of Proposition 3 now follows.

Proposition 4. *For a path P_n and a cycle C_n on $n \ge 3$ vertices,*

$$sd_{\gamma_w}(P_n) = sd_{\gamma_w}(C_n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Observation 5. *If G is a graph of order at least 3, then there exists a $\gamma_w(G)$-set containing no leaf and $\gamma_w(G) \ge n_S(G)$.*

Now we present two sufficient conditions for the weakly connected domination subdivision number of a connected graph to be equal one.

Proposition 6. *If G is a connected graph of order at least 3 and $\gamma_w(G) = 1$, then*

$$sd_{\gamma_w}(G) = 1.$$

Proof. If $\gamma_w(G) = 1$, then clearly G has a vertex of degree $n(G) - 1$, say u. Then $\{u\}$ is $\gamma_w(G)$-set. Let G' be a graph obtained from G by subdividing any edge of G. Since $n(G) \ge 3$, no vertex of G' has degree $n(G') - 1$ and thus $\gamma_w(G') > 1$. ■

A *strong support vertex* is a vertex adjacent to at least two leaves.

Proposition 7. *If a connected graph G has a strong support vertex, then*

$$sd_{\gamma_w}(G) = 1.$$

Proof. Let u be a strong support vertex and let v_1, v_2 be any two leaves adjacent to u. Let G' be the graph obtained from G by subdividing the edge uv_1 and let x_1 be the new vertex. It is easy to see that x_1 and u are elements of a $\gamma_w(G')$-set containing no leaf, say D'. Moreover, $D = D' - \{x_1\}$ is a WCDS of G, which implies that $\gamma_w(G) < \gamma_w(G')$. ■

Proposition 8. *If a connected graph G of order at least 3 has adjacent support vertices, then*

$$sd_{\gamma_w}(G) \leq 3.$$

Proof. Let u_1 and u_2 be two adjacent supports and let v_1 and v_2 be two leaves adjacent to u_1 and u_2, respectively. Denote by G' the graph obtained from G by subdividing edges u_1u_2, u_1v_1 and u_2v_2 and let x_1 and x_2 be the new vertices in G' adjacent to v_1 and v_2, respectively, and let y be the new vertex in G' adjacent to u_1 and u_2. Let D' be a $\gamma_w(G')$-set containing no leaf. Then x_1 and x_2 are elements of D'. Moreover, since D' is weakly connected and dominating, at least one vertex of $\{u_1, u_2, y\}$ belongs to D'. For this reasons, $D = (D' - \{x_1, x_2, y\}) \cup \{u_1, u_2\}$ is a WCDS of G of cardinality smaller than $\gamma_w(G')$. We conclude that $sd_{\gamma_w}(G) \leq 3$. ■

3. Weakly Connected Domination Subdivision Numbers of Trees

Proposition 9. *If T is a tree of order at least 3, D is a $\gamma_w(T)$-set of T and uv is an edge, then u or v is contained in D.*

Proof. Suppose both u and v belong to $V(G) - D$. Since D is dominating, we have that $d_T(u) > 1$ and $d_T(v) > 1$. Therefore there exists a path $(x_1, \ldots, u, v, \ldots, x_l)$ such that $x_1, x_l \in L(T)$. We conclude that there exist vertices $x, y \in D$ such that the unique $(x - y)$ path in T contains u and v. For this reason D is not weakly connected in T. ■

Theorem 10. *For any tree T of order at least* 3,

$$1 \leq sd_{\gamma_w}(T) \leq 2.$$

Proof. Let D be a $\gamma_w(T)$-set. Assume first that u is a strong support vertex of T. Then Proposition 7 implies that $sd_{\gamma_w}(T) = 1$.

Assume now that each support vertex of T is adjacent to exactly one leaf. Let $(s_0, s_1, \ldots, s_l)$ be a longest path of a tree T. Since T has no strong support vertex, $l \geq 3$. Let T' be the tree obtained from T by subdividing edges s_0s_1 and s_1s_2 such that $(s_0, x, s_1, y, s_2, \ldots, s_l)$ is the longest path of T'. Note that in this situation $d_T(s_1) = d_{T'}(s_1) = 2$. Obviously there exists a $\gamma_w(T')$-set, denoted D', containing x and y, and not containing s_0 and s_1. Consider the set $D = (D' - \{x, y\}) \cup \{s_1\}$. It is immediate that D is a WCDS of T. Obviously $|D| = |D'| - 1$, which implies that $\gamma_w(T') = |D'| > |D| \geq \gamma_w(T)$ and hence $sd_{\gamma_w}(T) \leq 2$. ■

Since the weakly connected domination subdivision number of a tree is either 1 or 2, trees can be classified as Class 1 or Class 2 depending on whether their weakly connected domination subdivision number is 1 or 2, respectively. In what follows, we characterize all trees in Class 2. To this aim we introduce some additional notation. A graph G is a *γ_w-excellent graph* if each vertex of G is contained in some $\gamma_w(G)$-set. If T is a tree and $e = uv$ is an edge of T, then by $T_{uv}(u)$ and $T_{uv}(v)$ we denote the two components of $T - uv$, where $u \in T_{uv}(u)$ and $v \in T_{uv}(v)$.

Theorem 11. *A tree T of order at least* 3 *is in Class* 2 *if and only if T is a γ_w-excellent tree.*

Proof. Assume first that T is a γ_w-excellent tree. It suffices to show that $\gamma_w(T) = \gamma_w(T')$ for any tree T' which can be obtained from T by subdividing one edge.

Let $e = uv \in E(T)$ be an edge. If there exists a $\gamma_w(T)$-set D containing u and v, then obviously D is a WCDS of the tree T' obtained from T by subdividing the edge e. Hence $\gamma_w(T') = \gamma_w(T)$. If not, then since T is a γ_w-excellent tree, there exists a $\gamma_w(T)$-set D_u which contains u and not contains v and there exists a $\gamma_w(T)$-set D_v which contains v and not contains u. Denote by x the vertex obtained by subdividing the edge e in order to obtain T'. It is immediate that $D'_u = (D_v \cap V(T_{uv}(u))) \cup \{x\}$ is a weakly connected dominating set of $T'_{xv}(x)$ and $D'_v = (D_u \cap V(T_{uv}(v))) \cup \{x\}$ is a weakly connected dominating set of $T'_{xu}(x)$. Hence, $D' = D'_u \cup D'_v$ is a WCDS of T'. Moreover, $D_u \cap V(T_{uv}(u))$ is a WCDS of $T_{uv}(u)$ and $D_v \cap V(T_{uv}(v))$ is a WCDS of $T_{uv}(v)$. In this way we conclude that $\overline{D} = (D_u \cap V(T_{uv}(u))) \cup (D_v \cap V(T_{uv}(v)))$ is a WCDS of T. Since a $\gamma_w(T)$-set containing u and v does not exist, $|\overline{D}| \geq \gamma_w(T) + 1$. Thus

$$2\gamma_w(T) = |D_u| + |D_v| = |\overline{D}| + |D' - \{x\}| \geq \gamma_w(T) + 1 + |D'| - 1$$

and therefore $|D'| \leq \gamma_w(T)$. Hence D' is a $\gamma_w(T')$-set and $\gamma_w(T') = \gamma_w(T)$.

Assume now that T is not a γ_w-excellent tree. We show that there exists an edge $e \in E(T)$ such that $\gamma_w(T') > \gamma_w(T)$ for the tree T' which can be obtained from T by subdividing e. Let u be a vertex which does not belong to any $\gamma_w(T)$-set. Let e be an edge incident with u and let x be the vertex of T' obtained by subdividing e. Denote by D' a $\gamma_w(T')$-set. Since T' is a tree, Proposition 9 implies that at least one of u and x belongs to D'. If $u, x \in D'$, then $D = D' - \{x\}$ is clearly a WCDS of T implying that $\gamma_w(T') > \gamma_w(T)$. If $u \in D'$ and $x \notin D'$, then D' is a WCDS of T and since $u \in D'$, we conclude that $\gamma_w(T') = |D'| > \gamma_w(T)$. If $x \in D'$ and $u \notin D'$, then $D = (D' - \{x\}) \cup \{u\}$ is a WCDS of T and since $u \in D$, we again conclude that $\gamma_w(T') = |D'| = |D| > \gamma_w(T)$. ■

A similar result for general connected graphs is not true. Since each vertex of the cycle C_3 is contained in some minimum weakly connected set, C_3 is a γ_w-excellent graph, but $sd_{\gamma_w}(C_3) = 1$. Moreover, vertex v of the graph G in Figure 1 does not belong to any $\gamma_w(G)$-set, however it is possible to verify that subdividing any one edge of G does not increase its weakly connected domination number.

Domke *et al.* [1] have defined the class $\mathcal{E}$ to be the class of trees obtained from P_2 by a finite sequence of the following operation: attach to any vertex a P_2. They have proved the following results.

Theorem 12 (Domke *et al.* [1]). *A nontrivial tree T is γ_w-excellent if and only if T belongs to the family $\mathcal{E}$.*

A set S of vertices of $G = (V, E)$ is an *independent set* if no two vertices of S are adjacent. The *vertex independence number*, denoted $\beta(G)$, is the maximum cardinality of an independent set of G.

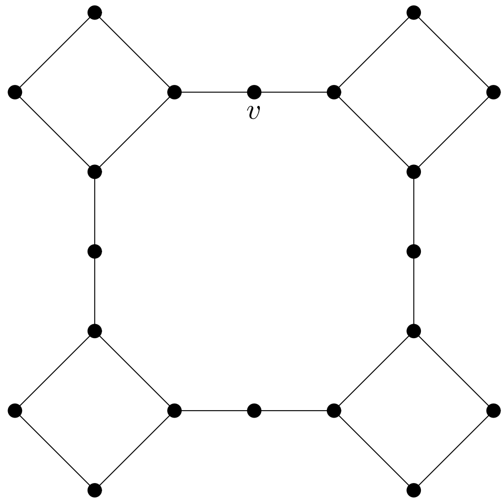

Figure 1. Graph G

Theorem 13 (Domke *et al.* [1]). *A nontrivial tree T is γ_w-excellent if and only if*

$$\beta(T) = \frac{n(T)}{2}.$$

The following result appears in [2].

Theorem 14 (Dunbar *et al.* [2]). *If T is a nontrivial tree, then*

$$\gamma_w(T) = n(T) - \beta(T).$$

Theorems 11, 12, 13 and 14 imply what follows.

Corollary 15. *Let T be a tree of order at least 3. Then the following conditions are equivalent:*

1. *T belongs to the family $\mathcal{E}$;*
2. *T is a γ_w-excellent tree;*
3. *$sd_{\gamma_w}(T) = 2$;*
4. *$\gamma_w(T) = \frac{n(T)}{2}$;*
5. *$\beta(T) = \frac{n(T)}{2}$.*

4. Weakly Connected Domination Subdivision Numbers of Graphs

In this section first we give an upper bound for the weakly connected domination subdivision number of an arbitrary connected graph.

Theorem 16. *Let G be a connected graph of order at least 3 and let $\delta_L(G)$ be the smallest degree among vertices of $V(G) - L(G)$. Then*

$$sd_{\gamma_w}(G) \leq \delta_L(G).$$

Proof. Let v be a vertex of G of degree $\delta_L(G)$. Denote by $u_1, \ldots, u_{d_G(v)}$ the neighbours of v in G and denote by $x_1, \ldots, x_{d_G(v)}$ the neighbours of v in the graph G' obtained from G by subdividing all edges incident with v, so that u_i and x_i, for $i = 1, \ldots, d_G(v)$, are adjacent in G'. For notational convience we let $U = \{u_1, \ldots, u_{d_G(v)}\}$ and $X = \{x_1, \ldots, x_{d_G(v)}\}$. It suffices to show that $\gamma_w(G') > \gamma_w(G)$.

Let D' be a $\gamma_w(G')$-set. We consider three cases.

Case 1. If $|X \cap D'| \geq 2$, then obviously $D = (D' - X) \cup \{v\}$ is a WCDS of G of cardinality smaller than $\gamma_w(G')$.

Case 2. If $|X \cap D'| = 1$, then without loss of generality we let $X \cap D' = \{x_1\}$. If $v \in D'$, then clearly $D = D' - \{x_1\}$ is a WCDS of G. Otherwise, if $v \notin D'$, then since D' is a WCDS of G', we obtain that $u_2, \ldots, u_{d_G(v)} \in D'$. Moreover, at least two vertices of $N_{G'}[u_1]$ belong to D', where one of them is x_1. For this reasons $D = D' - \{x_1\}$ is a WCDS of G of cardinality smaller than $\gamma_w(G')$.

Case 3. If $|X \cap D'| = 0$, then $v \in D'$ and since D' is a WCDS of G', $|D' \cap U| \geq 1$. In this situation $D = D' - \{v\}$ is a WCDS of G of cardinality smaller than $\gamma_w(G')$.

In all cases $\gamma_w(G) \leq |D| < \gamma_w(G')$ and we conclude that $sd_{\gamma_w}(G) \leq \delta_L(G)$. ■

Corollary 17. *The weakly connected subdivision number is defined for every connected graph G of order at least* 3.

Proof. Every such a graph G contains a vertex u with $d_G(u) > 1$. Hence $1 \leq sd_{\gamma_w}(G) \leq d_G(u) \leq |E(G)|$. ■

Corollary 18. *For any $r \times s$ grid graph $G_{r\times s}$ of order at least* 3,

$$1 \leq sd_{\gamma_w}(G_{r\times s}) \leq 2.$$

Proof. Corollary follows from the simple observation that if either $r = 1$ or $s = 1$, then $G_{r,s}$ is a path and $1 \leq sd_{\gamma_w}(G_{r\times s}) \leq 2$ by Theorem 10. Otherwise $G_{r,s}$ must contain a corner vertex of degree two and Theorem 16 implies the desired result. ■

Corollary 19. *For a k-regular connected graph G, where $k \geq 2$ and with $n(G) \geq 3$,*

$$1 \leq sd_{\gamma_w}(G) \leq k.$$

Corollary 20. *For any connected cubic graph G of order at least* 3,

$$1 \leq sd_{\gamma_w}(G) \leq 3.$$

A k-tree is any graph which can be obtained from a complete graph on k vertices, by repeatedly adding a new vertex and joining to it every vertex in a complete subgraph of order k of the existing graph. Since every k-tree contains a vertex of degree k, we obtain what follows.

Corollary 21. *For any k-tree T, $k \geq 2$, of order greater than* 2,

$$1 \leq sd_{\gamma_w}(G) \leq k.$$

Next we show that the weakly connected domination subdivision number of a graph can be arbitralily large. Our construction of a graph G is similar to the construction of a graph G of similar result for the total subdivision number of a graph by Haynes, Henning and Hopkins [4].

Theorem 22. *For any integer $k \geq 2$, there exists a connected graph G with $sd_{\gamma_w}(G) = k$.*

Proof. Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subseteq X : |Y| = k\}$. Thus, $\mathcal{Y}$ consists of all k-element subsets of X, and so $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let G be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$ add an edge joining x and Y if and only if $x \in Y$. Then G is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces in G a complete graph on $3(k-1)$ vertices, while the set $\mathcal{Y}$ is an independent set where each vertex has degree k in G.

To dominate $\mathcal{Y}$, any dominating set of G must contain at least $2(k-1)$ vertices of X, since otherwise there would be at least k vertices belonging to X and not belonging to a dominating set of G. Then by the construction of G, one vertex belonging to $\mathcal{Y}$ would not be dominated. Hence $\gamma_w(G) \geq 2(k-1)$. On the other hand, any subset of X of cardinality $2(k-1)$ is a WCDS of G, implying that $\gamma_w(G) \leq 2(k-1)$. Consequently, $\gamma_w(G) = 2(k-1)$.

Let $F = \{u_1v_1, \ldots, u_{k-1}v_{k-1}\}$ be an arbitrary subset of $k-1$ edges of G. Let H be the graph obtained from G by subdividing each edge of F. We show that $\gamma_w(H) = \gamma_w(G)$. Since every edge of G is incident with at least one vertex of X, we may assume that $u_i \in X$ for $i = 1, \ldots, k-1$. If $v_i \in \mathcal{Y}$, then since $d_G(v_i) = k$ and $|F| = k-1$, v_i is adjacent to a vertex $w_i \in X$ such that $w_iv_i \notin F$. If $v_i \in X$, then let w_i be any vertex of X such that $w_iv_i \notin F$. Let $D_F = \bigcup_{i=1}^{k-1}\{u_i, w_i\}$. Then $|D_F| \leq 2(k-1)$. If $|D_F| < 2(k-1)$, then let D be any subset of $2(k-1)$ vertices of X that contains D_F. If $|D_F| = 2(k-1)$, then let $D = D_F$. Then D is a WCDS of H, and so $\gamma_w(H) \leq 2(k-1) = \gamma_w(G)$. Since subdividing any number of edges of G cannot decrease its weakly connected domination number, $\gamma_w(H) \geq \gamma_w(G)$. Consequently, $\gamma_w(H) = \gamma_w(G)$, whence $sd_{\gamma_w}(G) \geq k$.

Since each vertex of $\mathcal{Y}$ has degree k, Theorem 16 implies that $sd_{\gamma_w}(G) \leq k$. Hence we conclude that $sd_{\gamma_w}(G) = k$. ■

Our last result gives a sufficient condition for the weakly connected domination subdivision number of an arbitrary graph to be equal to one.

Theorem 23. *If G is a connected graph of order at least 3 and G has exactly one minimum weakly connected dominating set, then*

$$sd_{\gamma_w}(G) = 1.$$

Proof. Suppose D is the unique minimum weakly connected dominating set of G and $sd_{\gamma_w}(G) > 1$. If u is a leaf in G and v is a support, then since D is the unique minimum weakly connected dominating set of G, it is easy to see that the weakly connected domination number of the graph obtained from G by subdividing the edge uv is greater than $\gamma_w(G) = |D|$, which contradicts with fact that $sd_{\gamma_w}(G) > 1$. Hence we conclude that G has no leaf. Suppose next that $uv \in E(G)$ is an edge such that $u, v \notin D$. Then the weakly connected domination number of the graph obtained from

G by subdividing the edge uv is clearly greater than $\gamma_w(G) = |D|$, which again contradicts $sd_{\gamma_w}(G) > 1$. Hence each edge of G is incident with at least one vertex of D and for this reason, for each vertex u belonging to $V(G) - D$ we have $N_G(u) \subseteq D$ and $|N_G(u)| \geq 2$.

Let $S = (s_0, s_1, \ldots, s_l)$ be a longest path in G. It is possible to observe that $G - s_0$ is a connected graph. If $s_0 \in D$ and u is a neighbour of s_0 which does not belong to D, then since each vertex of $V(G) - D$ has at least two neighbours in D, it is clear that $(D - \{s_0\}) \cup \{u\}$ is another weakly connected dominating set of G of size $\gamma_w(G)$, a contradiction. If $s_0 \in D$ and each neighbour of s_0 is in D, then we conclude that $D - \{s_0\}$ is a smaller weakly connected dominating set of G, again a contradiction. If $s_0 \notin D$, then $s_1 \in D$ and $v \in D$, where $v \neq s_1$ is a neighbour of s_0. If $G - s_1$ is disconnected, then since S is a longest path, s_1 and v are adjacent. In this situation $(D - \{v\}) \cup \{s_0\}$ is also a weakly connencted dominating set of G of cardinality $\gamma_w(G)$, a contradiction. If $G - s_1$ is connected, then since each edge of G is incident with at least one vertex of D, it is possible to verify that $(D - \{s_1\}) \cup \{s_0\}$ is also a weakly connencted dominating set of G of cardinality $\gamma_w(G)$, a contradiction. Therefore we conclude that $sd_{\gamma_w}(G) = 1$. ■

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