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# TREES WITH EQUAL TOTAL DOMINATION AND TOTAL RESTRAINED DOMINATION NUMBERS

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### Abstract

For a graph G = (V, E), a set  $S \subseteq V(G)$  is a *total dominating set* if it is dominating and both  $\langle S \rangle$  has no isolated vertices. The cardinality of a minimum total dominating set in G is the *total domination num*ber. A set  $S \subseteq V(G)$  is a *total restrained dominating set* if it is total dominating and  $\langle V(G) - S \rangle$  has no isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total re*strained domination number. We characterize all trees for which total domination and total restrained domination numbers are the same.

**Keywords:** total domination number, total restrained domination number, tree.

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#### 1. INTRODUCTION

By a graph we mean a finite, undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Arumugam [1].

Let G = (V, E) be a simple graph of order n. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by  $d_G(v)$ ,  $N_G(v)$  and  $N_G[v] = N_G(v) \cup \{v\}$ , respectively. For a subset S of V,  $N_G(S) = \bigcup_{v \in S} N_G(v)$  and  $N_G[S] = N_G(S) \cup S$ . The graph induced by  $S \subseteq V$  is denoted by  $\langle S \rangle$ . The minimum degree and maximum degree of the graph G are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. The diameter diam(G) of a connected graph G is the maximum distance between two vertices of G, that is  $diam(G) = \max_{u,v \in V(G)} d_G(u,v)$ . Let  $P_n$  denote a path with n vertices. Let  $K_{1,r}$  denote the star with r+1 vertices. Define  $K_{1,r,4}$  as follows: for each edge of  $K_{1,r}$ , we subdivide by two vertices. The vertex of degree r is called the central vertex of  $K_{1,r,4}$ . Let  $\eta$  be a family of graphs and  $\eta = \{K_{1,r,4} | r \geq 1$ and r is an integer  $\}$ .

A subset S of V is called a *dominating set* if every vertex in V - S is adjacent to some vertex in S. The *domination number*  $\gamma(G)$  of G is the minimum cardinality taken over all dominating sets of G. A set  $S \subseteq V(G)$  is a *total dominating set* if it is dominating and  $\langle S \rangle$  has no isolated vertices. The cardinality of a minimum total dominating set in G is the *total domination number* and is denoted by  $\gamma_t(G)$ . Cockayne *et al.* [6] studied total dominating functions in trees: minimality and convexity.

The total restrained domination number of a graph was defined by D. Ma *et al.* in [4]. A set  $S \subseteq V(G)$  is a *total restrained dominating set* if it is total dominating and  $\langle V(G) - S \rangle$  has no isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number* and is denoted by  $\gamma_r^t(G)$ .

A total dominating set S with cardinality  $\gamma_t(G)$  is called a  $\gamma_t$ -set. A total restrained dominating set S with cardinality  $\gamma_r^t$  is called a  $\gamma_r^t$ -set. Let  $S \subset V(G)$  and  $x \in S$ , we say that x has a private neighbour (with respect to S) if there is a vertex in V(G) - S whose only neighbour in S is x. Let PN(x, S) denote the private neighbours set of x with respect to S.

A vertex of degree one is called a *leaf*. A vertex v of G is called a *support* if it is adjacent to a leaf. If T is a tree, L(T) and S(T) denote the set of leaves and supports, respectively. Any vertex of degree greater than one is called an *internal vertex*.

For any graph theoretical parameters  $\lambda$  and  $\mu$ , we define G to be  $(\lambda, \mu)$ -graph if  $\lambda(G) = \mu(G)$ . In this paper we provide a constructive characterization of  $(\gamma_t, \gamma_r^t)$ -trees.

## 2. A CHARACTERIZATION OF $(\gamma_t, \gamma_r^t)$ -TREES

As a consequence of the definition of total restrained domination number, we have the following observations.

**Observation 1.** Let G be a graph without isolated vertices. Then

- (i) every leaf belongs to every  $\gamma_r^t$ -set;
- (ii) every support belongs to every  $\gamma_r^t$ -set;
- (iii)  $\gamma_t(G) \leq \gamma_r^t(G)$ .

**Observation 2.** Let T be a  $(\gamma_t, \gamma_r^t)$ -tree. Then each  $\gamma_r^t(T)$ -set is a  $\gamma_t(T)$ -set.

Let  $\tau_1$  and  $\tau_2$  be the following two operations defined on a tree T.

• Operation  $\tau_1$ . Assume  $x \in V(T)$  is a leaf or support. Then add one or more trees of  $\eta$  and the edges between x and each central vertex.

• Operation  $\tau_2$ . Assume  $x \in N(S(T)) - L(T)$ . Then add one or more paths  $P_3$  and the edges between x and one leaf of each  $P_3$ .

Let  $\tau$  be the family of trees such that  $\tau = \{T : T \text{ is obtained from } P_6 \text{ by a finite sequence of operations } \tau_1 \text{ or } \tau_2\} \cup \{P_2, P_6\}$ . We show first that each tree in the family  $\tau$  has equal total domination number and total restrained domination number.

**Lemma 1.** If T belongs to the family  $\tau$ , then T is a  $(\gamma_t, \gamma_r^t)$ -tree.

**Proof.** We proceed by induction on the number of operations s(T) required to construct the tree T. If s(T) = 0, then  $T \in \{P_2, P_6\}$  and clearly T is a  $(\gamma_t, \gamma_r^t)$ -tree. Assume now that T is a tree with s(T) = k for some positive integer k and each tree  $T' \in \tau$  with s(T') < k is a  $(\gamma_t, \gamma_r^t)$ -tree. Then T can be obtained from a tree T' belonging to  $\tau$  by operation  $\tau_1$  or  $\tau_2$ . We now consider two possibilities depending on whether T is obtained from T' by operation  $\tau_1$  or  $\tau_2$ . Case 1. T is obtained from T' by operation  $\tau_1$ . Without loss of generality, we can assume that T is obtained from T' by adding k trees  $K_{1,r_1,4}, K_{1,r_2,4}, \ldots, K_{1,r_k,4}$  of  $\eta$  and the edges between x and each central vertex, where  $r_1 \leq r_2 \leq \cdots \leq r_k$ . It is obvious that  $\gamma_t(T) \leq \gamma_t(T') + 2\sum_{1\leq i\leq k} r_i$ . Let D be a  $\gamma_t$ -set of T such that  $D \cap L(T) = \emptyset$ . Then  $|D \cap K_{1,r_i,4}| \geq 2r_i$  for each  $K_{1,r_i,4}$ . Let  $D' = D \cap V(T')$ .

Case 1.1. x is a support of T'. Then  $x \in D'$ . If  $N_{T'}(x) \cap D' \neq \emptyset$ , then D' is a total dominating set of T'. So  $\gamma_t(T') \leq |D'| \leq \gamma_t(T) - 2\sum_{1 \leq i \leq k} r_i$ . If  $N_{T'}(x) \cap D' = \emptyset$ , then there exists a tree  $K_{1,r_i,4}$  such that  $|D \cap K_{1,r_i,4}| \geq 2r_i+1$  and its central vertex belongs to D. Let  $y \in N_{T'}(x)$  and  $D'' = D' \cup \{y\}$ . Then D'' is a total dominating set of T'. So  $\gamma_t(T') \leq |D''| = |D'| + 1 \leq \gamma_t(T) - 2\sum_{1 \leq i \leq k} r_i$ .

Case 1.2. x is a leaf of T'. Let  $y \in N_{T'}(x)$ . If  $y \in D$ , then D' is a total dominating set of T'. Suppose  $y \notin D$ . Then there exists a tree  $K_{1,r_i,4}$  such that  $|D \cap K_{1,r_i,4}| \ge 2r_i + 1$  and its central vertex belongs to D. Let  $D'' = D' \cup \{y\}$ . Then D'' is a total dominating set of T'. So  $\gamma_t(T') \le |D''| = |D'| + 1 \le \gamma_t(T) - 2\sum_{1 \le i \le k} r_i$ .

By Case 1.1 and 1.2,  $\gamma_t(T') \leq \gamma_t(T) - 2\sum_{1 \leq i \leq k} r_i$ . Hence,  $\gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq k} r_i$ . It is obvious that  $\gamma_r^t(T) \leq \gamma_r^t(T') + 2\sum_{1 \leq i \leq k} r_i$ . Since  $\gamma_r^t(T') + 2\sum_{1 \leq i \leq k} r_i = \gamma_t(T') + 2\sum_{1 \leq i \leq k} r_i = \gamma_t(T') + 2\sum_{1 \leq i \leq k} r_i = \gamma_t(T) = \gamma_r^t(T)$ . Hence  $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1 \leq i \leq k} r_i$ . So  $\gamma_t(T) = \gamma_r^t(T)$ .

Case 2. T is obtained from T' by operation  $\tau_2$ . Without loss of generality, we can assume that T is obtained from T' by adding paths  $v_{1j}, v_{2j}, v_{3j}$  and the edges between x and  $v_{1j}$  for  $j = 1, 2, \cdots, k$ . It is obvious that  $\gamma_t(T) \leq \gamma_t(T') + 2k$ . Let D be a  $\gamma_t$ -set of T such that  $D \cap L(T) = \emptyset$ . Then  $v_{1j}, v_{2j} \in D$ . Let  $D' = D \cap V(T')$ . Then D' is a total dominating set of T'. So  $\gamma_t(T') \leq \gamma_t(T) - 2k$ . Hence  $\gamma_t(T) = \gamma_t(T') + 2k$ . Let D'' be a  $\gamma_r^t$ -set of T'. Since T' is a  $(\gamma_t, \gamma_r^t)$ -tree, it follows that  $x \notin D''$ . Otherwise, assume  $N_{T'}(x) \cap S(T') = \{y\}$  and  $N_{T'}(y) \cap L(T') = \{z\}$ . Then  $D'' - \{z\}$  is a total dominating set of T' with cardinality less than |D''|, which is a contradiction. So,  $\gamma_r^t(T) \leq \gamma_r^t(T') + 2k$ . Since  $\gamma_r^t(T') + 2k = \gamma_t(T') + 2k = \gamma_t(T) \leq \gamma_r^t(T)$ . Hence  $\gamma_r^t(T) = \gamma_r^t(T') + 2k$ .

We show next that every  $(\gamma_t, \gamma_r^t)$ -tree belongs to the family  $\tau$ .

**Lemma 2.** Let T be a  $(\gamma_t, \gamma_r^t)$ -tree. Then

(i) for each support  $v \in S(T)$ ,  $|N(v) \cap L(T)| = 1$ ;

(ii) for any two supports  $u, v \in S(T), d(u, v) \ge 3$ .

**Proof.** (i) Suppose that there exists a support v such that  $|N(v) \cap L(T)| \ge 2$ . Let  $N(v) \cap L(T) = \{v_1, \ldots, v_k\}$  where  $k \ge 2$ . Let D be a  $\gamma_r^t$ -set of T. Then, by Observation 1, it follows that  $D - \{v_2, \ldots, v_k\}$  is a total dominating set of T with cardinality less than  $\gamma_t(T)$ , which is a contradiction. Hence,  $|N(v) \cap L(T)| = 1$  for each support  $v \in S(T)$ .

(ii) Suppose that there exist two supports u and v such that  $d(u, v) \leq 2$ . Let  $u_1 \in N(u) \cap L(T)$  and  $v_1 \in N(v) \cap L(T)$ . Let D be a  $\gamma_r^t$ -set of T. If u is adjacent to v, then, by Observation 1, it follows that  $D - \{u_1\}$  is a total dominating set of T with cardinality less than  $\gamma_t(T)$ , which is a contradiction. Suppose d(u, v) = 2. Assume  $w \in N(u) \cap N(v)$ . Then by Observation 1, it follows that  $(D - \{u_1, v_1\}) \cup \{w\}$  is a total dominating set of T with cardinality less than  $\gamma_t(T)$ , which is a contradiction. Hence,  $d(u, v) \geq 3$  for any two supports  $u, v \in S(T)$ .

**Lemma 3.** If T is a  $(\gamma_t, \gamma_r^t)$ -tree, then T belongs to the family  $\tau$ .

**Proof.** Let T be a  $(\gamma_t, \gamma_r^t)$ -tree. If  $diam(T) \leq 5$ , then T is  $P_2$  or  $P_6$ . It is clear that the statement is true. For this reason, we only consider only trees T with  $diam(T) \geq 6$ .

Let T be a  $(\gamma_t, \gamma_r^t)$ -tree and assume that the result holds for all trees on n(T) - 1 and fever vertices. We proceed by induction on the number of vertices of a  $(\gamma_t, \gamma_r^t)$ -tree. Let  $P = (v_0, v_1, \ldots, v_l), l \ge 6$ , be a longest path in T and let D be a  $\gamma_r^t(T)$ -set. Then  $v_0, v_1 \in D$ . By Lemma 2, it follows that  $d(v_1) = d(v_2) = 2$ . It is obvious that  $v_2, v_3 \notin D$ . Otherwise  $D - \{v_0\}$  is a total dominating set with cardinality less than |D|, which is a contradiction.

Now we have the following claim.

Claim 1.  $|N_T(v_3) \cap D| = 1$ .

**Proof.** Without loss of generality, we can assume  $|N_T(v_3) \cap D| = t$  and t > 1. Then  $N_T(v_3) \cap D \subseteq S(T) \cup \{v_4\}$ . By Lemma 2,  $|N_T(v_3) \cap D \cap S(T)| = 1$ . So, t = 2. We can assume  $N_T(v_3) \cap D = \{v_{31}, v_4\}$ , where  $v_{31} \in S(T)$ . By Lemma 2, it is easy to prove that  $v_5 \in D$ . Let  $A_1 = N_T(v_5) - \{v_4\}$ . Then for any  $v \in A_1$ ,  $v \notin D$ . Otherwise, let  $T_1$  denote the component of  $T - \{v_5\}$  containing  $v_4$ . Then  $(D - (L(T_1) \cup \{v_4\})) \cup (N_{T_1}[S(T_1)] - L(T_1))$  is a total dominating set of T with cardinality less than |D|, which is a contradiction. Let  $B_1 = N_T(A_1) \cap (V(T) - D)$ ,  $A_2 = N_T(B_1) \cap D$  and  $B_2 = N_T(A_2) \cap D$ . For  $i \ge 1$ , let  $A_{2i+1} = N_T(B_{2i}) \cap (V(T) - D)$ ,  $B_{2i+1} = N_T(A_{2i+1}) \cap (V(T) - D)$ ,  $A_{2i+2} = N_T(B_{2i+1}) \cap D$  and  $B_{2i+2} = N_T(A_{2i+2}) \cap D$ . It is obvious that  $|B_{2i+1}| \le |A_{2i+2}| \le |B_{2i+2}|$  for  $i \ge 0$ .

Now we prove that if  $N_T(B_{2i+2}) \cap D - A_{2i+2} \neq \emptyset$ , then  $|N_T(v) \cap D| \ge 2$ for any  $v \in N_T(B_{2i+2}) \cap D - A_{2i+2}$ . Otherwise, we can assume t is the maximum i satisfying  $N_T(B_{2i+2}) \cap D - A_{2i+2} \neq \emptyset$  and there exists a vertex  $v \in N_T(B_{2i+2}) \cap D - A_{2i+2}$  such that  $|N_T(v) \cap D| = 1$ . Without loss of generality, we can assume that  $u \in B_{2t+2}$  and  $uv \in E(T)$ .

Define  $C_1 = N_T(v) \setminus \{u\}$ . Then for any  $w \in C_1$ ,  $w \notin D$ . Let  $D_1 = N_T(C_1) \cap (V(T) - D)$ . Let  $C_2 = N_T(D_1) \cap D$  and  $D_2 = N_T(C_2) \cap D$ . For  $i \ge 1$ , let  $C_{2i+1} = N_T(D_{2i}) \cap (V(T) - D)$ ,  $D_{2i+1} = N_T(C_{2i+1}) \cap (V(T) - D)$ ,  $C_{2i+2} = N_T(D_{2i+1}) \cap D$  and  $D_{2i+2} = N_T(C_{2i+2}) \cap D$ . It is obvious that  $|D_{2i+1}| \le |C_{2i+2}| \le |D_{2i+2}|$  for  $i \ge 0$ . Let  $D' = (D - \{v\} - \bigcup_{0 \le i \le t} D_{2i+2}) \cup \bigcup_{0 \le i \le t} D_{2i+1}$ . It is obvious that D' is a total dominating set of T with cardinality less than |D|, which is a contradiction.

Let  $w \in A_1$ . Let  $\overline{D} = (D - (L(T_1) \cup \{v_4, v_5\}) - \bigcup_{0 \le i \le t} B_{2i+2}) \cup \bigcup_{0 \le i \le t} B_{2i+1} \cup \{w\} \cup (N_{T_1}[S(T_1))] - L(T_1))$ . It is obvious that  $\overline{D}$  is a total dominating set of T with cardinality less than |D|, which is a contradiction. Hence,  $|N_T(v_3) \cap D| = 1$ .

By the above claim, we consider the following three cases. Assume  $d_T(v_4) = j$ .

Case 1.  $v_4 \in D$  and  $v_4 \in S(T)$ . Let  $T_1$  denote the component of  $T - \{v_4\}$  containing  $v_5$ . Let  $N_T(v_4) \cap L(T) = \{l\}$  and  $N_T(v_4) - \{v_5, l\} = \{v_{41}, \cdots, v_{4(j-2)}\}$ . Denote  $T' = \langle V(T_1) \cup \{v_4, l\} \rangle$ . Then it is easy to prove that  $\gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$ . It is obvious that  $\gamma_r^t(T') \leq \gamma_r^t(T) - 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$ . Since T is a  $(\gamma_t, \gamma_r^t)$ -tree, it follows that  $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1) \leq \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$ . Hence  $\gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-2)} (d_T(v_{4i}) - 1)$ . So  $\gamma_t(T') = \gamma_r^t(T')$ . Consequently, T' is a  $(\gamma_t, \gamma_r^t)$ -tree and by induction hypothesis,  $T' \in \tau$ . As  $v_4$  is a support in T', we deduce that T may be obtained from T' by operation  $\tau_1$ .

TREES WITH EQUAL TOTAL DOMINATION AND ...

Case 2.  $v_4 \in D$  and  $v_4 \notin S(T)$ . Let  $T_1$  denote the component of  $T - \{v_4\}$ containing  $v_5$ . Then  $v_5 \in D$ . Let  $N_T(v_4) - \{v_5\} = \{v_{41}, \cdots, v_{4(j-1)})\}$ . Denote  $T' = \langle V(T_1) \cup \{v_4\} \rangle$ . Then it is obvious that  $\gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-1)} (d(v_{4i}) - 1)$ . It is obvious that  $\gamma_r^t(T') \leq \gamma_r^t(T) - 2\sum_{1 \leq i \leq (j-1)} (d(v_{4i}) - 1)$ . Since T is a  $(\gamma_t, \gamma_r^t)$ -tree, it follows that  $\gamma_r^t(T) = \gamma_t(T) = \gamma_t(T') + 2\sum_{1 \leq i \leq (j-1)} (d(v_{4i}) - 1) \leq \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-1)} (d(v_{4i}) - 1) \leq \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-1)} (d(v_{4i}) - 1)$ . Hence  $\gamma_r^t(T) = \gamma_r^t(T') + 2\sum_{1 \leq i \leq (j-1)} (d(v_{4i}) - 1)$ . So  $\gamma_t(T') = \gamma_r^t(T')$ . Consequently, T' is a  $(\gamma_t, \gamma_r^t)$ -tree and by induction hypothesis,  $T' \in \tau$ . As  $v_4$  is a leaf in T', we deduce that T may be obtained from T' by operation  $\tau_1$ .

Case 3.  $v_4 \notin D$ . Then there exists exactly one vertex  $x \in N_T(v_3) \cap D$  and x is a support. Assume  $N_T(x) \cap L(T) = \{l\}$ . Let  $T_1$  denote the component of  $T - \{v_3\}$  containing  $v_4$ . Denote  $T' = \langle V(T_1) \cup \{v_3, x, l\} \rangle$ . It is obvious that  $\gamma_t(T) = \gamma_t(T') + 2(d_T(v_3) - 2)$ . It is obvious that  $x, l \in D$ . Hence  $\gamma_r^t(T') \leq \gamma_r^t(T) - 2(d_T(v_3) - 2)$ . Since T is a  $(\gamma_t, \gamma_r^t)$ -tree, it follows that  $\gamma_r^t(T) = \gamma_t(T') + 2(d_T(v_3) - 2) \leq \gamma_r^t(T') + 2(d_T(v_3) - 2)$ . Hence  $\gamma_r^t(T) = \gamma_t(T') + 2(d_T(v_3) - 2)$ . So  $\gamma_t(T') = \gamma_r^t(T')$ . Consequently, T' is a  $(\gamma_t, \gamma_r^t)$ -tree and by induction hypothesis,  $T' \in \tau$ . As  $v_3$  is a vertex adjacent to a support in T', we deduce that T may be obtained from T' by operation  $\tau_2$ .

As an immediate consequence of Lemmas 2 and 3 we have the following characterization of  $(\gamma_t, \gamma_r^t)$ -trees.

**Theorem 3.** A tree T is a  $(\gamma_t, \gamma_r^t)$ -tree if and only if T belongs to the family  $\tau$ .

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