Discussiones Mathematicae
Graph Theory 28 (2008) 91–96

AN UPPER BOUND FOR GRAPHS OF DIAMETER 3 AND GIVEN DEGREE OBTAINED AS ABELIAN LIFTS OF DIPOLES

Tomáš Vetrík

Department of Mathematics, SvF
Slovak University of Technology
Bratislava, Slovakia

e-mail: vetrik@math.sk

Abstract

We derive an upper bound on the number of vertices in graphs of diameter 3 and given degree arising from Abelian lifts of dipoles with loops and multiple edges.

Keywords: degree and diameter of a graph, dipole.

2000 Mathematics Subject Classification: 05C12, 05C35.

1. Introduction

McKay, Miller and Širáň [1] found a family of the current largest known graphs for the diameter $k = 2$ and degrees of the form $d = (3q-1)/2$, where q is a prime power congruent to 1 $(mod\ 4)$. The order of these graphs is $\frac{8}{9}(d+\frac{1}{2})^2$, which is about 89% of the Moore bound d^2+1 diameter two. The graphs were constructed as Abelian lifts of complete bipartite graphs with the same number of loops attached to each vertex. In addition, the authors proved that these graphs are vertex-transitive but non-Cayley. Šiagiová [3] showed that it is possible to lift the McKay-Miller-Širáň graphs from quotients having only two vertices, called dipoles, with an Abelian group as voltage group.

This provides a strong motivation for examining upper bounds on the number of vertices for graphs that arise as lifts of dipoles with Abelian voltage group. In [2] Šiagiová proved that the order of Abelian lifts of a

dipole of large degree d and diameter 2 is bounded above by approximately $0.93d^2$. This result shows how close the McKay-Miller-Širáň graphs are to the theoretical maximum, since their order is approximately $0.89d^2$.

Motivated by this, we considered the problem of deriving an upper bound on the number of vertices in graphs of diameter 3 and given degree arising from lifts of dipoles in Abelian groups. Our main result is that for large maximum degree d, the order of such a lift is less than $0.6d^3$.

2. Preliminaries

Let G be an undirected graph, let $D(G)$ be the set of all directed edges of G and let Γ be an arbitrary finite group. If $e \in D(G)$, then e^{-1} is the edge with the reverse direction. A mapping $\alpha : D(G) \to \Gamma$ is a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for each $e \in D(G)$.

A graph G with a voltage assignment α determines a new graph G^α, called a *lift* of G. The vertex set of G^α is $V(G^\alpha) = V(G) \times \Gamma$ and the set of directed edges of G^α is $D(G^\alpha) = D(G) \times \Gamma$. In G^α, the edge (e, g) is a directed edge from the vertex (u, g) to the vertex (v, h) if and only if e is a directed edge from u to v in G and $h = g\alpha(e)$. Since the directed edges (e, g) and $(e^{-1}, g\alpha(e))$ form an undirected edge of G^α, the lifted graph G^α is undirected. The lift G^α is *Abelian* if the group Γ is Abelian. For any vertex v of the base graph G, the set $\{(v, g), g \in \Gamma\}$ forms a *fiber* above v in the lifted graph G^α. For the walk $W = e_1e_2 \dots e_t$ of directed edges of length t in G we put $\alpha(W) = \alpha(e_1)\alpha(e_2) \dots \alpha(e_t)$ and call it the voltage of the walk. Note that if $t = 0$, then the voltage of a walk of length 0 is the identity element of Γ. In [1] McKay, Miller and Širáň proved the following lemma, which helps determine the diameter of G^α.

Lemma A. *Let α be a voltage assignment on a graph G in a group Γ. Then $diam(G^\alpha) \le k$ if and only if for each ordered pair of vertices u, v (possibly, $u = v$) of G and for each $g \in \Gamma$ there exists a $u \to v$ walk W of length $\le k$ such that $\alpha(W) = g$.*

3. Counting Voltages on Walks

Let $D_{m,l}$ denote the graph with two vertices u, v, joined by m parallel edges, and with l loops attached to each vertex. For brevity each such graph will be called a *dipole*. Now we state an upper bound on the number of vertices of Abelian lift of a dipole of diameter 3.

Lemma 1. *Let α be a voltage assignment on a dipole $D_{m,l}$ in an Abelian group Γ such that the lifted graph $D^{\alpha}_{m,l}$ is of diameter 3. Then the number of vertices in $D^{\alpha}_{m,l}$ is at most $w(m,l) = 2\min\{(4l+1)m(m-1) + 2l(2l^2 + 3l + 7)/3 + 1, m[m(m-1)/2 + 8l^2 + 4l + 1]\}$.*

Proof. Assume that the graph $D^{\alpha}_{m,l}$ is of diameter 3. By Lemma A, the number of vertices in the fiber above u cannot exceed the number of distinct voltages on the $u \to u$ walks in $D_{m,l}$ of length at most 3. Since the voltage group is Abelian, on $u \to u$ walks we have 1 voltage of a walk of length 0, at most $2l$ distinct voltages of a walk of length 1, and by [2] we know that there are at most $m(m-1) + 2l^2$ distinct voltages on $u \to u$ walks of length 2, which are different from the voltage on $u \to u$ walk of length 0.

We derive a bound on the number of distinct voltages on $u \to u$ walks of length 3. Here the voltage is a sum of three elements. First, let us estimate the number of distinct voltages on $u \to v \to v \to u$ walks. Let $d_1d_2d_3$ be a $u \to v \to v \to u$ walk of length 3 in $D_{m,l}$. The number of ordered triples of directed edges for such $u \to u$ walks is $2lm^2$. Therefore, the number of all ordered triples of voltages on the edges of the $u \to v \to v \to u$ walks is at most $2lm^2$. Consider the walks where $\alpha(d_1^{-1}) = \alpha(d_3)$. The number of ordered triples of directed edges for such walks is $2lm$. Since $\alpha(d_1)\alpha(d_2)\alpha(d_1^{-1}) = \alpha(d_2)$, the number of distinct voltages of the walks $d_1d_2d_1^{-1}$ is at most $2l$. Hence, we have at most $2lm^2 - 2lm + 2l = 2l(m^2 - m + 1)$ distinct voltages on $u \to v \to v \to u$ walks, which are not voltages of paths of length 1.

Let $e_1e_2e_3$ be a $u \to u \to v \to u$ walk. Note that the number of ordered triples of directed edges of the walk $e_1e_2e_2^{-1}$ is $2lm$, but $\alpha(e_1)\alpha(e_2)\alpha(e_2^{-1})$ is equal to $\alpha(e_1)$, which is the voltage of the $u \to u$ walk of length 1. Hence, the number of distinct voltages on $u \to u \to v \to u$ walks, which are not voltages of paths of length 1, is at most $2lm^2 - 2lm = 2lm(m-1)$.

Let $f_1f_2f_3$ be a $u \to v \to u \to u$ walk. Remark that the voltage $\alpha(f_1)\alpha(f_2)\alpha(f_3)$ is equal to $\alpha(f_3)\alpha(f_1)\alpha(f_2)$, which is the voltage on $u \to u \to v \to u$ walk $f_3f_1f_2$. It is easy to see that the voltages on $u \to v \to u \to u$ walks are the same as the voltages on $u \to u \to v \to u$ walks.

Let $k_1k_2k_3$ be a $u \to u \to u \to u$ walk. If $\alpha(k_i) = \alpha(k_j^{-1})$ for $i, j \in \{1,2,3\}, i \neq j$, then $\alpha(k_1)\alpha(k_2)\alpha(k_3)$ is equal to the voltage of some $u \to u$ walk of length 1. The number of distinct voltages on $u \to u \to u \to u$ walks where $k_1 = k_2 = k_3$ is at most $2l$. If some loop appears in the walk exactly twice, then the number of distinct voltages on such $u \to u \to u \to u$ walks

is at most $2l(2l-2)2/2!$. Finally, if all three loops of the walks are different, the number of distinct voltages is at most $2l(2l-2)(2l-4)/3!$.

Consequently, the number of vertices in the fiber above u can be at most

$$(4l+1)m(m-1)+2l(2l^2+3l+7)/3+1. \tag{1}$$

On the other hand, the number of vertices in $D^{\alpha}_{m,l}$ cannot exceed two times the number of distinct voltages on $u \to v$ walks in $D_{m,l}$ of length at most 3. There are at most m distinct voltages of a walk of length 1 and at most $4lm$ distinct voltages of a walk of length 2.

Consider now the estimates on the number of distinct voltages of a walk of length 3. Assume first that $d_1d_2d_3$ is a $u \to v \to u \to v$ walk. If $\alpha(d_1) = \alpha(d_2^{-1})$ or $\alpha(d_3) = \alpha(d_2^{-1})$, then the voltage on $d_1d_2d_3$ is the same as the voltage of some walk of length 1. Therefore, let $\alpha(d_1) \neq \alpha(d_2^{-1}) \neq \alpha(d_3)$. The number of ordered triples of such voltages is bounded above by $m(m-1)(m-1)$. The number of ordered triples of voltages where all three elements $\alpha(d_1)$, $\alpha(d_2)$ and $\alpha(d_3)$ are different is bounded by $m(m-1)(m-2)$. Observe that the voltage on the walk $d_1d_2d_3$ is the same as the voltage on the walk $d_3d_2d_1$. Hence, the number of distinct voltages on $u \to v \to u \to v$ walks is at most $m(m-1)(m-1) - m(m-1)(m-2)/2 = m^2(m-1)/2$.

Further, it can be easily shown that we have at most $2l^2m$ distinct voltages on $u \to u \to u \to v$ walks, at most $2l^2m$ distinct voltages on $u \to v \to v \to v$ walks, and at most $4l^2m$ distinct voltages on $u \to u \to v \to v$ walks.

It follows that the maximum number of distinct voltages on $u \to v$ walks in $D_{m,l}$ of length at most 3 is

$$m[m(m-1)/2+8l^2+4l+1]. \tag{2}$$

Since there are two fibers in the lifted graph, the number of vertices of $D^{\alpha}_{m,l}$ satisfies $\mid V(D^{\alpha}_{m,l}) \mid \leq w(m,l)$, where $w(m,l) = 2 \min \{(4l+1)m(m-1) + 2l(2l^2+3l+7)/3+1, m[m(m-1)/2+8l^2+4l+1]\}$. ■

4. The Main Result

We have now enough tools to present an upper bound on the number of vertices in graphs of diameter 3 and given degree that arise as lifts of dipoles in Abelian groups.

Theorem 2. *Let D be a dipole of degree d, let Γ be an Abelian group and let α be a voltage assignment on D in Γ, such that the lifted graph D^α has diameter* 3.

I. *If* $d \geq 36$, *then* $\mid V(D^\alpha) \mid < 0.608d^3$.

II. *If* $d \geq 378$, *then* $\mid V(D^\alpha) \mid < 0.6d^3$.

Proof. Let us suppose that the degree $d \geq 36$. Since $d = m + 2l$, according to (2) the number of vertices in the graph D^α is at most $p_1(l) = \frac{1}{2}[-40l^3 + (28d - 20)l^2 + (-6d^2 + 12d - 4)l + d^3 - d^2 + 2d]$. Similarly, by (1), $\mid V(D^\alpha) \mid \leq p_2(l) = \frac{1}{3}[52l^3 + (-48d + 42)l^2 + (12d^2 - 24d + 20)l + 3d^2 - 3d + 3]$. Then the bound $w(m, l)$ from Lemma 1 can be expressed as $w(d) = 2 \max \min\{p_1(l), p_2(l)\}$, where the maximum is taken over all l such that $1 \leq l < \frac{d}{2}$.

We first find the coordinates in which the polynomials $p_1(l)$ and $p_2(l)$ attain the local extremes, and then we estimate the coordinates of the points of intersection of the graphs of $p_1(l)$ and $p_2(l)$.

The function $p_1(l)$ has the local minimum at $m_1 = (-28d + 20 + \sqrt{D_1})/(-120)$, where $D_1 = 64d^2 + 320d - 80$, and the local maximum at $M_1 = (-28d + 20 - \sqrt{D_1})/(-120)$. It can be shown that for $d \geq 36$ we have $0.16d < m_1 < 0.17d$ and $0.29d < M_1 < 0.30d$. The local minimum of $p_2(l)$ is attained at $m_2 = (24d - 21 + \sqrt{D_2})/78$, where $D_2 = 108d^2 - 72d - 339$ and $0.43d < m_2 < 0.45d$, and the local maximum at $M_2 = (24d - 21 - \sqrt{D_2})/78$, where $0.16d < M_2 < 0.175d$.

Now we estimate the coordinates of the points of intersection of the graphs of $p_1(l)$ and $p_2(l)$. Let $p(l) = p_2(l) - p_1(l) = \frac{1}{6}[224l^3 + (-180d + 144)l^2 + (42d^2 - 84d + 52)l - 3d^3 + 9d^2 - 12d + 6]$. For the first root l_1 of the equation $p(l) = 0$ we obtain $0.148d < l_1 < 0.16d$. Since the function $p_1(l)$ is in this interval decreasing and $p_1(0.148d) < 0.3035d^3$, the value $0.3035d^3$ gives the upper bound for $p_1(l_1)$. For the second root of $p(l) = 0$ one has $0.175d < l_2 < 0.1974d$. Because here $p_2(l)$ is decreasing and $p_2(0.175d) < 0.304d^3$, obviously $0.304d^3$ is an upper bound for $p_2(l_2)$. Finally, $0.45d < l_3 < 0.46d$ and $p_2(0.46d) < 0.16d^3$, hence $w(d) < 0.608d^3$.

If $d \geq 378$, then for l_2 we have $0.1955d < l_2 < 0.1974d$. Since $p_2(0.1955d) < 0.3d^3$ and $p_1(0.148d) < 0.2985d^3$, one has $\mid V(D^\alpha) \mid < 0.6d^3$. ■

Comparing our result with the well-known Moore bound $d^3 - d^2 + d + 1$ for the diameter $k = 3$, we notice a considerable gap, explained by severe restrictions imposed by the fact that the graphs under consideration are Abelian lifts of dipoles.

Acknowledgment

The author thanks Jozef Širáň and Eyal Loz for valuable discussions. The research was supported by the VEGA Grant No. 1/2004/05 and the APVV Grant No. 0040–06.

References

[1] B.D. McKay, M. Miller and J. Širáň, *A note on large graphs of diameter two and given maximum degree*, J. Combin. Theory (B) **74** (1998) 110–118.

[2] J. Šiagiová, *A Moore-like bound for graphs of diameter* 2 *and given degree, obtained as Abelian lifts of dipoles*, Acta Math. Univ. Comenianae **71** (2002) 157–161.

[3] J. Šiagiová, *A note on the McKay-Miller-Širáň graphs*, J. Combin. Theory (B) **81** (2001) 205–208.

Received 29 September 2006
Revised 13 February 2007
Accepted 13 February 2007