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TREES WITH EQUAL TOTAL DOMINATION AND TOTAL RESTRAINED DOMINATION NUMBERS

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V(G)$ is a *total dominating set* if it is dominating and both $\langle S \rangle$ has no isolated vertices. The cardinality of a minimum total dominating set in G is the *total domination number*. A set $S \subseteq V(G)$ is a *total restrained dominating set* if it is total dominating and $\langle V(G) - S \rangle$ has no isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number*. We characterize all trees for which total domination and total restrained domination numbers are the same.

Keywords: total domination number, total restrained domination number, tree.

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1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Arumugam [1].

Let $G = (V, E)$ be a simple graph of order n. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. For a subset S of V, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. The graph induced by $S \subseteq V$ is denoted by $\langle S \rangle$. The minimum degree and maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *diameter diam(G)* of a connected graph G is the maximum distance between two vertices of G, that is $diam(G) = \max_{u,v \in V(G)} d_G(u, v)$. Let P_n denote a path with n vertices. Let $K_{1,r}$ denote the star with $r+1$ vertices. Define $K_{1,r,4}$ as follows: for each edge of $K_{1,r}$, we subdivide by two vertices. The vertex of degree r is called the central vertex of $K_{1,r,4}$. Let η be a family of graphs and $\eta = \{K_{1,r,4} | r \geq 1$ and r is an integer $\}$.

A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets of G. A set $S \subseteq V(G)$ is a *total dominating set* if it is dominating and $\langle S \rangle$ has no isolated vertices. The cardinality of a minimum total dominating set in G is the *total domination number* and is denoted by $\gamma_t(G)$. Cockayne *et al.* [6] studied total dominating functions in trees: minimality and convexity.

The total restrained domination number of a graph was defined by D. Ma *et al.* in [4]. A set $S \subseteq V(G)$ is a *total restrained dominating set* if it is total dominating and $\langle V(G) - S \rangle$ has no isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* and is denoted by $\gamma_r^t(G)$.

A total dominating set S with cardinality $\gamma_t(G)$ is called a γ_t-set. A total restrained dominating set S with cardinality γ_r^t is called a γ_r^t-set. Let $S \subset V(G)$ and $x \in S$, we say that x has a private neighbour (with respect to S) if there is a vertex in $V(G) - S$ whose only neighbour in S is x. Let $PN(x, S)$ denote the private neighbours set of x with respect to S.

A vertex of degree one is called a *leaf.* A vertex v of G is called a *support* if it is adjacent to a leaf. If T is a tree, $L(T)$ and $S(T)$ denote the set of leaves and supports, respectively. Any vertex of degree greater than one is called an *internal vertex.*

For any graph theoretical parameters λ and μ, we define G to be (λ, μ)-graph if $\lambda(G) = \mu(G)$. In this paper we provide a constructive characterization of (γ_t, γ_r^t)-trees.

2. A Characterization of (γ_t, γ_r^t)-trees

As a consequence of the definition of total restrained domination number, we have the following observations.

Observation 1. *Let G be a graph without isolated vertices. Then*

(i) *every leaf belongs to every γ_r^t-set;*

(ii) *every support belongs to every γ_r^t-set;*

(iii) $\gamma_t(G) \leq \gamma_r^t(G)$.

Observation 2. *Let T be a (γ_t, γ_r^t)-tree. Then each $\gamma_r^t(T)$-set is a $\gamma_t(T)$-set.*

Let τ_1 and τ_2 be the following two operations defined on a tree T.

• **Operation τ_1.** Assume $x \in V(T)$ is a leaf or support. Then add one or more trees of η and the edges between x and each central vertex.

• **Operation τ_2.** Assume $x \in N(S(T)) - L(T)$. Then add one or more paths P_3 and the edges between x and one leaf of each P_3.

Let τ be the family of trees such that $\tau = \{T : T$ is obtained from P_6 by a finite sequence of operations τ_1 or $\tau_2\} \cup \{P_2, P_6\}$. We show first that each tree in the family τ has equal total domination number and total restrained domination number.

Lemma 1. *If T belongs to the family τ, then T is a (γ_t, γ_r^t)-tree.*

Proof. We proceed by induction on the number of operations $s(T)$ required to construct the tree T. If $s(T) = 0$, then $T \in \{P_2, P_6\}$ and clearly T is a (γ_t, γ_r^t)-tree. Assume now that T is a tree with $s(T) = k$ for some positive integer k and each tree $T' \in \tau$ with $s(T') < k$ is a (γ_t, γ_r^t)-tree. Then T can be obtained from a tree T' belonging to τ by operation τ_1 or τ_2. We now consider two possibilities depending on whether T is obtained from T' by operation τ_1 or τ_2.

Case 1. T is obtained from T' by operation τ_1. Without loss of generality, we can assume that T is obtained from T' by adding k trees $K_{1,r_1,4}, K_{1,r_2,4}, \ldots, K_{1,r_k,4}$ of η and the edges between x and each central vertex, where $r_1 \le r_2 \le \cdots \le r_k$. It is obvious that $\gamma_t(T) \le \gamma_t(T') + 2\sum_{1\le i\le k} r_i$. Let D be a γ_t-set of T such that $D \cap L(T) = \emptyset$. Then $|D \cap K_{1,r_i,4}| \ge 2r_i$ for each $K_{1,r_i,4}$. Let $D' = D \cap V(T')$.

Case 1.1. x is a support of T'. Then $x \in D'$. If $N_{T'}(x) \cap D' \neq \emptyset$, then D' is a total dominating set of T'. So $\gamma_t(T') \le |D'| \le \gamma_t(T) - 2\sum_{1\le i\le k} r_i$. If $N_{T'}(x) \cap D' = \emptyset$, then there exists a tree $K_{1,r_i,4}$ such that $|D \cap K_{1,r_i,4}| \ge 2r_i+1$ and its central vertex belongs to D. Let $y \in N_{T'}(x)$ and $D'' = D' \cup \{y\}$. Then D'' is a total dominating set of T'. So $\gamma_t(T') \le |D''| = |D'| + 1 \le \gamma_t(T) - 2\sum_{1\le i\le k} r_i$.

Case 1.2. x is a leaf of T'. Let $y \in N_{T'}(x)$. If $y \in D$, then D' is a total dominating set of T'. Suppose $y \notin D$. Then there exists a tree $K_{1,r_i,4}$ such that $|D \cap K_{1,r_i,4}| \ge 2r_i + 1$ and its central vertex belongs to D. Let $D'' = D' \cup \{y\}$. Then D'' is a total dominating set of T'. So $\gamma_t(T') \le |D''| = |D'| + 1 \le \gamma_t(T) - 2\sum_{1\le i\le k} r_i$.

By Case 1.1 and 1.2, $\gamma_t(T') \le \gamma_t(T) - 2\sum_{1\le i\le k} r_i$. Hence, $\gamma_t(T) = \gamma_t(T') + 2\sum_{1\le i\le k} r_i$. It is obvious that $\gamma_r^t(T) \le \gamma_r^t(T') + 2\sum_{1\le i\le k} r_i$. Since $\gamma_r^t(T') + 2\sum_{1\le i\le k} r_i = \gamma_t(T') + 2\sum_{1\le i\le k} r_i = \gamma_t(T) \le \gamma_r^t(T)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1\le i\le k} r_i$. So $\gamma_t(T) = \gamma_r^t(T)$.

Case 2. T is obtained from T' by operation τ_2. Without loss of generality, we can assume that T is obtained from T' by adding paths v_{1j}, v_{2j}, v_{3j} and the edges between x and v_{1j} for $j = 1, 2, \cdots, k$. It is obvious that $\gamma_t(T) \le \gamma_t(T') + 2k$. Let D be a γ_t-set of T such that $D \cap L(T) = \emptyset$. Then $v_{1j}, v_{2j} \in D$. Let $D' = D \cap V(T')$. Then D' is a total dominating set of T'. So $\gamma_t(T') \le \gamma_t(T) - 2k$. Hence $\gamma_t(T) = \gamma_t(T') + 2k$. Let D'' be a γ_r^t-set of T'. Since T' is a (γ_t, γ_r^t)-tree, it follows that $x \notin D''$. Otherwise, assume $N_{T'}(x) \cap S(T') = \{y\}$ and $N_{T'}(y) \cap L(T') = \{z\}$. Then $D'' - \{z\}$ is a total dominating set of T' with cardinality less than $|D''|$, which is a contradiction. So, $\gamma_r^t(T) \le \gamma_r^t(T') + 2k$. Since $\gamma_r^t(T') + 2k = \gamma_t(T') + 2k = \gamma_t(T) \le \gamma_r^t(T)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2k$. So $\gamma_t(T) = \gamma_r^t(T)$. ■

We show next that every (γ_t, γ_r^t)-tree belongs to the family τ.

Lemma 2. *Let T be a (γ_t, γ_r^t)-tree. Then*

(i) *for each support $v \in S(T)$, $|N(v) \cap L(T)| = 1$;*

(ii) *for any two supports $u, v \in S(T)$, $d(u, v) \geq 3$.*

Proof. (i) Suppose that there exists a support v such that $|N(v) \cap L(T)| \geq 2$. Let $N(v) \cap L(T) = \{v_1, \ldots, v_k\}$ where $k \geq 2$. Let D be a γ_r^t-set of T. Then, by Observation 1, it follows that $D - \{v_2, \ldots, v_k\}$ is a total dominating set of T with cardinality less than $\gamma_t(T)$, which is a contradiction. Hence, $|N(v) \cap L(T)| = 1$ for each support $v \in S(T)$.

(ii) Suppose that there exist two supports u and v such that $d(u, v) \leq 2$. Let $u_1 \in N(u) \cap L(T)$ and $v_1 \in N(v) \cap L(T)$. Let D be a γ_r^t-set of T. If u is adjacent to v, then, by Observation 1, it follows that $D - \{u_1\}$ is a total dominating set of T with cardinality less than $\gamma_t(T)$, which is a contradiction. Suppose $d(u, v) = 2$. Assume $w \in N(u) \cap N(v)$. Then by Observation 1, it follows that $(D - \{u_1, v_1\}) \cup \{w\}$ is a total dominating set of T with cardinality less than $\gamma_t(T)$, which is a contradiction. Hence, $d(u, v) \geq 3$ for any two supports $u, v \in S(T)$. ■

Lemma 3. *If T is a (γ_t, γ_r^t)-tree, then T belongs to the family τ.*

Proof. Let T be a (γ_t, γ_r^t)-tree. If $diam(T) \leq 5$, then T is P_2 or P_6. It is clear that the statement is true. For this reason, we only consider only trees T with $diam(T) \geq 6$.

Let T be a (γ_t, γ_r^t)-tree and assume that the result holds for all trees on $n(T) - 1$ and fewer vertices. We proceed by induction on the number of vertices of a (γ_t, γ_r^t)-tree. Let $P = (v_0, v_1, \ldots, v_l)$, $l \geq 6$, be a longest path in T and let D be a $\gamma_r^t(T)$-set. Then $v_0, v_1 \in D$. By Lemma 2, it follows that $d(v_1) = d(v_2) = 2$. It is obvious that $v_2, v_3 \notin D$. Otherwise $D - \{v_0\}$ is a total dominating set with cardinality less than $|D|$, which is a contradiction.

Now we have the following claim.

Claim 1. $|N_T(v_3) \cap D| = 1$.

Proof. Without loss of generality, we can assume $|N_T(v_3) \cap D| = t$ and $t > 1$. Then $N_T(v_3) \cap D \subseteq S(T) \cup \{v_4\}$. By Lemma 2, $|N_T(v_3) \cap D \cap S(T)| = 1$. So, $t = 2$. We can assume $N_T(v_3) \cap D = \{v_{31}, v_4\}$, where $v_{31} \in S(T)$. By Lemma 2, it is easy to prove that $v_5 \in D$. Let $A_1 = N_T(v_5) - \{v_4\}$.

Then for any $v \in A_1$, $v \notin D$. Otherwise, let T_1 denote the component of $T - \{v_5\}$ containing v_4. Then $(D - (L(T_1) \cup \{v_4\})) \cup (N_{T_1}[S(T_1)] - L(T_1))$ is a total dominating set of T with cardinality less than $|D|$, which is a contradiction. Let $B_1 = N_T(A_1) \cap (V(T) - D)$, $A_2 = N_T(B_1) \cap D$ and $B_2 = N_T(A_2) \cap D$. For $i \geq 1$, let $A_{2i+1} = N_T(B_{2i}) \cap (V(T) - D)$, $B_{2i+1} = N_T(A_{2i+1}) \cap (V(T) - D)$, $A_{2i+2} = N_T(B_{2i+1}) \cap D$ and $B_{2i+2} = N_T(A_{2i+2}) \cap D$. It is obvious that $|B_{2i+1}| \leq |A_{2i+2}| \leq |B_{2i+2}|$ for $i \geq 0$.

Now we prove that if $N_T(B_{2i+2}) \cap D - A_{2i+2} \neq \emptyset$, then $|N_T(v) \cap D| \geq 2$ for any $v \in N_T(B_{2i+2}) \cap D - A_{2i+2}$. Otherwise, we can assume t is the maximum i satisfying $N_T(B_{2i+2}) \cap D - A_{2i+2} \neq \emptyset$ and there exists a vertex $v \in N_T(B_{2i+2}) \cap D - A_{2i+2}$ such that $|N_T(v) \cap D| = 1$. Without loss of generality, we can assume that $u \in B_{2t+2}$ and $uv \in E(T)$.

Define $C_1 = N_T(v) \setminus \{u\}$. Then for any $w \in C_1$, $w \notin D$. Let $D_1 = N_T(C_1) \cap (V(T) - D)$. Let $C_2 = N_T(D_1) \cap D$ and $D_2 = N_T(C_2) \cap D$. For $i \geq 1$, let $C_{2i+1} = N_T(D_{2i}) \cap (V(T) - D)$, $D_{2i+1} = N_T(C_{2i+1}) \cap (V(T) - D)$, $C_{2i+2} = N_T(D_{2i+1}) \cap D$ and $D_{2i+2} = N_T(C_{2i+2}) \cap D$. It is obvious that $|D_{2i+1}| \leq |C_{2i+2}| \leq |D_{2i+2}|$ for $i \geq 0$. Let $D' = (D - \{v\} - \bigcup_{0 \leq i \leq t} D_{2i+2}) \cup \bigcup_{0 \leq i \leq t} D_{2i+1}$. It is obvious that D' is a total dominating set of T with cardinality less than $|D|$, which is a contradiction.

Let $w \in A_1$. Let $\overline{D} = (D - (L(T_1) \cup \{v_4, v_5\}) - \bigcup_{0 \leq i \leq t} B_{2i+2}) \cup \bigcup_{0 \leq i \leq t} B_{2i+1} \cup \{w\} \cup (N_{T_1}[S(T_1))] - L(T_1))$. It is obvious that $\overline{D}$ is a total dominating set of T with cardinality less than $|D|$, which is a contradiction. Hence, $|N_T(v_3) \cap D| = 1$. ■

By the above claim, we consider the following three cases. Assume $d_T(v_4) = j$.

Case 1. $v_4 \in D$ and $v_4 \in S(T)$. Let T_1 denote the component of $T - \{v_4\}$ containing v_5. Let $N_T(v_4) \cap L(T) = \{l\}$ and $N_T(v_4) - \{v_5, l\} = \{v_{41}, \cdots, v_{4(j-2)}\}$. Denote $T' = \langle V(T_1) \cup \{v_4, l\} \rangle$. Then it is easy to prove that $\gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$. It is obvious that $\gamma_r^t(T') \leq \gamma_r^t(T) - 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$. Since T is a (γ_t, γ_r^t)-tree, it follows that $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1) \leq \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$. So $\gamma_t(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_t, γ_r^t)-tree and by induction hypothesis, $T' \in \tau$. As v_4 is a support in T', we deduce that T may be obtained from T' by operation τ_1.

Case 2. $v_4 \in D$ and $v_4 \notin S(T)$. Let T_1 denote the component of $T-\{v_4\}$ containing v_5. Then $v_5 \in D$. Let $N_T(v_4) - \{v_5\} = \{v_{41}, \cdots, v_{4(j-1))}\}$. Denote $T' = \langle V(T_1) \cup \{v_4\}\rangle$. Then it is obvious that $\gamma_t(T) = \gamma_t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. It is obvious that $\gamma_r^t(T') \le \gamma_r^t(T) - 2\sum_{1\le i\le (j-1)} (d(v_{4i}) - 1)$. Since T is a (γ_t, γ_r^t)-tree, it follows that $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1) \le \gamma_r^t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1\le i\le (j-1)}(d(v_{4i}) - 1)$. So $\gamma_t(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_t, γ_r^t)-tree and by induction hypothesis, $T' \in \tau$. As v_4 is a leaf in T', we deduce that T may be obtained from T' by operation τ_1.

Case 3. $v_4 \notin D$. Then there exists exactly one vertex $x \in N_T(v_3) \cap D$ and x is a support. Assume $N_T(x) \cap L(T) = \{l\}$. Let T_1 denote the component of $T - \{v_3\}$ containing v_4. Denote $T' = \langle V(T_1) \cup \{v_3, x, l\}\rangle$. It is obvious that $\gamma_t(T) = \gamma_t(T') + 2(d_T(v_3) - 2)$. It is obvious that $x, l \in D$. Hence $\gamma_r^t(T') \le \gamma_r^t(T) - 2(d_T(v_3) - 2)$. Since T is a (γ_t, γ_r^t)-tree, it follows that $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2(d_T(v_3) - 2) \le \gamma_r^t(T') + 2(d_T(v_3) - 2)$. Hence $\gamma_r^t(T) = \gamma_r^t(T') + 2(d_T(v_3) - 2)$. So $\gamma_t(T') = \gamma_r^t(T')$. Consequently, T' is a (γ_t, γ_r^t)-tree and by induction hypothesis, $T' \in \tau$. As v_3 is a vertex adjacent to a support in T', we deduce that T may be obtained from T' by operation τ_2. ∎

As an immediate consequence of Lemmas 2 and 3 we have the following characterization of (γ_t, γ_r^t)-trees.

Theorem 3. *A tree T is a (γ_t, γ_r^t)-tree if and only if T belongs to the family τ.*

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