# A CLASSIFICATION FOR MAXIMAL NONHAMILTONIAN BURKARD-HAMMER GRAPHS 

Ngo Dac Tan<br>Institute of Mathematics<br>18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam<br>e-mail: ndtan@math.ac.vn

AND

Chawalit Iamjaroen<br>Department of Mathematics, Mahasarakham University<br>Kamrieng, Kantarawichai, Mahasarakham 44150, Thailand<br>e-mail: chawalit.i@msu.ac.th


#### Abstract

A graph $G=(V, E)$ is called a split graph if there exists a partition $V=I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of $G$ induced by $I$ and $K$ are empty and complete graphs, respectively. In 1980, Burkard and Hammer gave a necessary condition for a split graph $G$ with $|I|<|K|$ to be hamiltonian. We will call a split graph $G$ with $|I|<|K|$ satisfying this condition a Burkard-Hammer graph. Further, a split graph $G$ is called a maximal nonhamiltonian split graph if $G$ is nonhamiltonian but $G+u v$ is hamiltonian for every $u v \notin E$ where $u \in I$ and $v \in K$. Recently, Ngo Dac Tan and Le Xuan Hung have classified maximal nonhamiltonian Burkard-Hammer graphs $G$ with minimum degree $\delta(G) \geq|I|-3$. In this paper, we classify maximal nonhamiltonian Burkard-Hammer graphs $G$ with $|I| \neq 6,7$ and $\delta(G)=|I|-4$.


Keywords: split graph, Burkard-Hammer condition, Burkard-Hammer graph, hamiltonian graph, maximal nonhamiltonian split graph.
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## 1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If $G$ is a graph, then $V(G)$ and $E(G)$ (or $V$ and $E$ for short) will denote its vertex-set and its edge-set, respectively. For a subset $W \subseteq V(G)$, the set of all neighbours of $W$ is denoted by $N_{G}(W)$ or $N(W)$ for short. For a vertex $v \in V(G)$, the degree of $v$, denoted by $\operatorname{deg}(v)$, is the number $|N(v)|$. The minimum degree of $G$, denoted by $\delta(G)$, is the number $\min \{\operatorname{deg}(v) \mid v \in V(G)\}$. By $N_{G, W}(v)$ or $N_{W}(v)$ for short we denote the set $W \cap N_{G}(v)$. The subgraph of $G$ induced by $W$ is denoted by $G[W]$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

A graph $G=(V, E)$ is called a split graph if there exists a partition $V=I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of $G$ induced by $I$ and $K$ are empty and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. Further, a split graph $G=S(I \cup K, E)$ is called a complete split graph if every $u \in I$ is adjacent to every $v \in K$. The notion of split graphs was introduced in 1977 by Földes and Hammer [4]. These graphs are interesting because they are related to many problems in combinatorics (see $[3,5,10]$ ) and in computer science (see $[6,7]$ ).

In 1980, Burkard and Hammer gave a necessary condition for a split graph $G=S(I \cup K, E)$ with $|I|<|K|$ to be hamiltonian [2] (see Section 2 for more detail). We will call this condition the Burkard-Hammer condition. Also, we will call a split graph $G=S(I \cup K, E)$ with $|I|<|K|$, which satisfies the Burkard-Hammer condition, a Burkard-Hammer graph.

Thus, by [2] any hamiltonian split graph $G=S(I \cup K, E)$ with $|I|<|K|$ is a Burkard-Hammer graph. In general, the converse is not true. The first nonhamiltonian Burkard-Hammer graph has been indicated in [2]. Further infinite families of nonhamiltonian Burkard-Hammer graphs have been constructed recently in [13].

A split graph $G=S(I \cup K, E)$ is called a maximal nonhamiltonian split graph if $G$ is nonhamiltonian but the graph $G+u v$ is hamiltonian for every $u v \notin E$ where $u \in I$ and $v \in K$. It is known from a result in [12] that any nonhamiltonian Burkard-Hammer graph is contained in a maximal nonhamiltonian Burkard-Hammer graph. So knowledge about maximal nonhamiltonian Burkard-Hammer graphs provides us certain information about nonhamiltonian Burkard-Hammer graphs.

It has been shown in [12] that there are no nonhamiltonian BurkardHammer graphs and therefore no maximal nonhamiltonian Burkard-Hammer
graphs $G=S(I \cup K, E)$ with $\delta(G) \geq|I|-2$. In the same paper [12], Ngo Dac Tan and Le Xuan Hung have classified maximal nonhamiltonian BurkardHammer graphs $G=S(I \cup K, E)$ with $\delta(G)=|I|-3$. Namely, they have proved in [12] that for every integer $n>5$ there exists up to isomorphisms exactly one maximal nonhamiltonian Burkard-Hammer graph $G=S(I \cup K, E)$ with $|K|=n$ and $\delta(G)=|I|-3$ which is the graph $H^{4, n}$ in their notation there (see the definition of $H^{4, n}$ in Section 2). Recently, Ngo Dac Tan and Iamjaroen have constructed in [14] a family of maximal nonhamiltonian Burkard-Hammer graphs $G=S(I \cup K, E)$ with $\delta(G)=|I|-4$. In this paper, we will show that if a maximal nonhamiltonian Burkard-Hammer graph $G=S(I \cup K, E)$ with $\delta(G)=|I|-4$ has $|I| \neq 6,7$, then $G$ must be a graph in the family constructed by Ngo Dac Tan and Iamjaroen in [14]. Namely, we will prove the following main result of the paper.

Theorem 1. Let $G=S(I \cup K, E)$ be a split graph with $|I| \neq 6,7$ and $\delta(G)=|I|-4$. Then $G$ is a maximal nonhamiltonian Burkard-Hammer graph if and only if $G$ is isomorphic to the expansion $H^{4, t}\left[G_{2}, v_{2}^{*}\right]$ where $t=|K|-|I|+5$ and $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ is a complete split graph with $\left|K_{2}\right|-1=\left|I_{2}\right|=|I|-5 \geq 3$.

The expansion graph $H^{4, t}\left[G_{2}, v_{2}^{*}\right]$ will be defined in Section 2.
Thus, we will get the classification of maximal nonhamiltonian BurkardHammer graphs $G=S(I \cup K, E)$ with $\delta(G)=|I|-4$ for the case $|I| \neq 6,7$.

We would like to note that there is an interesting discussion about the Burkard-Hammer condition in [9]. Concerning the hamiltonian problem for split graphs, readers can see also [8] and [11].

## 2. Preliminaries

Let $G=S(I \cup K, E)$ be a split graph and $I^{\prime} \subseteq I, K^{\prime} \subseteq K$. Denote by $B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ the graph $G\left[I^{\prime} \cup K^{\prime}\right]-E\left(G\left[K^{\prime}\right]\right)$. It is clear that $G^{\prime}=$ $B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ is a bipartite graph with the bipartition subsets $I^{\prime}$ and $K^{\prime}$. So we will call $B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ the bipartite subgraph of $G$ induced by $I^{\prime}$ and $K^{\prime}$. For a component $G_{j}^{\prime}=B_{G}\left(I_{j}^{\prime} \cup K_{j}^{\prime}, E_{j}^{\prime}\right)$ of $G^{\prime}=B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ we define

$$
k_{G}\left(G_{j}^{\prime}\right)=k_{G}\left(I_{j}^{\prime}, K_{j}^{\prime}\right)= \begin{cases}\left|I_{j}^{\prime}\right|-\left|K_{j}^{\prime}\right| & \text { if }\left|I_{j}^{\prime}\right|>\left|K_{j}^{\prime}\right|, \\ 0 & \text { otherwise } .\end{cases}
$$

If $G^{\prime}=B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ has $r$ components $G_{1}^{\prime}=B_{G}\left(I_{1}^{\prime} \cup K_{1}^{\prime}, E_{1}^{\prime}\right), \ldots, G_{r}^{\prime}=$ $B_{G}\left(I_{r}^{\prime} \cup K_{r}^{\prime}, E_{r}^{\prime}\right)$ then we define

$$
k_{G}\left(G^{\prime}\right)=k_{G}\left(I^{\prime}, K^{\prime}\right)=\sum_{j=1}^{r} k_{G}\left(G_{j}^{\prime}\right) .
$$

A component $G_{j}^{\prime}=B_{G}\left(I_{j}^{\prime} \cup K_{j}^{\prime}, E_{j}^{\prime}\right)$ of $G^{\prime}=B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ is called a $T$-component (resp., $H$-component, L-component) if $\left|I_{j}^{\prime}\right|>\left|K_{j}^{\prime}\right|$ (resp., $\left.\left|I_{j}^{\prime}\right|=\left|K_{j}^{\prime}\right|,\left|I_{j}^{\prime}\right|<\left|K_{j}^{\prime}\right|\right)$. Let $h_{G}\left(G^{\prime}\right)=h_{G}\left(I^{\prime}, K^{\prime}\right)$ denote the number of $H$-components of $G^{\prime}$.

In 1980, Burkard and Hammer proved the following necessary but not sufficient condition for hamiltonian split graphs [2].

Theorem 2 [2]. Let $G=S(I \cup K, E)$ be a split graph with $|I|<|K|$. If $G$ is hamiltonian, then

$$
k_{G}\left(I^{\prime}, K^{\prime}\right)+\max \left\{1, \frac{h_{G}\left(I^{\prime}, K^{\prime}\right)}{2}\right\} \leq\left|N_{G}\left(I^{\prime}\right)\right|-\left|K^{\prime}\right|
$$

holds for all $\emptyset \neq I^{\prime} \subseteq I, K^{\prime} \subseteq N_{G}\left(I^{\prime}\right)$ with $\left(k_{G}\left(I^{\prime}, K^{\prime}\right), h_{G}\left(I^{\prime}, K^{\prime}\right)\right) \neq(0,0)$.
We will shortly call the condition in Theorem 2 the Burkard-Hammer condition. We also call a split graph $G=S(I \cup K, E)$ with $|I|<|K|$, which satisfies the Burkard-Hammer condition, a Burkard-Hammer graph. Thus, by Theorem 2 any hamiltonian split graph $G=S(I \cup K, E)$ with $|I|<|K|$ is a Burkard-Hammer graph. For split graphs $G=S(I \cup K, E)$ with $|I|<|K|$ and $\delta(G) \geq|I|-2$ the converse is true [12]. But it is not true in general. The first example of a nonhamiltonian Burkard-Hammer graph has been indicated in [2]. Recently, Ngo Dac Tan and Le Xuan Hung have classified nonhamiltonian Burkard-Hammer graphs $G=S(I \cup K, E)$ with $\delta(G)=|I|-3$. Namely, they have proved the following result.

Theorem 3 [12]. Let $G=S(I \cup K, E)$ be a split graph with $|I|<|K|$ and the minimum degree $\delta(G) \geq|I|-3$. Then
(i) If $|I| \neq 5$, then $G$ has a Hamilton cycle if and only if $G$ satisfies the Burkard-Hammer condition;
(ii) If $|I|=5$ and $G$ satisfies the Burkard-Hammer condition, then $G$ has no Hamilton cycles if and only if $G$ is isomorphic to one of the graphs $H^{1, n}, H^{2, n}, H^{3, n}$ or $H^{4, n}$ listed in Table 1.

Table 1. The graphs $H^{1, n}, H^{2, n}, H^{3, n}$ and $H^{4, n}$.

| The graph | The vertex-set | The edge-set |
| :--- | :--- | :--- |
| $G$ | $V(G)=I^{*} \cup K^{*}$ | $E(G)=E_{1}^{*} \cup \ldots \cup E_{5}^{*} \cup E_{K^{*}}^{*}$ |
| $H^{1, n}$ | $I^{*}=\left\{u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}, u_{5}^{*}\right\}$, | $E_{1}^{*}=\left\{u_{1}^{*} v_{1}^{*}, u_{1}^{*} v_{2}^{*}\right\}$, |
| $(n>5)$ | $K^{*}=\left\{v_{1}^{*}, v_{2}^{*}, \ldots, v_{n}^{*}\right\}$. | $E_{2}^{*}=\left\{u_{2}^{*} v_{2}^{*}, u_{2}^{*} v_{4}^{*}\right\}$, |
|  |  | $E_{3}^{*}=\left\{u_{3}^{*} v_{2}^{*}, u_{3}^{*} v_{3}^{*}, u_{3}^{*} v_{6}^{*}\right\}$, |
|  |  | $E_{4}^{*}=\left\{u_{4}^{*} v_{1}^{*}, u_{4}^{*} v_{4}^{*}, u_{4}^{*} v_{6}^{*}\right\}$, |
|  |  | $E_{5}^{*}=\left\{u_{5}^{*} v_{5}^{*}, u_{5}^{*} v_{6}^{*}\right\}$, |
|  |  | $E_{K^{*}}^{*}=\left\{v_{i}^{*} v_{j}^{*} \mid i \neq j ; i, j=1, \ldots, n\right\}$. |
| $H^{2, n}$ | $V\left(H^{2, n}\right)=V\left(H^{1, n}\right)$ | $E\left(H^{2, n}\right)=E\left(H^{1, n}\right) \cup\left\{u_{4}^{*} v_{2}^{*}\right\}$ |
| $H^{3, n}$ | $V\left(H^{3, n}\right)=V\left(H^{1, n}\right)$ | $E\left(H^{3, n}\right)=E\left(H^{1, n}\right) \cup\left\{u_{5}^{*} v_{2}^{*}\right\}$ |
| $H^{4, n}$ | $V\left(H^{4, n}\right)=V\left(H^{1, n}\right)$ | $E\left(H^{4, n}\right)=E\left(H^{1, n}\right) \cup\left\{u_{4}^{*} v_{2}^{*}, u_{5}^{*} v_{2}^{*}\right\}$ |

Theorem 3 shows that there are up to isomorphisms only four nonhamiltonian Burkard-Hammer graphs $G=S(I \cup K, E)$ with $K=N(I)$ and $\delta(G)=|I|-3$, namely, the graphs $H^{1,6}, H^{2,6}, H^{3,6}$ and $H^{4,6}$. In contrast with this result, the number of nonhamiltonian Burkard-Hammer graphs $G=S(I \cup K, E)$ with $K=N(I)$ and $\delta(G)=|I|-4$ is infinite. This is a recent result of Ngo Dac Tan and Iamjaroen [13]. We remind now one of the constructions in this work, which is needed here.

Let $G_{1}=S\left(I_{1} \cup K_{1}, E_{1}\right)$ and $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ be split graphs with

$$
V\left(G_{1}\right) \cap V\left(G_{2}\right)=\emptyset
$$

and $v$ be a vertex of $K_{1}$. We say that a graph $G$ is an expansion of $G_{1}$ by $G_{2}$ at $v$ if $G$ is the graph obtained from $\left(G_{1}-v\right) \cup G_{2}$ by adding the set of edges

$$
E_{0}=\left\{x_{i} v_{j} \mid x_{i} \in V\left(G_{1}\right) \backslash\{v\}, v_{j} \in K_{2} \text { and } x_{i} v \in E_{1}\right\} .
$$

It is clear that such a graph $G$ is a split graph $S(I \cup K, E)$ with $I=I_{1} \cup I_{2}$, $K=\left(K_{1} \backslash\{v\}\right) \cup K_{2}$ and is uniquely determined by $G_{1}, G_{2}$ and $v \in K_{1}$.

Because of this, we will denote this graph $G$ by $G_{1}\left[G_{2}, v\right]$. Further, a graph $G$ is called an expansion of $G_{1}$ by $G_{2}$ if it is an expansion of $G_{1}$ by $G_{2}$ at some vertex $v \in K_{1}$.

As an example, we show in Figure 1 the expansion of the graph $H^{4, n}$ by the complete split graph $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ with $I_{2}=\left\{u_{1}, u_{2}\right\}$ and $K_{2}=\left\{v_{1}, v_{2}, v_{3}\right\}$ at the vertex $v_{2}^{*}$ of $H^{4, n}$.


Figure 1. The expansion $H^{4, n}\left[G_{2}, v_{2}^{*}\right]$.
The following results are needed later.
Lemma 4 [11]. Let $G=S(I \cup K, E)$ be a split graph with $|I|<|K|$. Then $G$ has a Hamilton cycle if and only if $|N(I)|>|I|$ and the subgraph $G^{\prime}=G[I \cup N(I)]$ has a Hamilton cycle.

Lemma 5 [12]. Let $G=S(I \cup K, E)$ be a Burkard-Hammer graph. Then for any $u \in I$ and $v \in K$ with $u v \notin E$, the graph $G+u v$ also is a BurkardHammer graph.

Lemma 6 [12]. Let $G=S(I \cup K, E)$ be a Burkard-Hammer graph. Then for any $\emptyset \neq I^{\prime} \subseteq I$, we have $\left|N\left(I^{\prime}\right)\right|>\left|I^{\prime}\right|$.

Theorem 7 [13]. Let $G_{1}=S\left(I_{1} \cup K_{1}, E_{1}\right)$ be a Burkard-Hammer graph and $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ be a complete split graph with $\left|I_{2}\right|<\left|K_{2}\right|$. Then any expansion of $G_{1}$ by $G_{2}$ is a Burkard-Hammer graph.

Theorem 8 [13]. Let $G_{1}=S\left(I_{1} \cup K_{1}, E_{1}\right)$ be an arbitrary split graph and $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ be a split graph with $\left|K_{2}\right|=\left|I_{2}\right|+1$. Then an expansion of $G_{1}$ by $G_{2}$ is a hamiltonian graph if and only if both $G_{1}$ and $G_{2}$ are hamiltonian graphs.

Let $G=S(I \cup K, E)$ be a split graph. Set

$$
B_{i}(G)=\left\{v \in K| | N_{G, I}(v) \mid=i\right\}
$$

If the graph $G$ is clear from the context then we also write $B_{i}$ instead of $B_{i}(G)$.

Theorem 9 [14]. Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $|I| \geq 7$ and $\delta(G)=|I|-4$. Then $B_{4}=B_{5}=$ $\cdots=B_{|I|-1}=\emptyset$ but $B_{3} \neq \emptyset$.

Theorem 10 [14]. Any expansion of the graph $H^{4, n}$ by a complete split graph $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ with $\left|I_{2}\right|=\left|K_{2}\right|-1 \geq 1$ at the vertex $v_{2}^{*}$ of $H^{4, n}$ is a maximal nonhamiltonian Burkard-Hammer graph $G=S(I \cup K, E)$ with $\delta(G)=|I|-4$.

Let $C$ be a cycle in a graph $G=(V, E)$. By $\vec{C}$ we denote the cycle $C$ with a given orientation and by $\overleftarrow{C}$ the cycle $C$ with the reverse orientation. If $w_{1}, w_{2} \in V(C)$, then $w_{1} \vec{C} w_{2}$ denotes the consecutive vertices of $C$ from $w_{1}$ to $w_{2}$ in the direction specified by $\vec{C}$. The same vertices in the reverse order are given by $w_{2} \overleftarrow{C} w_{1}$. We will consider $w_{1} \vec{C} w_{2}$ and $w_{2} \overleftarrow{C} w_{1}$ both as paths and as vertex sets. If $w \in V(C)$, then $w^{+}$denotes the successor of $w$ on $\vec{C}$, and $w^{-}$denotes its predecessor. The vertices $\left(w^{+}\right)^{+}$and $\left(w^{-}\right)^{-}$are written briefly by $w^{++}$and $w^{--}$, respectively. Similar notation as described above for a cycle is also used for a path.

We prove now the following lemma.
Lemma 11. Let $G=S(I \cup K, E)$ be a Burkard-Hammer graph with $|I| \geq 7$ and $\delta(G)=|I|-4$. Then $G$ is a maximal nonhamiltonian split graph if and only if $G^{\prime}=G\left[I \cup N_{G}(I)\right]$ is a maximal nonhamiltonian split graph.

Proof. Let $G=S(I \cup K, E)$ be a Burkard-Hammer graph with $|I| \geq 7$ and $\delta(G)=|I|-4$. Then by Lemma $6\left|N_{G}(I)\right|>|I|$.

First suppose that $G$ is a maximal nonhamiltonian split graph. Then by Lemma 4 it is not difficult to see that $G^{\prime}=G\left[I \cup N_{G}(I)\right]$ is a maximal nonhamiltonian split graph.

Conversely, suppose that $G^{\prime}=G\left[I \cup N_{G}(I)\right]=S\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ where $I^{\prime}=I$ and $K^{\prime}=N_{G}(I)$ is a maximal nonhamiltonian split graph. By Lemma $4, G$ is nonhamiltonian. So it remains to prove that for any $u \in I$
and any $v \in K$ with $u v \notin E$ the graph $H=G+u v$ is hamiltonian. We consider separately two cases.

Case 1. $v \in N_{G}(I)$.
Then $u \in I^{\prime}, v \in K^{\prime}$ and $u v \notin E^{\prime}$. Therefore, $H^{\prime}=G^{\prime}+u v$ has a Hamilton cycle because $G^{\prime}$ is a maximal nonhamiltonian split graph. Since $H^{\prime}=$ $H\left[I \cup N_{H}(I)\right]$, by Lemma $4 H$ also has a Hamilton cycle.

Case 2. $v \in K \backslash N_{G}(I)$.
First assume that $u$ is adjacent in $G$ to all vertices of $N_{G}(I)$. Then we consider the graph $G-u$ which is a Burkard-Hammer graph $S\left(I_{u} \cup K, E_{u}\right)$ with $I_{u}=I \backslash\{u\}$ and $E_{u}=E \backslash\left\{u w \mid w \in N_{G}(I)\right\}$. Since $\left|I_{u}\right| \neq 5$ and $\delta(G-u) \geq\left|I_{u}\right|-3$, by Theorem $3, G-u$ has a Hamilton cycle $C_{u}$. We fix an orientation for $C_{u}$. Since $v \in K \backslash N_{G}(I)$ both $v^{-}$and $v^{+}$are in $K$. By going from $v$ along $C_{u}$ in the direction specified by $\overrightarrow{C_{u}}$ we can find a vertex $w$ such that $w \in N_{G}(I)$ but $w^{-} \in K \backslash N_{G}(I)$. Then $w$ is adjacent in $G$ to $u$ by our assumption. Therefore, $C=v u w \overrightarrow{C_{u}} v$ is a Hamilton cycle of $G+u v=H$ if $w=v^{+}$and $C=v u w \overrightarrow{C_{u}} v^{-} w^{-} \overleftarrow{C_{u}} v$ is a Hamilton cycle of $G+u v=H$ if $w \neq v^{+}$.

Now assume that there is a vertex $v_{1} \in N_{G}(I)$ such that $u$ is not adjacent in $G$ to $v_{1}$. By Case $1, G+u v_{1}$ has a Hamilton cycle $C^{\prime}$ that must contain the edge $u v_{1}$ because $G$ is nonhamiltonian. We fix an orientation for $C^{\prime}$ so that $u^{+}=v_{1}$. Since $v \in K \backslash N_{G}(I)$, we have $v^{+} \in K$. Therefore, $C=u v \overleftarrow{C^{\prime}} v_{1} v^{+} \overrightarrow{C^{\prime}} u$ is a Hamilton cycle of $G+u v=H$.

The proof of the lemma is complete.

## 3. Classification for Case $|I| \neq 6,7$

First of all, we prove the following Lemmas 12 and 13.
Lemma 12. Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian BurkardHammer graph with $m=|I| \neq 6, n=|K|$ and $\delta(G)=|I|-4$. Then $|I| \geq 7$ and $G$ possesses a Hamilton path $P$ with the endvertices $u_{1}$ and $v_{n}$ such that $u_{1} \in I, v_{n} \in B_{3}$ and if $\vec{P}=u_{1} \ldots v_{n}$ is the path $P$ with the orientation from $u_{1}$ to $v_{n}$, then $v_{n}^{-} \in I$.

Proof. Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian BurkardHammer graph with $m=|I| \neq 6, n=|K|$ and $\delta(G)=|I|-4$. By Lemma 6,
for any vertex $u \in I$ we have $|N(u)|>|\{u\}|=1$. So, $\delta(G)=|I|-4 \geq 2$ and therefore we must have $|I| \geq 6$. This implies that $|I| \geq 7$ because we assume that $|I| \neq 6$. Now by Theorem $9, B_{4}=B_{5}=\cdots=B_{m-1}=\emptyset$ but $B_{3} \neq \emptyset$. Choose a vertex $v_{n} \in B_{3}$. Since $m=|I| \geq 7$ we can find a vertex $u_{1} \in I \backslash N_{I}\left(v_{n}\right)$.

Since $u_{1} v_{n} \notin E$ and $G$ is a maximal nonhamiltonian split graph, $G+u_{1} v_{n}$ has a Hamilton cycle $D$ which must contain the edge $u_{1} v_{n}$. So $P=D-u_{1} v_{n}$ is a Hamilton path in $G$ with $u_{1}$ and $v_{n}$ its endvertices.

Let $\vec{P}=u_{1} \ldots v_{n}$ be the path $P$ with the orientation from $u_{1}$ to $v_{n}$. If $v_{n}^{-} \in I$, then $P$ already is a Hamilton path required in the lemma. So we suppose that in $\vec{P}$ the vertex $v_{n}^{-}$is in $K$. Since $\left|N_{I}\left(v_{n}\right)\right|=3$, there exists $u \in N_{I}\left(v_{n}\right)$. Then $\overrightarrow{P^{\prime}}=u_{1} \vec{P} u^{-} v_{n}^{-} \overleftrightarrow{P} u v_{n}$ is also a Hamilton path of $G$ with the endvertices $u_{1}$ and $v_{n}$. But in $\overrightarrow{P^{\prime}}$ the predecessor of $v_{n}$ is $u$ which is in $I$. Thus, the path $P^{\prime}$ is a Hamilton path required in the lemma. The proof of Lemma 12 is complete.

Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian Burkard-Hammer graph with $m=|I| \neq 6, n=|K|$ and $\delta(G)=|I|-4$ and let $P, u_{1}$ and $v_{n}$ be as in Lemma 12. Set $\overline{N_{I}\left(v_{n}\right)}=I \backslash N_{I}\left(v_{n}\right)$. Then we have $\left|\overline{N_{I}\left(v_{n}\right)}\right|=$ $|I|-\left|N_{I}\left(v_{n}\right)\right|=m-3$. Let

$$
u_{1}, u_{2}, \ldots, u_{m-3}
$$

be the vertices of $\overline{N_{I}\left(v_{n}\right)}$ occurring on $\vec{P}$ in the order of their indices. Set

$$
P_{1}=u_{1} \vec{P} u_{2}^{-}, P_{2}=u_{2} \vec{P} u_{3}^{-}, \ldots, P_{m-4}=u_{m-4} \vec{P} u_{m-3}^{-}, P_{m-3}=u_{m-3} \vec{P} v_{n}
$$

Then these subpaths of $P$ appear on $\vec{P}$ in the order of their indices. Because of this we will call the subpath $P_{j}, j=1, \ldots, m-3$, the $j$-th subpath of $P$. If $v$ is a neighbour of $u_{1}$ and $v^{-}$is adjacent to $v_{n}$, then $C=u_{1} \vec{P} v^{-} v_{n} \overleftarrow{P} v u_{1}$ is a Hamilton cycle of $G$, a contradiction. Thus, $v^{-}$is not adjacent to $v_{n}$, i.e., $v^{-} \in \overline{N_{I}\left(v_{n}\right)}=\left\{u_{1}, \ldots, u_{m-3}\right\}$. Hence, $v \in\left\{u_{1}^{+}, \ldots, u_{m-3}^{+}\right\}$. We have proved the following lemma.

Lemma 13. $N\left(u_{1}\right) \subseteq\left\{u_{1}^{+}, \ldots, u_{m-3}^{+}\right\}$.
The following Lemmas 14, 15 and 16 help us to know the structure of $G$ in more detail.

Lemma 14. (i) If $v \in N\left(u_{j}\right) \cap V\left(P_{i}\right)$ with $j \leq i$ where $i \in\{1,2, \ldots, m-3\}$, $j \in\{2, \ldots, m-3\}$ and $u_{j}^{+} \in N\left(u_{1}\right)$, then $v^{-} \notin N\left(v_{n}\right)$;
(ii) If $v \in N\left(u_{j}\right) \cap V\left(P_{i}\right)$ with $j>i$ where $i \in\{1,2, \ldots, m-3\}, j \in$ $\{2, \ldots, m-3\}$ and $u_{j}^{+} \in N\left(u_{1}\right)$, then $v^{+} \notin N\left(v_{n}\right)$.

Proof. First assume that $v \in N\left(u_{j}\right) \cap V\left(P_{i}\right)$ with $j \leq i$ where $i \in\{1,2, \ldots$, $m-3\}, j \in\{2, \ldots, m-3\}$ and $u_{j}^{+} \in N\left(u_{1}\right)$. If $v=u_{j}^{+}$then $v^{-}=u_{j} \notin N\left(v_{n}\right)$. If $v \neq u_{j}^{+}$and $v^{-} \in N\left(v_{n}\right)$, then $C=u_{j} \overleftarrow{P} u_{1} u_{j}^{+} \vec{P} v^{-} v_{n} \overleftarrow{P} v u_{j}$ is a Hamilton cycle of $G$, a contradiction.

Now assume that $v \in N\left(u_{j}\right) \cap V\left(P_{i}\right)$ with $j>i$ where $i \in\{1,2, \ldots$, $m-3\}, j \in\{2, \ldots, m-3\}$ and $u_{j}^{+} \in N\left(u_{1}\right)$. If $v^{+} \in N\left(v_{n}\right)$, then $C=u_{j} v \overleftarrow{P} u_{1} u_{j}^{+} \vec{P} v_{n} v^{+} \vec{P} u_{j}$ is a Hamilton cycle of $G$, contradicting the nonhamiltonicity of $G$ again. This completes the proof of Lemma 14.

By Lemma $14, \overline{N_{I}\left(v_{n}\right)}=\left\{u_{1}, u_{2}, \ldots, u_{m-3}\right\}$ and $\delta(G)=|I|-4$, we have immediately the following Lemma 15.

Lemma 15. If $u_{j}^{+} \in N\left(u_{1}\right)$ for $j \in\{2, \ldots, m-3\}$, then $N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}\right.$, $\left.\ldots, u_{j}^{-}, u_{j}^{+}, u_{j+1}^{+}, \ldots, u_{m-3}^{+}\right\}$.

Lemma 16. If integers $a$ and $b$ with $2 \leq a<b \leq m-3$ are such that $u_{a}^{+} \in N\left(u_{1}\right), u_{a}^{-}$is adjacent to $u_{b}$ and $u_{a}$ is adjacent to $u_{b}^{+}$, then both $u_{a-1}^{+}=u_{a}^{-}$and $u_{a}^{+}=u_{a+1}^{-}$hold.
$\boldsymbol{P r o o f}$. Suppose, on the contrary, that $u_{a-1}^{+} \neq u_{a}^{-}$. Then $u_{a}^{--} \notin \overline{N_{I}\left(v_{n}\right)}$ and therefore it is adjacent to $v_{n}$. Further, since $m \geq 7$, we have $\operatorname{deg}\left(u_{j}\right) \geq$ $m-4 \geq 3$ for every $j \in\{1,2, \ldots, m-3\}$. Therefore, since $u_{1}$ is adjacent to $u_{a}^{+}, C=u_{1} u_{a}^{+} \vec{P} u_{b} u_{a}^{-} u_{a} u_{b}^{+} \vec{P} v_{n} u_{a}^{--} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, a contradiction. Similarly, if $u_{a}^{+} \neq u_{a+1}^{-}$, then $u_{a}^{++}$is adjacent to $v_{n}$. So, since $\operatorname{deg}\left(u_{j}\right) \geq m-4 \geq 3$ for every $j \in\{1,2, \ldots, m-3\}$ and $u_{1}$ is adjacent to $u_{a}^{+}$, $C=u_{1} u_{a}^{+} u_{a} u_{b}^{+} \vec{P} v_{n} u_{a}^{++} \vec{P} u_{b} u_{a}^{-} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, contradicting the nonhamiltonicity of $G$ again. The proof of Lemma 16 is complete.
Now we prove the following two Lemmas 17 and 18 which are crucial for the classification.

Lemma 17. Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian BurkardHammer graph with $|I| \neq 6,7$ and $\delta(G)=|I|-4$. Then $|I| \geq 8$ and $G$ possesses a vertex $v \in B_{3}$ such that some vertex $u \in \overline{N_{I}(v)}=I \backslash N_{I}(v)$ has $\operatorname{deg}(u) \geq|I|-3$.

Proof. Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian BurkardHammer graph with $m=|I| \notin\{6,7\}, n=|K|$ and $\delta(G)=|I|-4$. By Lemma $12, m=|I| \geq 8$ and $G$ possesses a Hamilton path $P$ with the endvertices $u_{1}$ and $v_{n}$ such that $u_{1} \in I, v_{n} \in B_{3}$ and if $\vec{P}=u_{1} \ldots v_{n}$ be the path $P$ with the orientation from $u_{1}$ to $v_{n}$ then $v_{n}^{-} \in I$.

Suppose, on the contrary, that $G$ does not satisfy the last conclusion of the lemma. This means that for any vertex $v \in B_{3}$ and for any vertex $u \in \overline{N_{I}(v)}=I \backslash \underline{N_{I}(v)}$, we have $\operatorname{deg}(u) \leq m-4$. But $\delta(G)=m-4$. So for any vertex $u \in \overline{N_{I}(v)}, \operatorname{deg}(u)=m-4$.

We already noticed before Lemma 13 that $\left|\overline{N_{I}\left(v_{n}\right)}\right|=m-3$. There we also denoted the vertices of $\overline{N_{I}\left(v_{n}\right)}$ in the order of their appearing on $\vec{P}$ by $u_{1}, u_{2}, \ldots, u_{m-3}$ and defined the subpaths $P_{1}, P_{2}, \ldots, P_{m-3}$ of $P$.

By Lemma $13, N\left(u_{1}\right) \subseteq\left\{u_{1}^{+}, \ldots, u_{m-3}^{+}\right\}$.
From this and $\operatorname{deg}\left(u_{1}\right)=m-4$ it follows that there exists $r_{0} \in\{2,3, \ldots$, $m-3\}$ such that vertices $u_{j}^{+}$with $j \in\{1,2, \ldots, m-3\} \backslash\left\{r_{0}\right\}$ and only these vertices are neighbours of $u_{1}$.

By Lemma 15 we have

$$
N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, u_{j+1}^{+}, \ldots, u_{m-3}^{+}\right\}
$$

for any $j \in\{2, \ldots, m-3\} \backslash\left\{r_{0}\right\}$.
Claim 3.1. (i) If $3 \leq r_{0} \leq m-4$, then either some of $u_{2}, \ldots, u_{r_{0}-1}$ is adjacent to $u_{r_{0}}^{+}$or some of $u_{r_{0}+1}, \ldots, u_{m-3}$ is adjacent to $u_{r_{0}}^{-}$.
(ii) If $r_{0}=2$, then some of $u_{3}, \ldots, u_{m-3}$ is adjacent to $u_{2}^{-}$.
(iii) If $r_{0}=m-3$, then some of $u_{2}, \ldots, u_{m-4}$ is adjacent to $u_{m-3}^{+}$.

Proof. First we prove the assertion (i). So we assume now that $3 \leq r_{0} \leq$ $m-4$. Suppose, on the contrary, that $u_{r_{0}}^{+} \notin N\left(u_{j}\right)$ for every $j \in\{2, \ldots$, $\left.r_{0}-1\right\}$ and $u_{r_{0}}^{-} \notin N\left(u_{j}\right)$ for every $j \in\left\{r_{0}+1, \ldots, m-3\right\}$. Then by Lemma 13, Lemma 15 and $\operatorname{deg}\left(u_{j}\right)=m-4$, we have
$N\left(u_{1}\right)=\left\{u_{1}^{+}, \ldots, u_{r_{0}-1}^{+}, u_{r_{0}+1}^{+}, \ldots, u_{m-3}^{+}\right\}$,
$N\left(u_{j}\right)=\left\{u_{2}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{r_{0}-1}^{+}, u_{r_{0}+1}^{+}, \ldots, u_{m-3}^{+}\right\}$for $2 \leq j \leq r_{0}-1$ and
$N\left(u_{j}\right)=\left\{u_{2}^{-}, \ldots, u_{r_{0}-1}^{-}, u_{r_{0}+1}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-3}^{+}\right\}$for $r_{0}+1 \leq j \leq$ $m-3$.

If all equalities $u_{1}^{+}=u_{2}^{-}, \ldots, u_{r_{0}-2}^{+}=u_{r_{0}-1}^{-}, u_{r_{0}+1}^{+}=u_{r_{0}+2}^{-}, \ldots, u_{m-4}^{+}=$ $u_{m-3}^{-}$hold, then all vertices $u_{1}, \ldots, u_{r_{0}-1}, u_{r_{0}+1}, \ldots, u_{m-3}$ are adjacent to
each of $u_{1}^{+}, \ldots, u_{r_{0}-2}^{+}, u_{r_{0}+1}^{+}, \ldots, u_{m-3}^{+}$. This implies that each of vertices $u_{1}^{+}, \ldots, u_{r_{0}-2}^{+}, u_{r_{0}+1}^{+}, \ldots, u_{m-3}^{+}$has at least 4 neighbours in $I$ because $m \geq 8$. Therefore, by Theorem 9 , each of these vertices is adjacent to all vertices in $I$. In particular, they are adjacent to $u_{r_{0}}$. But $u_{r_{0}}^{-}$and $u_{r_{0}}^{+}$are also neighbours of $u_{r_{0}}$. So $\operatorname{deg}\left(u_{r_{0}}\right) \geq m-3$, contradicting the assumption about $G$.

Thus, there exists the number $j_{0} \in\{1, \ldots, m-4\} \backslash\left\{r_{0}-1, r_{0}\right\}$ such that $u_{j_{0}}^{+} \neq u_{j_{0}+1}^{-}$. So both $u_{j_{0}}^{++}$and $u_{j_{0}+1}^{--}$are adjacent to $v_{n}$. If $j_{0} \neq 1$, then $C=u_{j_{0}+1} u_{j_{0}}^{-} \overleftarrow{P} u_{1} u_{j_{0}}^{+} u_{j_{0}} u_{j_{0}+1}^{+} \vec{P} v_{n} u_{j_{0}}^{++} \vec{P} u_{j_{0}+1}$ is a Hamilton cycle of $G$. If $j_{0}=1$, then $C=u_{1} u_{2}^{+} \vec{P} u_{m-3} u_{2}^{-} u_{2} u_{m-3}^{+} \vec{P} v_{n} u_{2}^{--\overleftarrow{P}} u_{1}$ is a Hamilton cycle of $G$. We have got a contradiction in all possible situations. So the assertion (i) of the claim must be true.

Assertions (ii) and (iii) can be proved by similar arguments. We leave it to the reader to carry out the proofs of (ii) and (iii) in detail.

The proof of Claim 3.1 is complete.
Now if $r_{0}=m-3$, then $u_{m-3}^{+}$must be adjacent to some vertex $u_{j}$ with $j \in\{2, \ldots, m-4\}$ by Claim 3.1. Let $P^{\prime}=u_{j} \overleftarrow{P} u_{1} u_{j}^{+} \vec{P} v_{n}$. Then $\underline{P^{\prime} \text { is a }}$ Hamilton path of $G$ with the endvertices $u_{j}$ and $v_{n}$ and all vertices of $\overline{N_{I}\left(v_{n}\right)}$ are in $u_{j} \overrightarrow{P^{\prime}} u_{m-3}^{+}$. Moreover, in $P^{\prime}$ the vertex $u_{j}$ is adjacent to $u_{m-3}^{+}$. So by considering $u_{j}$ instead of $u_{1}$ and $P^{\prime}$ instead of $P$, if necessary, we may assume that

$$
2 \leq r_{0} \leq m-4
$$

Claim 3.2. There exists $j_{0} \in\{1,2, \ldots, m-4\}$ such that $u_{j_{0}}^{+} \neq u_{j_{0}+1}^{-}$.
Proof. Suppose, on the contrary, that $u_{j}^{+}=u_{j+1}^{-}$for every $j \in\{1,2, \ldots$, $m-4\}$. If $N\left(u_{r_{0}}\right) \cap u_{m-3}^{++} \vec{P} v_{n}=\emptyset$, then by Lemmas 13 and 15 we have $N\left(\left\{u_{1}, u_{2}, \ldots, u_{m-3}\right\}\right)=\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}\right\}$. It follows that $\mid N\left(\left\{u_{1}, u_{2}\right.\right.$, $\left.\left.\ldots, u_{m-3}\right\}\right)\left|=\left|\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}\right\}\right|=\left|\left\{u_{1}, u_{2}, \ldots, u_{m-3}\right\}\right|\right.$, contradicting Lemma 6. Thus, $N\left(u_{r_{0}}\right) \cap u_{m-3}^{++} \vec{P} v_{n} \neq \emptyset$. Let $w$ be a vertex of $N\left(u_{r_{0}}\right) \cap$ $u_{m-3}^{++} \vec{P} v_{n}$. Then $w^{-} \notin \overline{N_{I}\left(v_{n}\right)}$ and therefore $w^{-}$is adjacent to $v_{n}$.

By Claim 3.1, if $3 \leq r_{0} \leq m-4$ then either some of $u_{2}, \ldots, u_{r_{0}-1}$ is adjacent to $u_{r_{0}}^{+}$or some of $u_{r_{0}+1}, \ldots, u_{m-3}$ is adjacent to $u_{r_{0}}^{-}$and if $r_{0}=2$ then some of $u_{3}, \ldots, u_{m-3}$ is adjacent to $u_{2}^{-}$. If some of $u_{2}, \ldots, u_{r_{0}-1}$ is adjacent to $u_{r_{0}}^{+}$, say $u_{i_{0}}$, then $C=u_{1} \vec{P} u_{i_{0}} u_{r_{0}}^{+} \vec{P} w^{-} v_{n} \overleftarrow{P} w u_{r_{0}} \overleftarrow{P} u_{i_{0}}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction. If some of $u_{r_{0}+1}, \ldots, u_{m-3}$ is adjacent
to $u_{r_{0}}^{-}$, say $u_{j_{0}}$, then $C=u_{1} \vec{P} u_{r_{0}}^{-} u_{j_{0}} \overleftarrow{P} u_{r_{0}} w \vec{P} v_{n} w^{-} \overleftarrow{P} u_{j_{0}}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction again.

Thus, there must exist a subscript $j_{0} \in\{1,2, \ldots, m-4\}$ such that $u_{j_{0}}^{+} \neq u_{j_{0}+1}^{-}$.
Claim 3.3. $u_{m-3}^{++} \in I$.
Proof. By Claim 3.2 there exists $j_{0} \in\{1,2, \ldots, m-4\}$ such that $u_{j_{0}}^{+} \neq$ $u_{j_{0}+1}^{-}$. Then $u_{j_{0}+1}^{--}$is adjacent to $v_{n}$. Therefore, if $u_{m-3}^{++} \in K$, then $C=$ $u_{1} u_{m-3}^{+} \overleftarrow{P} u_{j_{0}+1}^{-} u_{m-3}^{++} \vec{P} v_{n} u_{j_{0}+1}^{--} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, a contradiction.

Claim 3.4. $u_{m-3}^{+}$is adjacent to all vertices of $G$.
Proof. Assume that $u_{m-3}^{+}$is not adjacent to $u_{j}$ for each $j \in\{2,3, \ldots$, $m-4\}$. Then by Lemma 15 and $\operatorname{deg}\left(u_{j}\right)=m-4$ we have

$$
N\left(u_{j}\right)=\left\{u_{2}^{-}, u_{3}^{-} \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-4}^{+}\right\}
$$

for every $j \in\{2,3, \ldots, m-4\} \backslash\left\{r_{0}\right\}$.
If $r_{0}=m-4$, then by applying Lemma 16 for $a=2, \ldots, m-6$ and $b=m-5$ we get $u_{1}^{+}=u_{2}^{-}, \ldots, u_{m-6}^{+}=u_{m-5}^{-}$. In particular, since $m \geq 8$, we always have $u_{1}^{+}=u_{2}^{-}$and $u_{2}^{+}=u_{3}^{-}$. Suppose that $u_{m-5}^{+} \neq u_{m-4}^{-}$. Then $u_{m-5}^{++}$is adjacent to $v_{n}$. Now if $u_{m-3}$ is adjacent to $u_{2}^{-}=u_{1}^{+}$, then $C=u_{1} u_{m-5}^{+} \overleftarrow{P} u_{2}^{-} u_{m-3} \overleftarrow{P} u_{m-5}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction. Thus, $u_{m-3}$ is not adjacent to $u_{2}^{-}$. Together with Lemma 15 and $\operatorname{deg}\left(u_{m-3}\right)=m-4$, this implies that $u_{m-3}$ is adjacent to $u_{3}^{-}=u_{2}^{+}$. Therefore, $C=u_{1} u_{1}^{+} u_{2} u_{m-5}^{+} \overleftarrow{P} u_{3}^{-} u_{m-3} \overleftarrow{P} u_{m-5}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction again. Thus, we also have $u_{m-5}^{+}=u_{m-4}^{-}$if $r_{0}=m-4$.

If $r_{0}=m-5$, then by applying Lemma 16 for $a=2, \ldots, m-6$ and $b=m-4$ we get $u_{1}^{+}=u_{2}^{-}, \ldots, u_{m-6}^{+}=u_{m-5}^{-}$. In particular, we have $u_{1}^{+}=u_{2}^{-}$. Since $m \geq 8$, we have $r_{0}=m-5 \geq 3$. So $u_{1}$ is adjacent to $u_{2}^{+}$. Now if $u_{m-5}^{+} \neq u_{m-4}^{-}$, then $u_{m-5}^{++}$is adjacent to $v_{n}$ and therefore $C=u_{1} u_{2}^{+} \vec{P} u_{m-5}^{+} u_{2} u_{2}^{-} u_{m-4} \overleftarrow{P} u_{m-5}^{++} v_{n} \overleftarrow{P} u_{m-4}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction. Thus, we also have $u_{m-5}^{+}=u_{m-4}^{-}$if $r_{0}=m-5$.

If $r_{0}=2$, then by applying Lemma 16 for $a=3, \ldots, m-5$ and $b=m-4$ we get $u_{2}^{+}=u_{3}^{-}, \ldots, u_{m-5}^{+}=u_{m-4}^{-}$. In particular, we have $u_{3}^{+}=u_{4}^{-}$. Since $m \geq 8$, we have $m-4 \geq 4$. Hence, $u_{4}$ is adjacent to $u_{2}^{-}$. Now if $u_{1}^{+} \neq u_{2}^{-}$, then $u_{2}^{--}$is adjacent to $v_{n}$ and therefore $C=u_{1} u_{3}^{+} u_{3} u_{2}^{+} u_{2} u_{2}^{-} u_{4} \vec{P} v_{n} u_{2}^{--} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, a contradiction. Thus, we also have $u_{1}^{+}=u_{2}^{-}$if $r_{0}=2$.

If $2<r_{0}<m-5$, then by applying Lemma 16 for $a=2, \ldots, r_{0}-1$ and $b=m-4$ we get $u_{1}^{+}=u_{2}^{-}, \ldots, u_{r_{0}-1}^{+}=u_{r_{0}}^{-}$and by applying Lemma 16 for $a=r_{0}+1, \ldots, m-5$ and $b=m-4$ we also get $u_{r_{0}}^{+}=u_{r_{0}+1}^{-}, \ldots, u_{m-5}^{+}=u_{m-4}^{-}$.

Thus, we always have $u_{1}^{+}=u_{2}^{-}, \ldots, u_{m-5}^{+}=u_{m-4}^{-}$for any value of $r_{0}$. By Claim 3.2, we must have $u_{m-4}^{+} \neq u_{m-3}^{-}$. Hence, $u_{m-4}^{++}$is adjacent to $v_{n}$. Since $m \geq 8, \operatorname{deg}\left(u_{m-3}\right)=m-4 \geq 4$. It follows that there exists $j_{0} \in\{3, \ldots, m-4\}$ such that $u_{m-3}$ is adjacent to $u_{j_{0}}^{-}$because by Lemma $15 N\left(u_{m-3}\right) \subseteq\left\{u_{2}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\}$. Therefore, $C=$ $u_{1} u_{m-3}^{+} \vec{P} v_{n} u_{m-4}^{++} \vec{P} u_{m-3} u_{j_{0}}^{-} \vec{P} u_{m-4}^{+} u_{j_{0}-1} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$. This final contradiction shows the assumption that $u_{m-3}^{+}$is not adjacent to $u_{j}$ for each $j \in\{2,3, \ldots, m-4\}$ is false.

So $u_{m-3}^{+}$must be adjacent to a vertex $u_{j}$ with $j \in\{2,3, \ldots, m-4\}$. By Claim 3.3, $u_{m-3}^{++}$is in $I$. Hence, $\left|N_{I}\left(u_{m-3}^{+}\right)\right| \geq 4$ because $u_{1}, u_{j}, u_{m-3}$ and $u_{m-3}^{++}$are in $N_{I}\left(u_{m-3}^{+}\right)$. By Theorem $9, u_{m-3}^{+}$must be adjacent to all vertices of $G$.

The proof of Claim 3.4 is complete.
Claim 3.5. $u_{m-4}^{+}=u_{m-3}^{-}$.
Proof. Suppose, on the contrary, that $u_{m-4}^{+} \neq u_{m-3}^{-}$. Then $u_{m-4}^{++} \notin \overline{N_{I}\left(v_{n}\right)}$ and therefore it is adjacent to $v_{n}$. Further, since $m \geq 8$, $\operatorname{deg}\left(u_{m-3}\right)=$ $m-4 \geq 4$. Together with $N\left(u_{m-3}\right) \subseteq\left\{u_{2}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\}$, it follows that there exists $s_{0} \in\{2, \ldots, m-4\}$ such that $N\left(u_{m-3}\right)=\left\{u_{2}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\} \backslash$ $\left\{u_{s_{0}}^{-}\right\}$. Now if $r_{0}<m-4$, then by taking $x \in\{2, \ldots, m-4\}$ with $x \neq s_{0}$ we have $C=u_{1} u_{m-4}^{+} \overleftarrow{P} u_{x} u_{m-3}^{+} \vec{P} v_{n} u_{m-4}^{++} \vec{P} u_{m-3} u_{x}^{-} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, a contradiction.

Thus, $r_{0}=m-4$ must hold. Now we consider separately the following cases.

Case 1. There exists a vertex $u_{t} \in\left\{u_{2}, \ldots, u_{m-5}\right\}$ adjacent to $u_{m-4}^{+}$. We have $N\left(u_{m-3}\right)=\left\{u_{2}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\} \backslash\left\{u_{s_{0}}^{-}\right\}$. Since $m \geq 8$ there exists $x \in\{2, \ldots, m-4\} \backslash\left\{t, s_{0}\right\}$. Then $u_{x}^{-}$is adjacent to $u_{m-3}$ because $x \neq s_{0}$. If $2 \leq x \leq t-1$, then $C=u_{t} u_{m-4}^{+} \overleftarrow{P} u_{t}^{+} u_{1} \vec{P} u_{x}^{-} u_{m-3} \overleftarrow{P} u_{m-4}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+}$ $u_{x} \vec{P} u_{t}$ is a Hamilton cycle of $G$. If $t+1 \leq x \leq m-4$, then $C=$ $u_{t} u_{m-4}^{+} \overleftarrow{P} u_{x} u_{m-3}^{+} \vec{P} v_{n} u_{m-4}^{++} \vec{P} u_{m-3} u_{x}^{-} \overleftarrow{P} u_{t}^{+} u_{1} \vec{P} u_{t}$ is a Hamilton cycle of $G$ We have got a contradiction in all possible situations. Thus, this case cannot occur.

Case 2. No vertices $u_{j} \in\left\{u_{2}, \ldots, u_{m-5}\right\}$ are adjacent to $u_{m-4}^{+}$.
In this case, for $j \in\{2, \ldots, m-5\}$, since $\operatorname{deg}\left(u_{j}\right)=m-4$ and $N\left(u_{j}\right) \subseteq$ $\left\{u_{2}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-3}^{+}\right\}$by Lemma 15 , we have

$$
N\left(u_{j}\right)=\left\{u_{2}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-5}^{+}, u_{m-3}^{+}\right\} .
$$

By applying Lemma 16 for $a=2, \ldots, m-6$ and $b=m-5$ we get $u_{1}^{+}=$ $u_{2}^{-}, \ldots, u_{m-6}^{+}=u_{m-5}^{-}$.

We show now that $u_{m-5}^{+}=u_{m-4}^{-}$also holds. We have $N\left(u_{m-3}\right)=$ $\left\{u_{2}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\} \backslash\left\{u_{s_{0}}^{-}\right\}$. If $s_{0} \neq m-5$, then by applying Lemma 16 for $a=m-5$ and $b=m-3$ we get $u_{m-5}^{+}=u_{m-4}^{-}$. So we may assume now that $s_{0}=m-5$. With this assumption we have $u_{m-3}$ is adjacent to $u_{2}^{-}=u_{1}^{+}$. Therefore, if $u_{m-5}^{+} \neq u_{m-4}^{-}$, then $u_{m-5}^{++}$is adjacent to $v_{n}$ and therefore $C=u_{1} u_{m-5}^{+} \overleftarrow{P} u_{2}^{-} u_{m-3} \overleftarrow{P} u_{m-5}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction. So, $u_{m-5}^{+}=u_{m-4}^{-}$always holds.

Thus, $N\left(u_{j}\right)=\left\{u_{2}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-5}^{+}, u_{m-3}^{+}\right\}=\left\{u_{1}^{+}, \ldots, u_{m-5}^{+}, u_{m-3}^{+}\right\}$ for $j \in\{2, \ldots, m-5\}$. Hence, each of the vertices $u_{1}, u_{2}, \ldots, u_{m-5}$ is adjacent to each of the vertices $u_{1}^{+}, \ldots, u_{m-5}^{+}$. Since $N\left(u_{m-3}\right)=\left\{u_{2}^{-}, \ldots\right.$, $\left.u_{m-3}^{-}, u_{m-3}^{+}\right\} \backslash\left\{u_{s_{0}}^{-}\right\}$, the vertex $u_{m-3}$ is adjacent to each of the vertices $u_{1}^{+}, \ldots, u_{s_{0}-2}^{+}, u_{s_{0}}^{+}, \ldots, u_{m-5}^{+}$. It follows that $\left|N_{I}\left(u_{1}^{+}\right)\right| \geq 4, \ldots,\left|N_{I}\left(u_{s_{0}-2}^{+}\right)\right| \geq$ $4,\left|N_{I}\left(u_{s_{0}}^{+}\right)\right| \geq 4, \ldots,\left|N_{I}\left(u_{m-5}^{+}\right)\right| \geq 4$. By Theorem $9, u_{1}^{+}, \ldots, u_{s_{0}-2}^{+}, u_{s_{0}}^{+}, \ldots$, $u_{m-5}^{+}$are adjacent to all vertices of $G$. In particular, they are adjacent to $u_{m-4}$. Further, since $m \geq 8$ and all $u_{1}, \ldots, u_{m-5}$ are adjacent to $u_{s_{0}-1}^{+}$, we have $\left|N_{I}\left(u_{s_{0}-1}^{+}\right)\right| \geq 3$. Now if $\left|N_{I}\left(u_{s_{0}-1}^{+}\right)\right|>3$, then again by Theorem 9 the vertex $u_{s_{0}-1}^{+}$is adjacent to all vertices of $G$. In particular, it is adjacent to $u_{m-4}$. So $\left\{u_{1}^{+}, \ldots, u_{m-3}^{+}\right\} \subseteq N\left(u_{m-4}\right)$. (We recall that $u_{m-3}^{+}$is adjacent to all vertices of $G$ by Claim 3.4.) Therefore $\operatorname{deg}\left(u_{m-4}\right) \geq m-3$, contradicting our assumption about $G$. Thus, $\left|N_{I}\left(u_{s_{0}-1}^{+}\right)\right|=3$. Since $\left\{u_{1}, \ldots, u_{m-5}\right\} \subseteq$ $N_{I}\left(u_{s_{0}-1}^{+}\right)$, this can happen only if $m=8$ and $N_{I}\left(u_{s_{0}-1}^{+}\right)=\left\{u_{1}, u_{2}, u_{3}\right\}$. Hence, $u_{s_{0}-1}^{+} \in B_{3}$. Let $u \in N_{I}\left(v_{n}\right)$ be such that $u \neq u_{m-3}^{++}$. Then $u \in \overline{N_{I}\left(u_{s_{0}-1}^{+}\right)}$and therefore $\operatorname{deg}(u)=m-4$ by our assumption about $G$. On the other hand, since $u_{1}^{+}=u_{2}^{-}, \ldots, u_{m-5}^{+}=u_{m-4}^{-}$as we have shown above, $u$ is either in $u_{m-4}^{+} \vec{P} u_{m-3}^{-}$or, if $u_{m-3}^{+++} \neq v_{n}$, in $u_{m-3}^{+++} \vec{P} v_{n}^{-}$. So, both $u^{-}$and $u^{+}$are different from $u_{1}^{+}, \ldots, u_{s_{0}-2}^{+}, u_{s_{0}}^{+}, \ldots, u_{m-5}^{+}$and $u_{m-3}^{+}$. But all the vertices $u_{1}^{+}, \ldots, u_{s_{0}-2}^{+}, u_{s_{0}}^{+}, \ldots, u_{m-5}^{+}, u_{m-3}^{+}, u^{-}$and $u^{+}$are in $N(u)$. (Recall that $u_{1}^{+}, \ldots, u_{s_{0}-2}^{+}, u_{s_{0}}^{+}, \ldots, u_{m-5}^{+}$and $u_{m-3}^{+}$are adjacent to all vertices of $G$.) Hence, $\operatorname{deg}(u) \geq m-3$, contradicting $\operatorname{deg}(u)=m-4$ obtained before.

Thus, Case 2 also cannot occur. This means our assumption that $u_{m-4}^{+} \neq$ $u_{m-3}^{-}$is false.

The proof of Claim 3.5 is complete.
Claim 3.6. $u_{1}^{+}=u_{2}^{-}$.
Proof. Let $Q=u_{m-3} \overleftarrow{P} u_{1} u_{m-3}^{+} \vec{P} v_{n}$. Then $Q$ is a Hamilton path in $G$ with the endvertices $u_{m-3}$ and $v_{n}$. In $\vec{Q}=u_{m-3} \ldots v_{n}$, the vertices $u_{1}, \ldots, u_{m-3}$ of $\overline{N_{I}\left(v_{n}\right)}$ appear in the reverse order of their indices. So, if $v^{+}$ and $v^{-}$are still the successor and the predecessor of a vertex $v$, respectively, with respect to $\vec{P}$, then the first subpath $Q_{1}$ of $Q$ is $u_{m-3} \overleftarrow{P} u_{m-4}^{+}, \ldots$, the $(m-4)$-th subpath $Q_{m-4}$ of $Q$ is $u_{2} \overleftarrow{P} u_{1}^{+}$and the $(m-3)$-th subpath $Q_{m-3}$ of $Q$ is $u_{1} u_{m-3}^{+} \vec{P} v_{n}$. Since $u_{m-3}$ is adjacent to $u_{m-3}^{+}$, the path $Q$ can play the role of $P$ in the discussion of Claim 3.5. Therefore, by exchanging the roles of $u_{j}$ and $u_{(m-3)-(j-1)}, u_{j}^{+}$and $u_{(m-3)-(j-1)}^{-}, u_{j}^{++}$and $u_{(m-3)-(j-1)}^{--}$, $r_{0}$ and $s_{0}$, respectively, we can repeat arguments in Claim 3.5 to show that $u_{2}^{-}=u_{1}^{+}$.
Now we complete the proof of Lemma 17. By Claims 3.5 and 3.6, $u_{m-4}^{+}=$ $u_{m-3}^{-}$and $u_{1}^{+}=u_{2}^{-}$. Therefore, by Claim 3.2 there is a subscript $j_{0}$ such that $2 \leq j_{0} \leq m-5$ and $u_{j_{0}}^{+} \neq u_{j_{0}+1}^{-}$. Then both $u_{j_{0}}^{++}$and $u_{j_{0}+1}^{--}$are adjacent to $v_{n}$.

Assume that $u_{1}$ is adjacent to $u_{m-4}^{+}$. If $u_{j_{0}+1}^{-}$is adjacent to $u_{m-3}$, then $C=u_{1} u_{m-4}^{+} \overleftarrow{P} u_{j_{0}+1}^{-} u_{m-3} \vec{P} v_{n} u_{j_{0}+1}^{-} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, a contradiction. Thus, $u_{j_{0}+1}^{-}$cannot be adjacent to $u_{m-3}$. Therefore, since $\operatorname{deg}\left(u_{m-3}\right)=m-4$ and $N\left(u_{m-3}\right) \subseteq\left\{u_{2}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\}$, the vertex $u_{m-3}$ must be adjacent to $u_{j_{0}}^{-}$. Now $Q=u_{1} \vec{P} u_{j_{0}}^{-} u_{m-3} \overleftarrow{P} u_{j_{0}} u_{m-3}^{+} \vec{P} v_{n}$ can play the role of $P$ in Claim 3.5. So we can get a contradiction as in the proof of Claim 3.5. Hence, the assumption that $u_{1}$ is adjacent to $u_{m-4}^{+}$is false.

Thus, $u_{1}$ is not adjacent to $u_{m-4}^{+}$, i.e., $r_{0}=m-4$. This means that $u_{1}$ is adjacent to each of $u_{1}^{+}, \ldots, u_{m-5}^{+}$and $u_{m-3}^{+}$. By Claim 3.6, $u_{1}^{+}=u_{2}^{-}$. Therefore, $u_{m-3}$ cannot be adjacent to $u_{2}^{-}$because otherwise $C=$ $u_{1} u_{j_{0}}^{+} \overleftarrow{P} u_{1}^{+} u_{m-3} \overleftarrow{P} u_{j_{0}}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+} u_{1}$ would be a Hamilton cycle of $G$, a contradiction. Thus, $u_{m-3}$ is adjacent to each of vertices $u_{3}^{-}, \ldots, u_{m-3}^{-}$and $u_{m-3}^{+}$. Therefore, if $j_{0}>2$, then $C=u_{1} \vec{P} u_{j_{0}}^{-} u_{m-3} \overleftarrow{P} u_{j_{0}}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+} u_{j_{0}} u_{j_{0}}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction; and if $j_{0}=2$, then $C=u_{1} u_{3}^{+} \vec{P} u_{m-3}$ $u_{3}^{-} u_{3} u_{m-3}^{+} \vec{P} v_{n} u_{3}^{--\overleftarrow{P}} u_{1}$ is a Hamilton cycle of $G$, a contradiction again.

This final contradiction shows the assumption that $G$ does not satisfy the last conclusion of Lemma 17 is false.

The proof of Lemma 17 is complete.
Lemma 18. Let $G=S(I \cup K, E)$ be a maximal nonhamiltonian BurkardHammer graph with $m=|I| \geq 7, n=|K|$ and $\delta(G)=|I|-4$. Furthermore, let $G$ possess a vertex $v \in B_{3}$ such that some vertex $u \in \overline{N_{I}(v)}=I \backslash N_{I}(v)$ has $\operatorname{deg}(u) \geq|I|-3$. Then $G$ is isomorphic to the expansion $H^{4, t}\left[G_{2}, v_{2}^{*}\right]$ where $t=|K|-|I|+5$ and $G_{2}=S\left(I_{2} \cup K_{2}, E_{2}\right)$ is a complete split graph with $\left|K_{2}\right|-1=\left|I_{2}\right|=|I|-5$.

Proof. By Lemma 11, without loss of generality, we may assume here that $K=N(I)$. Let $P, u_{1}$ and $v_{n}$ be as in Lemma 12. Set $u_{m}=v_{n}^{-} \in I$ and let the vertices $u_{1}, u_{2}, \ldots, u_{m-3}$ of $I$ and the subpaths $P_{1}, P_{2}, \ldots, P_{m-3}$ of $P$ be defined as before Lemma 13. By the assumption of our lemma, without loss of generality, we may assume that $u_{1}$ and $v_{n}$ are such that

$$
\operatorname{deg}\left(u_{1}\right) \geq m-3
$$

Together with Lemma 13, this implies the following Claim.
Claim 3.7. $N\left(u_{1}\right)=\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}\right\}$.
By Lemma 14 and Claim 3.7, for any $j \in\{1,2, \ldots, m-3\}$ we have $\operatorname{deg}\left(u_{j}\right) \leq$ $m-3$. But $\operatorname{deg}\left(u_{j}\right) \geq \delta(G)=m-4$. It follows that $\operatorname{deg}\left(u_{j}\right)=m-4$ or $m-3$ for any $j \in\{1,2, \ldots, m-3\}$. By Lemma 15 ,

$$
N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, u_{j+1}^{+}, \ldots, u_{m-3}^{+}\right\}
$$

for $j=2,3, \ldots, m-3$.
Claim 3.8. There exists a number $j_{0} \in\{1,2, \ldots, m-4\}$ such that $u_{j_{0}}^{+} \neq$ $u_{j_{0}+1}^{-}$but $u_{j_{0}+1}^{+}=u_{j_{0}+2}^{-}, \ldots, u_{m-4}^{+}=u_{m-3}^{-}$.

Proof. Suppose that $u_{j}^{+}=u_{j+1}^{-}$for each $j \in\{1,2, \ldots, m-4\}$. Then for $I^{\prime}=\left\{u_{1}, u_{2}, \ldots, u_{m-3}\right\}$, by Claim 3.7 and $N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}\right.$, $\left.u_{j+1}^{+}, \ldots, u_{m-3}^{+}\right\}$for each $j=2,3, \ldots, m-3$ just proved above, we have $N\left(I^{\prime}\right)=\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}\right\}$. So $\left|N\left(I^{\prime}\right)\right|=\left|I^{\prime}\right|$, contradicting Lemma 6. This means that Claim 3.8 must hold.

We have $u_{j_{0}}^{++}, u_{j_{0}+1}^{--} \notin \overline{N_{I}\left(v_{n}\right)}=\left\{u_{1}, \ldots, u_{m-3}\right\}$. Therefore, both $u_{j_{0}}^{++}$and $u_{j_{0}+1}^{--}$are adjacent to $v_{n}$. The reader should remember this because later we frequently use it, without mentioning it, to construct a Hamilton cycle in a graph $G$.

Claim 3.9. At least one of vertices $u_{2}$ or $u_{m-4}$ is adjacent to $u_{m-3}^{+}$.
Proof. Suppose, on the contrary, that neither $u_{2}$ nor $u_{m-4}$ is adjacent to $u_{m-3}^{+}$. Then since $\operatorname{deg}\left(u_{j}\right) \geq m-4$ and $N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, u_{j+1}^{+}\right.$, $\left.\ldots, u_{m-3}^{+}\right\}$for $j=2,3, \ldots, m-3$, we have

$$
\begin{aligned}
N\left(u_{2}\right) & =\left\{u_{2}^{-}, u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-4}^{+}\right\} \\
N\left(u_{m-4}\right) & =\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{m-4}^{-}, u_{m-4}^{+}\right\} .
\end{aligned}
$$

By applying Lemma 16 with $a=2$ and $b=m-4$ we get $u_{1}^{+}=u_{2}^{-}, u_{2}^{+}=u_{3}^{-}$. If there exists $x \in\{3, \ldots, m-5\}$ such that $u_{x}^{+} \neq u_{x+1}^{-}$, then $u_{x}^{++}$is adjacent to $v_{n}$. So since $\operatorname{deg}\left(u_{j}\right) \geq m-4 \geq 3$ for every $j \in\{1,2, \ldots, m-3\}$, $C=u_{1} u_{x}^{+} \overleftarrow{P} u_{2} u_{m-4}^{+} \vec{P} v_{n} u_{x}^{++} \vec{P} u_{m-4} u_{2}^{-} u_{1}$ is a Hamilton cycle of $G$, contradicting the nonhamiltonicity of $G$. Thus, we also have $u_{3}^{+}=u_{4}^{-}, u_{4}^{+}=$ $u_{5}^{-}, \ldots, u_{m-5}^{+}=u_{m-4}^{-}$. It follows that $j_{0}=m-4$ and therefore $u_{m-4}^{++}$is adjacent to $v_{n}$.

If $u_{m-3}$ is adjacent to $u_{2}^{-}$, then since $u_{1}^{+}=u_{2}^{-}, C=u_{m-3} u_{2}^{-} \vec{P} u_{m-4}^{+} u_{1}$ $u_{m-3}^{+} \vec{P} v_{n} u_{m-4}^{++} \vec{P} u_{m-3}$ is a Hamilton cycle of $G$, a contradiction. So $u_{m-3}$ is not adjacent to $u_{2}^{-}$. It follows that $N\left(u_{m-3}\right)=\left\{u_{3}^{-}, u_{4}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\}$ because $\operatorname{deg}\left(u_{m-3}\right) \geq m-4$ and $N\left(u_{m-3}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{m-3}^{-}, u_{m-3}^{+}\right\}$. Since $m \geq 7$ and $u_{1}^{+}=u_{2}^{-}, \ldots, u_{m-5}^{+}=u_{m-4}^{-}$, we always have $u_{1}^{+}=u_{2}^{-}$and $u_{2}^{+}=u_{3}^{-}$. Therefore, $C=u_{m-3} u_{3}^{-} \vec{P} u_{m-4}^{+} u_{2} u_{1}^{+} u_{1} u_{m-3}^{+} \vec{P} v_{n} u_{m-4}^{++} \vec{P} u_{m-3}$ is a Hamilton cycle of $G$, a contradiction again. The proof of Claim 3.9 is complete.

We continue the proof of Lemma 18. If $u_{m-3}^{++} \in K$, then $C=u_{1} u_{m-3}^{+} \overleftarrow{P}$ $u_{j_{0}+1}^{-} u_{m-3}^{++} \vec{P} v_{n} u_{j_{0}+1}^{--} \overleftarrow{P} u_{1}$ is a Hamilton cycle of $G$, contradicting the nonhamiltonicity of $G$. So $u_{m-3}^{++} \in I$. It follows that $\left|N_{I}\left(u_{m-3}^{+}\right)\right| \geq 4$ because $u_{1}, u_{m-3}, u_{m-3}^{++}$and at least one of vertices $u_{2}$ or $u_{m-4}$ by Claim 3.9 are in $N_{I}\left(u_{m-3}^{+}\right)$. By Theorem $9, N_{I}\left(u_{m-3}^{+}\right)=I$, i.e., $u_{m-3}^{+}$is adjacent to all vertices of $G$.

If $u_{m-3}$ is adjacent to $u_{2}^{-}$then by applying Lemma 16 with $a=2$ and $b=m-3$ we get $u_{1}^{+}=u_{2}^{-}, u_{2}^{+}=u_{3}^{-}$. Therefore, $j_{0} \geq 3$ and $C=$
$u_{1} u_{j_{0}}^{+} \overleftarrow{P} u_{2}^{-} u_{m-3} \overleftarrow{P} u_{j_{0}}^{++} v_{n} \overleftarrow{P} u_{m-3}^{+} u_{1}$ is a Hamilton cycle of $G$, a contradiction Thus, $u_{m-3}$ is not adjacent to $u_{2}^{-}$. Hence $N\left(u_{m-3}\right)=\left\{u_{3}^{-}, u_{4}^{-}, \ldots, u_{m-3}^{-}\right.$, $\left.u_{m-3}^{+}\right\}$because $\operatorname{deg}\left(u_{m-3}\right) \geq m-4$ and $N\left(u_{m-3}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, u_{4}^{-}, \ldots, u_{m-3}^{-}\right.$, $\left.u_{m-3}^{+}\right\}$. By applying now Lemma 16 with $a=3, \ldots, m-4$ and $b=m-3$ we get $u_{2}^{+}=u_{3}^{-}, u_{3}^{+}=u_{4}^{-}, \ldots, u_{m-4}^{+}=u_{m-3}^{-}$.

Thus, $j_{0}=1$. If $u_{2}^{-}$is adjacent to some $u_{j}$ with $j \in\{3,4, \ldots, m-3\}$ then $C=u_{j} u_{2}^{-} \vec{P} u_{j-1}^{+} u_{1} \vec{P} u_{2}^{--} v_{n} \overleftarrow{P} u_{j}$ is a Hamilton cycle of $G$, a contradiction. So $u_{2}^{-}$is not adjacent to any vertices $u_{3}, u_{4}, \ldots, u_{m-3}$. It follows that for $j=3,4, \ldots, m-3$,

$$
N\left(u_{j}\right)=\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}
$$

because $\operatorname{deg}\left(u_{j}\right) \geq m-4$ and $N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, u_{3}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-3}^{+}\right\}$.
We have proved before that $u_{m-3}^{++} \in I$. If $u_{m-3}^{++}=u_{m}=v_{n}^{-}$and $u_{m}$ has no neighbours in $P_{1}=u_{1} \vec{P} u_{2}^{-}$, then $B_{G}\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ with $I^{\prime}=\left\{u_{1}, u_{2}, \ldots\right.$, $\left.u_{m-3}, u_{m}\right\}$ and $K^{\prime}=N\left(I^{\prime}\right) \backslash\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}$has three $H$-components, namely $B_{G}\left(\left\{u_{1}\right\} \cup\left\{u_{1}^{+}\right\},\left\{u_{1} u_{1}^{+}\right\}\right), B_{G}\left(\left\{u_{2}\right\} \cup\left\{u_{2}^{-}\right\},\left\{u_{2} u_{2}^{-}\right\}\right)$and $B_{G}\left(\left\{u_{m}\right\} \cup\right.$ $\left.\left\{v_{n}\right\},\left\{u_{m} v_{n}\right\}\right)$ and $m-5 T$-components, each of which consists of a single vertex from $\left\{u_{3}, \ldots, u_{m-3}\right\}$. Therefore, $k\left(I^{\prime}, K^{\prime}\right)+\max \left\{1, \frac{h\left(I^{\prime}, K^{\prime}\right)}{2}\right\}=m-$ $5+\frac{3}{2}$. But $\left|N\left(I^{\prime}\right)\right|-\left|K^{\prime}\right|=\left|\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}\right|=m-4$. This contradicts the fact that $G$ is a Burkard-Hammer graph. Thus, if $u_{m-3}^{++}=u_{m}$ then $u_{m}$ has to have a neighbour $v$ in $P_{1}$. If $v \neq u_{1}^{+}$then $v^{-}$is adjacent to $v_{n}$ and therefore $C=u_{m} v \overrightarrow{\mathrm{P}} u_{m-3}^{+} u_{1} \vec{P} v^{-} v_{n} u_{m}$ is a Hamilton cycle of $G$. If $v=u_{1}^{+}$then $v^{+}$is adjacent to $v_{n}$ and therefore $C=u_{m} v u_{1} u_{m-3}^{+} \overleftarrow{P} v^{+} v_{n} u_{m}$ is a Hamilton cycle of $G$. We have got a contradiction in any situations.

Thus, $u_{m-3}^{++} \neq u_{m}$. Set $u_{m-2}=u_{m-3}^{++}, \overrightarrow{R_{1}}=u_{1}^{+} \vec{P} u_{2}^{-}$and $\overrightarrow{R_{2}}=u_{m-2} \vec{P}$ $v_{n} u_{m-2}$. Then $\overrightarrow{R_{1}}$ has at least two vertices and $\overrightarrow{R_{2}}$ is a cycle of length at least 4.

Claim 3.10. If there exist a vertex $y$ of the path $\overrightarrow{R_{1}}$ and a vertex $z$ of the cycle $\overrightarrow{R_{2}}$ such that either both $y z$ and $y^{+} z^{+}$are edges of $G$ or both $y z^{+}$and $y^{+} z$ are edges of $G$, where $y^{+}$and $z^{+}$are the successor of $y$ and the successor of $z$ with respect to $\overrightarrow{R_{1}}$ and $\overrightarrow{R_{2}}$, respectively, then $G$ has a Hamilton cycle.

Proof. Suppose that both $y z$ and $y^{+} z^{+}$are edges of $G$. If $z \neq v_{n}$, then $C=y \overleftarrow{P} u_{1} u_{m-3}^{+} \overleftarrow{P} y^{+} z^{+} \vec{P} v_{n} u_{m-2} \vec{P} z y$ is a Hamilton cycle of $G$. If $z=v_{n}$,
then $z^{+}=u_{m-2}$. Therefore, $C=y \overleftarrow{P} u_{1} u_{m-3}^{+} \overleftarrow{P} y^{+} u_{m-2} \vec{P} v_{n} y$ is a Hamilton cycle of $G$.

If both $y z^{+}$and $y^{+} z$ are edges of $G$, then Claim 3.10 can be proved similarly. The proof of Claim 3.10 is complete.
Let $u_{m-1}$ be the remaining vertex of $I$. Then either $u_{m-1} \in \overrightarrow{R_{1}}$ or $u_{m-1} \in$ $\overrightarrow{R_{2}}$.

If $u_{m-1} \in \overrightarrow{R_{2}}$ then all vertices of $\overrightarrow{R_{1}}$ are in $K$. Therefore, by using Claim 3.10, it is not difficult to see that $u_{m-2}^{+}=u_{m-1}^{-}, u_{m-1}^{+}=u_{m}^{-}$and $u_{m-2}, u_{m-1}, u_{m}$ have no neighbours in $\overrightarrow{R_{1}}$. Take $I^{\prime}=I, K^{\prime}=N\left(I^{\prime}\right) \backslash$ $\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}$. Then it is not difficult to see as before that $k\left(I^{\prime}, K^{\prime}\right)=$ $m-5$ and $h\left(I^{\prime}, K^{\prime}\right)=3$. Therefore, $k\left(I^{\prime}, K^{\prime}\right)+\max \left\{1, \frac{h\left(I^{\prime}, K^{\prime}\right)}{2}\right\}=m-$ $5+\frac{3}{2},\left|N\left(I^{\prime}\right)\right|-\left|K^{\prime}\right|=\left|\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}\right|=m-4$. It follows that $k\left(I^{\prime}, K^{\prime}\right)+\max \left\{1, \frac{h\left(I^{\prime} K^{\prime}\right)}{2}\right\}>\left|N\left(I^{\prime}\right)\right|-\left|K^{\prime}\right|$, contradicting the fact that $G$ is a Burkard-Hammer graph. Thus, $u_{m-1}$ cannot be a vertex of $\overrightarrow{R_{2}}$ and therefore $u_{m-1} \in \overrightarrow{R_{1}}$.

Suppose that $u_{m-2}^{+}=u_{m}^{-}$. Since $v_{n}$ is adjacent to every vertex of $\overrightarrow{R_{1}}$, again by using Claim 3.10, we see that $u_{m-2}$ and $u_{m}$ are not adjacent to any vertices of $\overrightarrow{R_{1}}$. Take $I^{\prime}=\left\{u_{1}, u_{2}, \ldots, u_{m-3}, u_{m-2}, u_{m}\right\}$ and $K^{\prime}=N\left(I^{\prime}\right) \backslash$ $\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}$. Then as before it is not difficult to check that $G$ does not satisfy the Burkard-Hammer condition with respect to these $I^{\prime}$ and $K^{\prime}$, a contradiction. Thus, $u_{m-2}^{+} \neq u_{m}^{-}$and therefore again by Claim 3.10 we must have $u_{1}^{+}=u_{m-1}^{-}, u_{m-1}^{+}=u_{2}^{-}$and $u_{m-1}$ is not adjacent to any vertices in $u_{m-2}^{+} \vec{P} u_{m}^{-}$. Further, if $u_{m-2}$ is adjacent to a vertex $v \in u_{m-2}^{++} \vec{P} u_{m}^{-}$then since $v^{-} \in K$,

$$
C=u_{1} u_{m-3}^{+} \overleftarrow{P} u_{m-1} v_{n} \overleftarrow{P} v u_{m-2} \vec{P} v^{-} u_{1}^{+} u_{1}
$$

is a Hamilton cycle of $G$. Similarly, if $u_{m}$ is adjacent to a vertex $v \in$ $u_{m-2}^{+} \vec{P} u_{m}^{--}$then since $v^{+} \in K$,

$$
C=u_{1} u_{m-3}^{+} \overleftarrow{P} u_{m-1} v_{n} u_{m-2} \vec{P} v u_{m} \overleftarrow{P} v^{+} u_{1}^{+} u_{1}
$$

is a Hamilton cycle of $G$. We have got a contradiction in both situations. Thus, in $\overrightarrow{R_{2}}$ the vertex $u_{m-2}$ is adjacent to only $u_{m-2}^{+}$and $v_{n}$ and the vertex $u_{m}$ is adjacent to only $u_{m}^{-}$and $v_{n}$. It follows that if $u_{m-2}^{+} \vec{P} u_{m}^{-}$has more than two vertices, then since $K=N(I)$ (by our assumption), $N\left(u_{1}\right)=$
$\left\{u_{1}^{+}, \ldots, u_{m-3}^{+}\right\}$(by Claim 3.7) and $N\left(u_{j}\right) \subseteq\left\{u_{2}^{-}, \ldots, u_{j}^{-}, u_{j}^{+}, \ldots, u_{m-3}^{+}\right\}$ for $j=2,3, \ldots, m-3$ (by Claim 3.7 and Lemma 15) are true, the vertex $u_{m-2}^{++}$must be adjacent to $u_{m-1}$. By Claim 3.10, $G$ has a Hamilton cycle, contradicting the nonhamiltonicity of $G$. Thus, $u_{m-2}^{++}=u_{m}^{-}$. It follows that $n=|K|=|N(I)|=m+1$ and

$$
\begin{aligned}
I & =\left\{u_{1}, u_{2}, \ldots, u_{m}\right\} \\
K & =\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}, u_{m-2}^{+}, u_{m-1}^{+}, u_{m}^{-}, v_{m+1}\right\} .
\end{aligned}
$$

Let $H=S(I \cup K, E(H))$ be a split graph with

$$
\begin{aligned}
N_{H}\left(u_{1}\right) & =\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}\right\}, \\
N_{H}\left(u_{2}\right) & =\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}, u_{m-1}^{+}\right\}, \\
N_{H}\left(u_{3}\right) & =N_{H}\left(u_{4}\right)=\cdots=N_{H}\left(u_{m-3}\right)=\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}, \\
N_{H}\left(u_{m-2}\right) & =\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}, u_{m-2}^{+}, v_{m+1}\right\}, \\
N_{H}\left(u_{m-1}\right) & =\left\{u_{1}^{+}, u_{2}^{+}, \ldots, u_{m-3}^{+}, u_{m-1}^{+}, v_{m+1}\right\}, \\
N_{H}\left(u_{m}\right) & =\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}, u_{m}^{-}, v_{m+1}\right\} .
\end{aligned}
$$

Set $I_{2}=\left\{u_{3}, u_{4}, \ldots, u_{m-3}\right\}, K_{2}=\left\{u_{2}^{+}, u_{3}^{+}, \ldots, u_{m-3}^{+}\right\}$and $G_{2}=H\left[I_{2} \cup K_{2}\right]$. Then $G_{2}$ is a complete split graph $S\left(I_{2} \cup K_{2}, E_{2}\right)$ with $\left|K_{2}\right|-1=\left|I_{2}\right|=$ $|I|-5$. Further, let $H^{4,6}=S\left(I^{*} \cup K^{*}, E\left(H^{4,6}\right)\right)$ with $I^{*}=\left\{u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}, u_{5}^{*}\right\}$ and $K^{*}=\left\{v_{1}^{*}, v_{2}^{*}, v_{3}^{*}, v_{4}^{*}, v_{5}^{*}, v_{6}^{*}\right\}$ be a split graph defined in Table 1 and $H^{\prime}=H^{4,6}\left[G_{2}, v_{2}^{*}\right]$. Then $H^{\prime}$ is a split graph $S\left(I^{\prime} \cup K^{\prime}, E^{\prime}\right)$ with $I^{\prime}=$ $\left\{u_{1}^{*}, u_{2}^{*}, u_{3}^{*}, u_{4}^{*}, u_{5}^{*}, u_{3}, u_{4}, \ldots, u_{m-3}\right\}$ and $K^{\prime}=\left\{v_{1}^{*}, v_{3}^{*}, v_{4}^{*}, v_{5}^{*}, v_{6}^{*}, u_{2}^{+}, u_{3}^{+}, \ldots\right.$, $\left.u_{m-3}^{+}\right\}$. Consider the following mapping $\varphi: V(H) \rightarrow V\left(H^{\prime}\right)$ with

$$
\begin{aligned}
\varphi\left(u_{1}\right) & =u_{1}^{*}, \varphi\left(u_{2}\right)=u_{2}^{*}, \varphi\left(u_{j}\right)=u_{j} \text { for } j=3,4, \ldots, m-3, \\
\varphi\left(u_{m-2}\right) & =u_{3}^{*}, \varphi\left(u_{m-1}\right)=u_{4}^{*}, \varphi\left(u_{m}\right)=u_{5}^{*}, \\
\varphi\left(u_{1}^{+}\right) & =v_{1}^{*}, \varphi\left(u_{j}^{+}\right)=u_{j}^{+} \text {for } j=2,3, \ldots, m-3, \\
\varphi\left(u_{m-2}^{+}\right) & =v_{3}^{*}, \varphi\left(u_{m-1}^{+}\right)=v_{4}^{*}, \varphi\left(u_{m}^{-}\right)=v_{5}^{*}, \varphi\left(v_{m+1}\right)=v_{6}^{*} .
\end{aligned}
$$

It is not difficult to see that $\varphi$ is an isomorphism between the graphs $H$ and $H^{\prime}$. By Theorem 10, $H^{\prime}$ is a maximal nonhamiltonian Burkard-Hammer graph. So, by $H \cong H^{\prime}, H=S(I \cup K, E(H))$ also is a maximal nonhamiltonian Burkard-Hammer graph.

By considerations before we see that $N_{G}\left(u_{i}\right) \subseteq N_{H}\left(u_{i}\right)$ for every $i=$ $1,2, \ldots, m$, i.e., $G=S(I \cup K, E)$ is a spanning subgraph of $H=S(I \cup K$, $E(H))$. But $G$ is a maximal nonhamiltonian Burkard-Hammer graph by our assumption. So $G$ must coincide with $H$ and therefore $G$ is isomorphic to $H^{\prime}=H^{4,6}\left[G_{2}, v_{2}^{*}\right]$.

The proof of Lemma 18 is complete.
From Theorem 10 and Lemmas 17 and 18 we can obtain immediately Theorem 1 formulated in the introduction, which gives us the classification of maximal nonhamiltonian Burkard-Hammer graphs $G=S(I \cup K, E)$ with $|I| \neq 6,7$ and $\delta(G)=|I|-4$.

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