

## PARTITIONS OF A GRAPH INTO CYCLES CONTAINING A SPECIFIED LINEAR FOREST

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### Abstract

In this note, we consider the partition of a graph into cycles containing a specified linear forest. Minimum degree and degree sum conditions are given, which are best possible.

**Keywords:** partition of a graph, vertex-disjoint cycle, 2-factor, linear forest.

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### 1. INTRODUCTION

In this paper, we consider only finite undirected graphs without loops or multiple edges. We will generally follow notation and terminology of [2]. For a vertex  $x$  of a graph  $G$ , the neighborhood of  $x$  is denoted by  $N_G(x)$  and  $d_G(x) = |N_G(x)|$  is the degree of  $x$  in  $G$ . For a subgraph  $H$  of  $G$  and

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a vertex  $x \in V(G) - V(H)$ , we also denote  $N_H(x) = N_G(x) \cap V(H)$  and  $d_H(x) = |N_H(x)|$ . For a subset  $S$  of  $V(G)$ , we write  $\langle S \rangle$  for the subgraph induced by  $S$ . For a subgraph  $H$  of  $G$  and a subset  $S$  of  $V(G)$ ,  $d_H(S) = \sum_{x \in S} d_H(x)$ ,  $N_H(S) = \bigcup_{x \in S} N_H(x)$  and define  $G - H = \langle V(G) - V(H) \rangle$  and  $G - S = \langle V(G) - S \rangle$ . For a graph  $G$ ,  $|G| = |V(G)|$  is the order of  $G$ ,  $\delta(G)$  is the minimum degree of  $G$ , and

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) \mid xy \notin E(G), x, y \in V(G), x \neq y\}$$

is the minimum degree sum of nonadjacent vertices. (When  $G$  is complete, we define  $\sigma_2(G) = \infty$ .)

A forest is a graph each of whose components is a tree and a linear forest is a forest consisting of paths. We regard a single vertex as a path of order 1. For a path  $P = v_1 v_2 \cdots v_p$ , we call  $v_i$  an internal vertex for  $2 \leq i \leq p-1$ . If  $P$  is contained in a cycle  $C$  as a subgraph, we denote it by  $P \subset C$ .

For graphs  $G$  and  $H$ ,  $G \cup H$  is the union of  $G$  and  $H$ , and  $G + H$  is the join of  $G$  and  $H$ .  $K_n$  is a complete graph of order  $n$ .

Suppose that  $H_1, \dots, H_k$  are vertex-disjoint subgraphs such that  $V(G) = \bigcup_{i=1}^k V(H_i)$ . Then we say  $G$  can be partitioned into  $H_1, \dots, H_k$  and  $\{H_1, \dots, H_k\}$  is a partition of  $G$ .

Research on partitions of a graph into cycles with a specified number of components was started by Brandt *et al.*

**Theorem 1** (Brandt *et al.* [1]). *Suppose that  $|G| \geq 4k$  and  $\sigma_2(G) \geq |G|$ . Then  $G$  can be partitioned into  $k$  cycles.*

In this paper, we consider partitions into cycles each of which contains exactly one component of a specified linear forest as a subgraph. In the following,  $n$  always denotes the order of a graph  $G$ , and ‘disjoint’ means ‘vertex-disjoint’ because we only deal with partitions of the vertex set.

The special cases where each component of a specified linear forest is a vertex or an edge were considered in several papers [3–11]. In particular, the following theorem was obtained in [7].

**Theorem 2** (Enomoto and Matsumura [7]). *Suppose that  $n \geq 10p + 10q$ ,  $p + q \geq 1$  and either*

$$\delta(G) \geq \max \left\{ \frac{n+q}{2}, \frac{n+p+2q-3}{2} \right\},$$

or

$$\sigma_2(G) \geq \max\{n + q, n + 2p + 2q - 2\}.$$

Then for any linear forest with components  $P_1, \dots, P_{p+q}$  such that  $|P_i| = 1$  for  $1 \leq i \leq p$  and  $|P_i| = 2$  for  $p+1 \leq i \leq p+q$ ,  $G$  can be partitioned into cycles  $H_1, \dots, H_{p+q}$  such that  $P_i \subset H_i$ .

In this paper, we consider a more general case, that is, we specify not only vertices and edges but also paths of order at least 3. The main result of this paper is the following.

**Theorem 3.** Suppose that  $n \geq 10p + 10q'$ ,  $p + q \geq 1$ ,  $p \geq 0$ ,  $q' \geq q \geq 0$ , and either

$$\delta(G) \geq \max\left\{\frac{n + q'}{2}, \frac{n + p + q + q' - 3}{2}\right\},$$

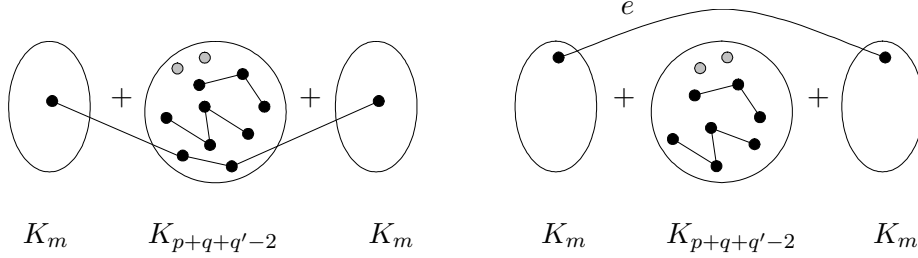
or

$$\sigma_2(G) \geq \max\{n + q', n + 2p + q + q' - 2\}.$$

Then for any linear forest with components  $P_1, \dots, P_{p+q}$  such that  $|P_i| = 1$  for  $1 \leq i \leq p$ ,  $|P_i| \geq 2$  for  $p+1 \leq i \leq p+q$  and  $\sum_{i=p+1}^{p+q} |E(P_i)| = q'$ ,  $G$  can be partitioned into cycles  $H_1, \dots, H_{p+q}$  such that  $P_i \subset H_i$ .

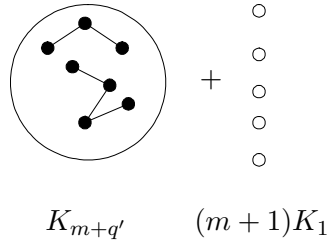
The minimum degree condition in Theorem 3 is sharp in the following sense. (In the following five examples, we let  $m$  be a sufficiently large integer.)

**Example 1.** Suppose that  $q' \geq q \geq 1$  and  $p + q \geq 2$ . Let  $G_1 = (K_m^1 \cup K_m^2) + K_{p+q+q'-2}$ , where  $K_m^i$  is a complete graph of order  $m$  for  $i = 1, 2$ . Take  $p$  distinct vertices  $P_1, \dots, P_p$  and  $q - 1$  disjoint paths  $P_{p+1}, \dots, P_{p+q-1}$  in  $K_{p+q+q'-2}$  such that  $|E(P_i)| \geq 1$  and  $\sum_{i=p+1}^{p+q-1} |E(P_i)| = q_0 < q'$ . Moreover, we take a path  $P_{p+q}$  which connects  $K_m^1$  and  $K_m^2$ ,  $|E(P_{p+q})| = q' - q_0$  and all internal vertices are contained in  $K_{p+q+q'-2}$ . (If  $q' - q_0 = 1$ , we add an edge  $e$  which connects  $K_m^1$  and  $K_m^2$  directly and let  $P_{p+q} = e$ .) Then we cannot take a cycle passing through  $P_{p+q}$  without using vertices in  $\bigcup_{i=1}^{p+q-1} V(P_i)$ . Hence  $G_1$  cannot have the desired partition, while  $\delta(G_1) = (|G_1| + p + q + q' - 4)/2$ .

Figure 1. The graph  $G_1$ .

**Example 2.** Suppose that  $q = 0$  and let  $G_2 = K_{m,m+1}$ , a complete bipartite graph with partite sets of order  $m$  and  $m + 1$ . Clearly,  $G_2$  cannot have the desired partition, while  $\delta(G) = (|G_2| - 1)/2$ .

**Example 3.** Suppose that  $p = 0$  and  $q' \geq q \geq 1$  and let  $G_3 = K_{m+q'} + (m + 1)K_1$ . Take  $q$  disjoint paths  $P_1, \dots, P_q$  in  $K_{m+q'}$  so that  $|E(P_i)| \geq 1$  and  $\sum_{i=1}^q |E(P_i)| = q'$ . Then  $G_3$  does not have the desired partition, while  $\delta(G_3) = (|G_3| + q' - 1)/2$ .

Figure 2. The graph  $G_3$ .

The degree sum condition in Theorem 3 is also sharp when there exists some component  $P_i$  such that  $|P_i| \leq 2$ .

**Example 4.** Suppose that  $p \geq 1$ . Let  $G_4 = (K_p \cup K_m) + K_{2p+q+q'-1}$ . Take  $p$  distinct vertices  $P_1, \dots, P_p$  in  $K_p$  and  $q$  disjoint paths  $P_{p+1}, \dots, P_{p+q}$  in  $K_{2p+q+q'-1}$  so that  $\sum_{i=p+1}^{p+q} |E(P_i)| = q'$ . To make a cycle through  $P_i$  for

$1 \leq i \leq p$ , we have to use at least 2 vertices in  $K_{2p+q+q'-1}$  but only  $2p-1$  vertices are available. Then  $G_4$  cannot have the desired partition, while  $\sigma_2(G_4) = |G_4| + 2p + q + q' - 3$ .

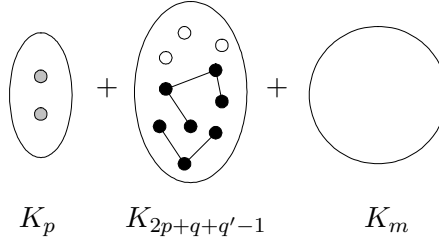


Figure 3. The graph  $G_4$ .

**Example 5.** Suppose that  $p = 0$  and let  $G_5 = (K_1 \cup K_m) + K_{q+q'-1}$ . Take  $q-1$  disjoint paths  $P_1, \dots, P_{q-1}$  in  $K_{q+q'-1}$  so that  $\sum_{i=1}^{q-1} |E(P_i)| = q' - 1$  and an edge  $P_q$  connecting  $K_1$  and  $K_{q+q'-1}$ . Then we cannot take a cycle through  $P_q$  without using the vertices of other specified paths. Hence  $G_5$  cannot be partitioned into cycles  $H_1, \dots, H_{p+q}$  such that  $P_i \subset H_i$ , while  $\sigma_2(G_5) = |G_5| + q + q' - 3$ .

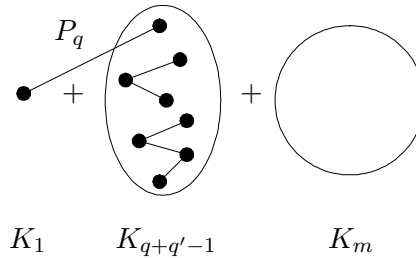


Figure 4. The graph  $G_5$ .

The graphs  $G_2$  and  $G_3$  show that the condition ' $\sigma_2(G) \geq n + q'$ ' cannot be dropped because  $\sigma_2(G_2) = |G_2| - 1$  and  $\sigma_2(G_3) = |G_3| + q' - 1$ .

For the case where each component of a specified linear forest is a path of order at least 3, the degree sum condition of Theorem 3 is not sharp and we prove the following.

**Theorem 4.** *Suppose that  $n \geq 3q + q'$ ,  $q \geq 1$ ,  $q' \geq 2q$  and*

$$\sigma_2(G) \geq \max\{n + q', n + q + q' - 3\}.$$

*Then for any disjoint paths of order at least 3  $P_1, \dots, P_q$  such that  $\sum_{i=1}^q |E(P_i)| = q'$ ,  $G$  can be partitioned into cycles  $H_1, \dots, H_q$  such that  $P_i \subset H_i$ .*

The graph  $G_1$  shows the sharpness of the degree sum condition in Theorem 4, because  $\sigma_2(G_1) = |G_1| + p + q + q' - 4$ .

To prove Theorem 4, we prove the following theorem, which deals with the case where all paths are of order 3.

**Theorem 5.** *Suppose that  $n \geq 5q$ ,  $q \geq 1$  and*

$$\sigma_2(G) \geq \max\{n + 2q, n + 3q - 3\}.$$

*Then for any disjoint paths of order 3  $P_1, \dots, P_q$ ,  $G$  can be partitioned into cycles  $H_1, \dots, H_q$  such that  $P_i \subset H_i$ .*

We can prove Theorems 3 and 4 similarly. The proof of Theorem 3 is given in the next section. Before proving Theorem 4, we will give a proof of Theorem 5 in Section 3. We will prove Theorem 4 in Section 4.

## 2. PROOF OF THEOREM 3

Let  $\{p_i\} = V(P_i)$  for  $1 \leq i \leq p$  and  $x_i$  and  $y_i$  be endvertices of  $P_i$  for  $p+1 \leq i \leq p+q$ .

We generate a new graph  $G'$  from  $G$  by deleting all internal vertices of  $P_i$  and adding the edge  $x_i y_i$  if  $x_i y_i \notin E(G)$  for  $p+1 \leq i \leq p+q$ . Then

$$\begin{aligned} \delta(G') &\geq \max \left\{ \frac{n + q'}{2}, \frac{n + p + q + q' - 3}{2} \right\} - (q' - q) \\ &= \max \left\{ \frac{(n - q' + q) + q}{2}, \frac{(n - q' + q) + p + 2q - 3}{2} \right\} \\ &= \max \left\{ \frac{|G'| + q}{2}, \frac{|G'| + p + 2q - 3}{2} \right\}, \end{aligned}$$

and

$$\begin{aligned}
\sigma_2(G') &\geq \max\{n + q', n + 2p + q + q' - 2\} - 2(q' - q) \\
&= \max\{(n - q' + q) + q, (n - q' + q) + 2p + 2q - 2\} \\
&= \max\{|G'| + q, |G'| + 2p + 2q - 2\}.
\end{aligned}$$

Moreover,  $|G'| \geq 10p + 10q' - (q' - q) = 10p + 9q' + q \geq 10p + 10q$ . Hence by Theorem 2,  $G'$  can be partitioned into cycles  $H'_1, \dots, H'_{p+q}$  such that  $p_i \in V(H'_i)$  for  $1 \leq i \leq p$  and  $x_i y_i \in E(H'_i)$  for  $p+1 \leq i \leq p+q$ .

If we replace  $x_i y_i$  by  $P_i$ , then we get a cycle  $H_i$  from  $H'_i$  for  $p+1 \leq i \leq p+q$  and  $\{H_1, \dots, H_{p+q}\}$  is the desired partition of  $G$ .

### 3. PROOF OF THEOREM 5

To prove Theorem 5, we first prove the following theorem.

**Theorem 6.** *Suppose that  $n \geq 5q$ ,  $q \geq 1$  and  $\sigma_2(G) \geq n + 3q - 3$ . Then for any disjoint paths of order 3  $P_1, \dots, P_q$ ,  $G$  contains  $q$  disjoint cycles  $C_1, \dots, C_q$  such that  $P_i \subset C_i$  and  $|C_i| \leq 5$ .*

To complete the proof of Theorem 5, we use the following theorem.

**Theorem 7** (Egawa et al. [4]). *Suppose that  $q \geq 1$ ,  $\sigma_2(G) \geq n + q$  and  $C_1, \dots, C_q$  are disjoint subgraphs such that  $C_i$  is a cycle or  $K_2$  and  $e_i \in E(C_i)$  for  $1 \leq i \leq q$ . Then there exist disjoint subgraphs  $H_1, \dots, H_q$  such that  $V(G) = \bigcup_{i=1}^q V(H_i)$ ,  $e_i \in E(H_i)$  and  $H_i$  is a cycle if  $C_i$  is a cycle and  $H_i$  is a cycle or  $K_2$  if  $C_i$  is  $K_2$  for  $1 \leq i \leq q$ .*

#### 3.1. Proof of Theorem 6

A cycle  $C$  is called *admissible* if  $P_i \subset C$  for some  $i$ ,  $1 \leq i \leq q$ ,  $|V(C) \cap \bigcup_{i=1}^q V(P_i)| = 3$  and  $|C| \leq 5$ . For  $1 \leq r \leq q$ , a set of cycles  $\{C_1, \dots, C_r\}$  is *admissible* if each  $C_i$  is admissible, and  $C_i$  and  $C_j$  are disjoint if  $i \neq j$ . If we say ' $r$  admissible cycles', then it means that the set of these  $r$  cycles is admissible. A set of admissible cycles  $\{C_1, \dots, C_r\}$  is *minimal* if there exist no  $r$  admissible cycles  $D_1, \dots, D_r$  such that  $|\bigcup_{i=1}^r V(D_i)| < |\bigcup_{i=1}^r V(C_i)|$ .

Let  $G$  be an edge-maximal counterexample and  $P_i = x_i y_i z_i$  for  $1 \leq i \leq q$ . Clearly,  $G$  is not complete. Let  $x$  and  $y$  be nonadjacent vertices of  $G$  and

define  $G' = G + xy$ , the graph obtained from  $G$  by adding the edge  $xy$ . Then  $G'$  is no longer a counterexample and  $G'$  has  $q$  admissible cycles. Since  $G$  is a counterexample, the edge  $xy$  is contained in some admissible cycle. This implies that  $G$  contains  $q - 1$  admissible cycles and we take minimal admissible cycles  $C_1, \dots, C_{q-1}$ . Without loss of generality, we may assume that  $P_i \subset C_i$  for  $1 \leq i \leq q - 1$ . Let  $L = \langle \bigcup_{i=1}^{q-1} V(C_i) \rangle$ ,  $M = G - L$  and  $D = M - V(P_q)$ . Note that  $x_q z_q \notin E(G)$  and  $N_D(x_q) \cap N_D(z_q) = \emptyset$ . If possible, we take  $C_1, \dots, C_{q-1}$  so that  $d_D(x_q) > 0$  and  $d_D(z_q) > 0$ .

**Claim 1.** We have  $d_D(x_q) > 0$  and  $d_D(z_q) > 0$ .

**Proof.** We first remark that we can take  $C_1, \dots, C_{q-1}$  so that  $d_D(x_q) > 0$ . To see this, suppose that  $d_D(x_q) = 0$  and take any  $y \in V(D)$ . Since

$$d_M(x_q) + d_M(y) \leq 1 + |M| - 2 = |M| - 1,$$

we have

$$\begin{aligned} d_L(x_q) + d_L(y) &\geq n + 3q - 3 - (|M| - 1) = |L| + 3q - 2 \\ &= \sum_{i=1}^{q-1} |C_i| + 3q - 2 > \sum_{i=1}^{q-1} (|C_i| + 3). \end{aligned}$$

Hence

$$d_{C_i}(x_q) + d_{C_i}(y) \geq |C_i| + 4$$

holds for some  $i$ ,  $1 \leq i \leq q - 1$ .

If  $|C_i| = 3$ , then this inequality cannot hold. Hence  $|C_i| \geq 4$ . Without loss of generality, we may assume that  $i = 1$ .

Suppose that  $|C_1| = 4$  and let  $C_1 = x_1 y_1 z_1 v x_1$ . Note that  $N_{C_1}(x_q) = N_{C_1}(y) = V(C_1)$ . If we take  $D_1 = x_1 y_1 z_1 y x_1$  and let  $D_i = C_i$  for  $2 \leq i \leq q - 1$ , then  $\{D_1, \dots, D_{q-1}\}$  is also minimal admissible and  $x_q$  can have a neighbor in  $G - \bigcup_{i=1}^{q-1} V(D_i)$  because  $x_q v \in E(G)$ .

Next suppose that  $|C_1| = 5$  and let  $C_1 = x_1 y_1 z_1 v u x_1$ . If  $\{x_1, z_1\} \subset N_{C_1}(y)$ , then we can find a shorter admissible cycle passing through  $P_1$ . Hence we have  $d_{C_1}(y) = 4$ . By symmetry, we may assume that  $N_{C_1}(y) = \{y_1, z_1, v, u\}$ . Then  $N_{C_1}(x_q) = V(C_1)$ . If we take  $D_1 = z_1 y_1 x_1 u y z_1$  and let  $D_i = C_i$  for  $2 \leq i \leq q - 1$ , then  $\{D_1, \dots, D_{q-1}\}$  is minimal admissible and



$x_q$  can have a neighbor in  $G - \bigcup_{i=1}^{q-1} V(D_i)$  because  $x_q v \in E(G)$ . Hence we may assume that  $d_D(x_q) > 0$ .

Now suppose that the claim is false. In view of the remark made at the beginning of the proof, we may assume that  $d_D(x_q) > 0$  and  $d_D(z_q) = 0$ . Take  $z \in N_D(x_q)$  and  $y \in V(D) - \{z\}$ . Arguing as above, we see that there exists  $j$  such that  $d_{C_j}(z_q) + d_{C_j}(y) \geq |C_j| + 4$  and we can take admissible cycles  $D_1, \dots, D_{q-1}$  so that  $\{D_1, \dots, D_{q-1}\}$  is minimal admissible and  $z_q$  can have a neighbor in  $G - \bigcup_{i=1}^{q-1} V(D_i)$ . But this contradicts the choice of  $C_1, \dots, C_{q-1}$  mentioned immediately before the statement of Claim 1.

Take any  $z \in N_D(x_q)$  and  $w \in N_D(z_q)$ . Note that  $\{zw, x_q w, z_q z\} \cap E(G) = \emptyset$ ,  $N_D(x_q) \cap N_D(w) = \emptyset$ , and  $N_D(z_q) \cap N_D(z) = \emptyset$ . (It may occur  $\{y_q z, y_q w\} \cap E(G) \neq \emptyset$ .)

Let  $S = \{x_q, z_q, z, w\}$ . Since

$$d_M(S) \leq 8 + 2(|M| - 5) = 2|M| - 2,$$

we have

$$\begin{aligned} d_L(S) &\geq 2(n + 3q - 3) - (2|M| - 2) = 2|L| + 6q - 4 \\ &= \sum_{i=1}^{q-1} 2|C_i| + 6q - 4 > \sum_{i=1}^{q-1} (2|C_i| + 6). \end{aligned}$$

This means that

$$d_{C_i}(S) \geq 2|C_i| + 7$$

for some  $i$ ,  $1 \leq i \leq q$ .

If  $|C_i| = 3$ , then this inequality cannot hold. Hence  $|C_i| \geq 4$ .

Suppose that  $|C_i| = 4$  and let  $C_i = x_i y_i z_i v x_i$ . By symmetry, we may assume that  $N_{C_i}(x_q) = N_{C_i}(z) = V(C_i)$ . Then  $v \notin N_{C_i}(z_q) \cup N_{C_i}(w)$ , because otherwise we can find two admissible cycles. But this means that  $d_{C_i}(S) \leq 14$ , a contradiction.

Next, suppose that  $|C_i| = 5$  and let  $C_i = x_i y_i z_i v u x_i$ . If  $d_{C_i}(z) = 5$ , then we can find an admissible cycle  $x_i y_i z_i z x_i$ , which is shorter than  $C_i$ . Hence  $d_{C_i}(z) \leq 4$ . Similarly,  $d_{C_i}(w) \leq 4$ . If  $(N_{C_i}(x_q) \cap N_{C_i}(z_q)) \cap \{v, u\} \neq \emptyset$ , we can also find shorter admissible cycle passing through  $P_q$ . Hence  $d_{C_i}(x_q) + d_{C_i}(z_q) \leq 8$ . But this implies that  $d_{C_i}(S) \leq 16$ , a contradiction.

This completes the proof of Theorem 6.

### 3.2. Proof of Theorem 5

By Theorem 6, there exist disjoint cycles  $C_1, \dots, C_q$  such that  $P_i \subset C_i$ . Let  $P_i = x_i y_i z_i$  for  $1 \leq i \leq q$ .

We make  $G'$  from  $G$  by deleting  $\{y_1, \dots, y_q\}$  and adding the edge  $x_i z_i$  for  $1 \leq i \leq q$  if  $x_i z_i \notin E(G)$ . Then we have disjoint subgraphs  $C'_1, \dots, C'_q$  of  $G'$  such that  $x_i z_i \in E(C'_i)$ , and  $C'_i$  is a cycle if  $|C_i| \geq 4$ , and  $C'_i$  is  $K_2$  if  $|C_i| = 3$ . Moreover,

$$\begin{aligned} \sigma_2(G') &\geq \max\{n + 3q - 3, n + 2q\} - 2q \\ &= \max\{(n - q) + 2q - 3, (n - q) + q\} \\ &= \max\{|G'| + 2q - 3, |G'| + q\} \geq |G'| + q. \end{aligned}$$

Hence by Theorem 7, there exist disjoint subgraphs  $H'_1, \dots, H'_q$  satisfying  $V(G') = \bigcup_{i=1}^q V(H'_i)$ ,  $x_i z_i \in E(H'_i)$  for  $1 \leq i \leq q$  and  $H'_i$  is a cycle if  $C'_i$  is a cycle and  $H'_i$  is a cycle or  $K_2$  if  $C'_i$  is  $K_2$ .

By replacing the edge  $x_i z_i$  by  $P_i$ , we make a cycle  $H_i$  from  $H'_i$  for  $1 \leq i \leq q$ . Then  $\{H_1, \dots, H_q\}$  is the desired partition of  $G$ .

This completes the proof of Theorem 5.

### 4. PROOF OF THEOREM 4

Let  $P_i = x_i z_i \dots y_i$  for  $1 \leq i \leq q$ . We make  $G'$  from  $G$  by deleting all internal vertices except  $z_i$  of  $P_i$  and adding the edge  $z_i y_i$  if  $z_i y_i \notin E(G)$  for  $1 \leq i \leq q$ . Then

$$\begin{aligned} \sigma_2(G) &\geq \max\{n + q', n + q + q' - 3\} - 2(q' - 2q) \\ &\geq \max\{(n - q' + 2q) + 2q, (n - q' + 2q) + 3q - 3\} \\ &\geq \max\{|G'| + 2q, |G'| + 3q - 3\}. \end{aligned}$$

Moreover,  $|G'| \geq 3q + q' - (q' - 2q) = 5q$ . Hence by Theorem 5,  $G'$  can be partitioned into cycles  $H'_1, \dots, H'_q$  such that  $P'_i \subset H'_i$  for  $1 \leq i \leq q$ , where  $P'_i = x_i z_i y_i$ .

We replace  $P'_i$  by  $P_i$  and get a cycle  $H_i$  from  $H'_i$  for  $1 \leq i \leq q$ . Then  $\{H_1, \dots, H_q\}$  is the desired partition of  $G$ .

This completes the proof of Theorem 4.

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