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RECOGNIZABLE COLORINGS OF GRAPHS

Gary Chartrand

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

Linda Lesniak

Department of Mathematics
and Computer Science, Drew University
Madison, NJ 07940, USA

Donald W. VanderJagt

Department of Mathematics
Grand Valley State University
Allendale, MI 49401, USA

and

Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

Dedicated to the memory of Frank Harary (1921–2005)

Abstract

Let G be a connected graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a coloring of the vertices of G for some positive integer k (where adjacent vertices may be colored the same). The color code of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $\mathrm{code}(v) = (a_0, a_1, \ldots, a_k)$ where a_0 is the color assigned to v and for $1 \le i \le k$, a_i is the number of vertices adjacent to v that are colored i. The coloring c is called recognizable if distinct vertices have distinct color codes and the recognition number $\mathrm{rn}(G)$ of G is the minimum positive integer k for which G has a recognizable k-coloring. Recognition numbers of complete multipartite graphs are determined and characterizations of connected graphs of order n having recognition numbers n or $n-1$ are established. It is shown that for each pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph of order n having recognition number k. Recognition numbers of cycles, paths, and trees are investigated.

Keywords: recognizable coloring, recognition number.

2000 Mathematics Subject Classification: 05C15, 05C70.

Seventy-five student leaders, 15 freshmen, 15 sophomores, 15 juniors, 15 seniors, and 15 graduate students, have been invited to a banquet. Is it possible to seat all 75 students around a 75-seat circular table in such a way that no two students belonging to the same class are seated next to two students belonging to the same class or the same two classes? Thus no two freshmen can both be seated next to two juniors, to two freshmen, to a senior and a graduate student, or to a freshmen and a sophomore, for example. This question has an affirmative answer, as is shown in Figure 1,

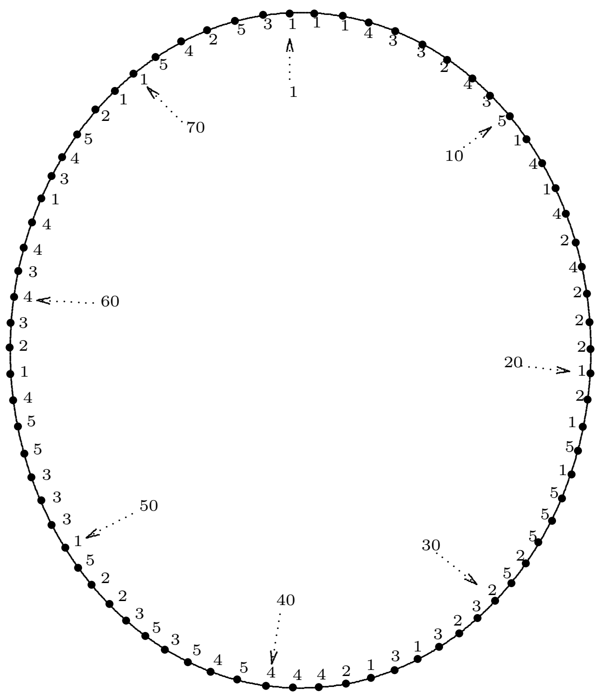

Figure 1. Seating 75 students around a table.

where 1, 2, 3, 4, 5 represent a freshman, sophomore, junior, senior, graduate student, respectively. Consequently, every student is uniquely determined by the class to which he or she belongs and the classes of his or her two neighbors at the banquet table.

The question asked above suggests a concept and some problems in graph theory.

1. Introduction

A basic problem in graph theory concerns finding means to distinguish the vertices of a connected graph. This has often been accomplished by means of an edge coloring, which is then sometime referred to as a vertex-distinguishing or irregular edge coloring.

One way to distinguish the vertices of a graph G was introduced by Harary and Plantholt [11] where colors were assigned to the edges of G in such a way that for every two vertices of G, one of the vertices is incident with an edge assigned one of these colors that the other vertex is not. Essentially then each vertex is assigned the set of colors of its incident edges and no two vertices of G are assigned the same set. Harary and Plantholt referred to the minimum number of colors needed to accomplish this as the *point-distinguishing chromatic index* of G.

Another way of distinguishing the vertices of a graph G is by assigning each vertex of G a color code from a given edge coloring of G. Let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of G for some positive integer k. The *color code* of a vertex v of G with respect to a k-edge coloring c of the edges of G is the ordered k-tuple $(a_1, a_2, \ldots, a_k)$ where a_i is the number of edges incident with v that are colored i for $1 \leq i \leq k$. The edge-coloring c is *vertex-distinguishing* (or *irregular*) if distinct vertices have distinct color codes. The minimum positive integer k for which G has a vertex-distinguishing k-coloring has been the primary topic of interest. These colorings have been studied in [2, 3, 5, 6, 7].

Yet another vertex-distinguishing edge coloring of a graph G that has been the object of study is that obtained from a coloring $c : E(G) \to \{1, 2, \ldots, k\}$, where each vertex is assigned the sum of the colors of its incident edges. This concept was introduced in [8].

A related problem deals with neighbor-distinguishing edge colorings of a graph. For example, Balister, Győri, Lehel, and Schelp [4] investigated edge colorings in which each vertex is assigned the set of colors of its incident edges and every two adjacent vertices are assigned distinct sets. Karoński, Łuczak, and Thomason [12] studied edge colorings in which each vertex is assigned the color which is the sum of the colors of its incident edges and adjacent vertices have different colors; while Addario-Berry, Aldred, K. Dalal, and Reed [1] studied edge colorings in which each vertex is assigned the resulting color code and adjacent vertices have different color codes. The problem discussed in [1] as to whether a neighbor-distinguishing 3-edge coloring exists for every graph was independently mentioned in [10] as well.

We now introduce a new method of uniquely recognizing the vertices of a graph that combines a number of the features of the methods mentioned above. Let G be a graph and let $c : V(G) \to \{1, 2, \ldots, k\}$ be a coloring of the vertices of G for some positive integer k (where adjacent vertices may be colored the same). The *color code* of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple

$$\text{code}_c(v) = (a_0, a_1, \ldots, a_k) \quad \text{(or simply, code}(v) = a_0a_1a_2 \cdots a_k),$$

where a_0 is the color assigned to v (that is, $c(v) = a_0$) and for $1 \le i \le k$, a_i is the number of vertices adjacent to v that are colored i. Therefore, $\sum_{i=1}^{k} a_i = \deg_G v$. The coloring c is called *recognizable* if distinct vertices have distinct color codes and the *recognition number* $\text{rn}(G)$ of G is the minimum positive integer k for which G has a recognizable k-coloring. Such a coloring is called a *minimum recognizable coloring*. A graph G and its complement $\overline{G}$ have the same recognition number.

Proposition 1.1. *For every graph G,* $\text{rn}(G) = \text{rn}(\overline{G})$.

Proof. Suppose that $\text{rn}(G) = k$ and $\text{rn}(\overline{G}) = \overline{k}$. Let c be a recognizable k-coloring of G. Define a k-coloring $\overline{c}$ of $\overline{G}$ by $\overline{c}(v) = c(v)$ for each $v \in V(\overline{G}) = V(G)$. Suppose, in the coloring c of G, there are n_i vertices of G colored i for $1 \le i \le k$. Let x and y be two vertices of $\overline{G}$ that have same color code with respect to $\overline{c}$. We may assume that $\overline{c}(x) = \overline{c}(y) = 1$ and that

$$\text{code}_{\overline{c}}(x) = (1, a_1, a_2, \ldots, a_k) = \text{code}_{\overline{c}}(y).$$

Consequently,

$$\text{code}_c(x) = (1, n_1 - a_1 - 1, n_2 - a_2, \ldots, n_k - a_k) = \text{code}_c(y).$$

Since c is recognizable, $x = y$, which implies that $\bar{c}$ is a recognizable k-coloring of $\overline{G}$. Since $\mathrm{rn}(\overline{G}) = \overline{k}$, it follows that $\overline{k} \leq k$. By a similar argument, $k \leq \overline{k}$ and so $k = \overline{k}$. Thus $\mathrm{rn}(G) = \mathrm{rn}(\overline{G})$. ∎

In particular, $\mathrm{rn}(K_n) = \mathrm{rn}(\overline{K}_n) = n$ for every positive integer n. Since the complement of every disconnected graph is connected, it follows by Proposition 1.1 that we may restrict our attention to connected graphs in our study of recognizable colorings of graphs.

Since every coloring that assigns distinct colors to the vertices of a connected graph is recognizable, the recognition number is always defined. On the other hand, it is well-known that every nontrivial graph contains at least two vertices having the same degree. Thus if all vertices of a nontrivial graph are assigned the same color, then any two vertices of the same degree will have the same color code. Therefore, if G is a nontrivial connected graph of order n, then

$$2 \leq \mathrm{rn}(G) \leq n.$$

There are some observations that will be useful to us.

Observation 1.2. *Let c be a coloring of the vertices of a graph G. If u and v are two vertices of G with $\deg_G u \neq \deg_G v$, then $\mathrm{code}(u) \neq \mathrm{code}(v)$.*

In particular, to show that a coloring of a graph G is recognizable, it is necessary and sufficient to show that every two vertices of the same degree and same color have distinct codes.

The *neighborhood* of a vertex u in a graph G is $N(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$.

Observation 1.3. *Let c be a recognizable coloring of a graph G. If u and v are distinct vertices of G with $N[u] = N[v]$, then $c(u) \neq c(v)$.*

The following result, dealing with combinations with repetition, is well-known in discrete mathematics.

Theorem A. *Let A be a set containing k different kinds of elements, where there are at least r elements of each kind. The number of different selections of r elements from A is $\binom{r+k-1}{r}$.*

In terms of graphs, Theorem A can be stated as follows.

Theorem 1.4. *Let c be a k-coloring of the vertices of a graph G. The number of different possible color codes of the vertices of degree r in G is $k\binom{r+k-1}{r}$.*

The following result is a consequence of Theorem 1.4.

Corollary 1.5. *If c is a recognizable k-coloring of a nontrivial connected graph G, then G contains at most $k\binom{r+k-1}{r}$ vertices of degree r.*

We now consider some examples of recognizable colorings of some cubic (3-regular) graphs. The following is a consequence of Corollary 1.5 for cubic graphs.

Corollary 1.6. *If G is a connected cubic graph of order n having recognition number k, then*

$$n \le \frac{k^4 + 3k^3 + 2k^2}{6}.$$

The Petersen graph P (shown in Figure 2) is a cubic graph of order 10. By Corollary 1.6, $\mathrm{rn}(P) \ge 3$. A 3-coloring of the Petersen graph is given in Figure 1 along with the corresponding color codes of its vertices. Since distinct vertices have distinct color codes, this coloring is recognizable. Thus $\mathrm{rn}(P) \le 3$ and so $\mathrm{rn}(P) = 3$.

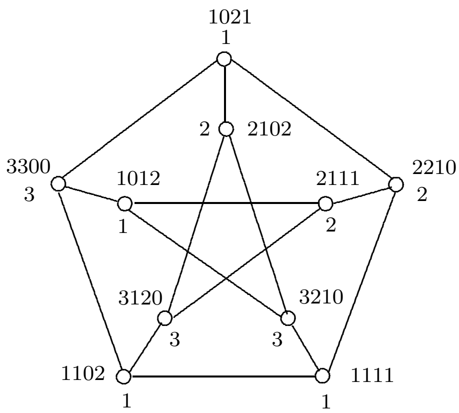

Figure 2. A minimum recognizable coloring of the Petersen graph.

According to Corollary 1.6, if G is a connected cubic graph of order n having recognition number 2, then $n \le 8$. There is no connected cubic graph of order 8 with recognition number 2, however.

Proposition 1.7. *There exists no connected cubic graph of order* 8 *having recognition number* 2.

Proof. Assume, to the contrary, that there exists a cubic graph G of order 8 with $\mathrm{rn}(G) = 2$. Therefore, there is a recognizable 2-coloring of G. Thus we may assume that $V(G) = \{v_1, v_2, \ldots, v_8\}$ and that the codes of the vertices of G are

$$\mathrm{code}(v_1) = 130,\ \mathrm{code}(v_2) = 103,\ \mathrm{code}(v_3) = 121,\ \mathrm{code}(v_4) = 112,$$
$$\mathrm{code}(v_5) = 230,\ \mathrm{code}(v_6) = 203,\ \mathrm{code}(v_7) = 221,\ \mathrm{code}(v_8) = 212.$$

Since v_1 is colored 1 and is adjacent to three vertices colored 1, namely v_2, v_3, and v_4, it follows that each of v_2, v_3, and v_4 is adjacent to at least one vertex colored 1. Since $\mathrm{code}(v_2) = 103$, the vertex v_2 is adjacent to no vertex colored 1, producing a contradiction. ■

A well-known cubic graph of order 8 is the 3-cube, often denoted by Q_3, which is shown in Figure 3. According to Proposition 1.7, $\mathrm{rn}(Q_3) \geq 3$. A 3-coloring of Q_3 is given in Figure 3 along with the corresponding codes of its vertices. Since distinct vertices have distinct color codes, this coloring is recognizable. Thus $\mathrm{rn}(Q_3) \leq 3$ and so $\mathrm{rn}(Q_3) = 3$.

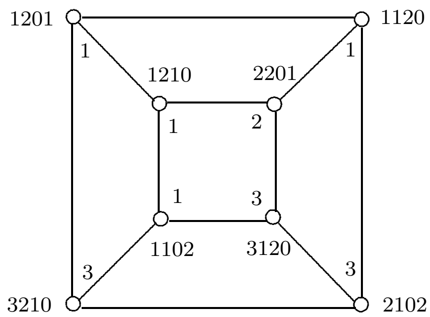

Figure 3. A minimum recognizable coloring of the 3-cube.

Another well-known cubic graph is $K_3 \times K_2$, the Cartesian product of K_3 and K_2. This graph is shown in Figure 4. If $\mathrm{rn}(K_3 \times K_2) = 2$, then there exists a recognizable 2-coloring c of $K_3 \times K_2$ whose vertices have six of the following eight color codes: 130, 103, 121, 112, 230, 203, 221, 212.

Suppose first that three vertices of $K_3 \times K_2$ are colored 1 and three vertices of $K_3 \times K_2$ are colored 2. Then no vertex has code 130 or 203. Hence some vertex u of $K_3 \times K_2$ has code 103 and so u is adjacent to three vertices x, y, and z, all of which are colored 2. Also, one of x, y, and z has code 230, say x has code 230. Thus x is adjacent only to vertices colored 1, namely u and two other vertices, say v and w. Since neither v nor w is adjacent to u, the vertices v and w are both adjacent to no or to exactly one vertex colored 1, contradicting the fact that 121 and 112 are both codes of vertices of $K_3 \times K_2$. Thus $\mathrm{rn}(K_3 \times K_2) \geq 3$. The 3-coloring of $K_3 \times K_2$ shown in Figure 4 is recognizable and so $\mathrm{rn}(K_3 \times K_2) \leq 3$. Thus $\mathrm{rn}(K_3 \times K_2) = 3$.

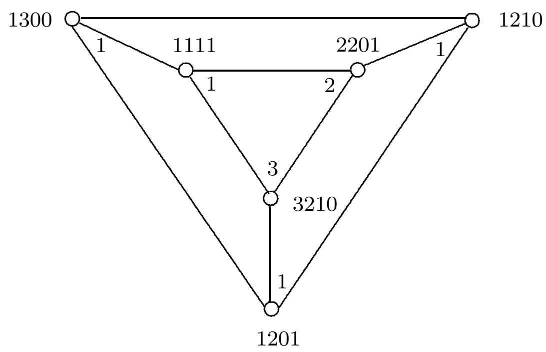

Figure 4. A minimum recognizable coloring of $K_3 \times K_2$.

We refer to the book [9] for graph-theoretic notation and terminology not described in this paper.

2. Recognizable Colorings of Cycles

It is straightforward to show that no 2-coloring of any cycle is recognizable. Therefore, $\mathrm{rn}(C_n) \geq 3$ for every integer $n \geq 3$. There are many cycles having recognition number 3, however. Recognizable 3-colorings of the cycles C_n, $3 \leq n \leq 9$, are shown in Figure 5.

By Corollary 1.5, the number of vertices of degree 2 in a graph having recognition number k is at most $(k^3 + k^2)/2$. In particular, if $\mathrm{rn}(C_n) = 3$, then $n \leq 18$. Since the 3-coloring of C_{18} shown in Figure 6 is recognizable, it follows that $\mathrm{rn}(C_{18}) = 3$.

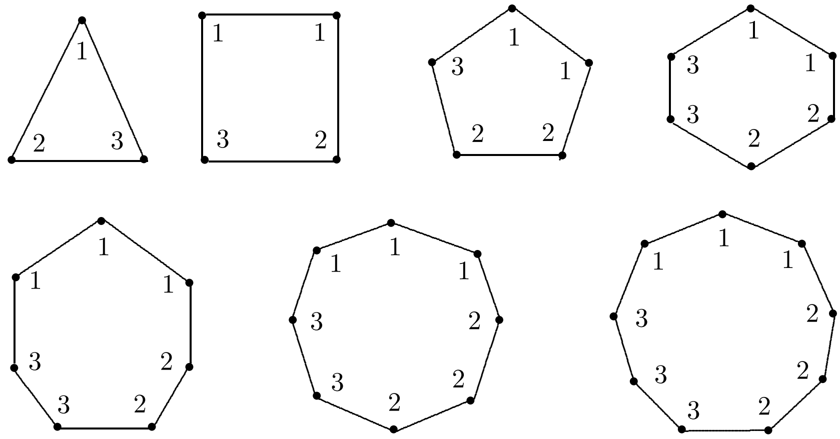

Figure 5. Recognizable 3-colorings of cycles C_n, $3 \leq n \leq 9$.

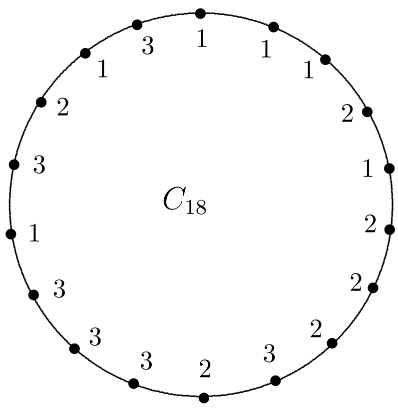

Figure 6. A minimum recognizable coloring of C_{18}.

Again, by Corollary 1.5, if $\mathrm{rn}(C_n) = 4$, then $n \leq 40$. There is no recognizable 4-coloring of C_{40}, however, for assume, to the contrary, that such a 4-coloring c of C_{40} exists. Then there exists a vertex v_1 C_{40} such that $c(v_1) = 111$. The vertices u_1 and w_1 adjacent to v_1 on C_{40} have codes $11a$ and $11b$, where $a \neq b$ and $a, b \in \{2, 3, 4\}$, say $\mathrm{code}(u_1) = 112$ and $\mathrm{code}(w_1) = 113$. The two vertices following u_1 (and w_1) about C_{40} cannot both be colored 1 since no two vertices of C_{40} have the same code. There exists some vertex y_1 on C_{40} with $\mathrm{code}(y_1) = 114$, however. Thus $c(y_1) = 1$.

Let x_1 and z_1 be the neighbors of y_1, where $c(x_1) = 1$ and $c(z_1) = 4$. However then, $\mathrm{code}(x_1) \in \{111, 112, 113\}$, which is impossible. In general, in any recognizable 3-coloring of C_{40}, at most two elements in each of the sets

$$\{112, 113, 114\},\ \{221, 223, 224\},\ \{331, 332, 334\},\ \{441, 442, 443\},$$

can be color codes. Therefore, if $\mathrm{rn}(C_n) = 4$, then $n \leq 36$. That $\mathrm{rn}(C_{36}) = 4$ is shown in Figure 7.

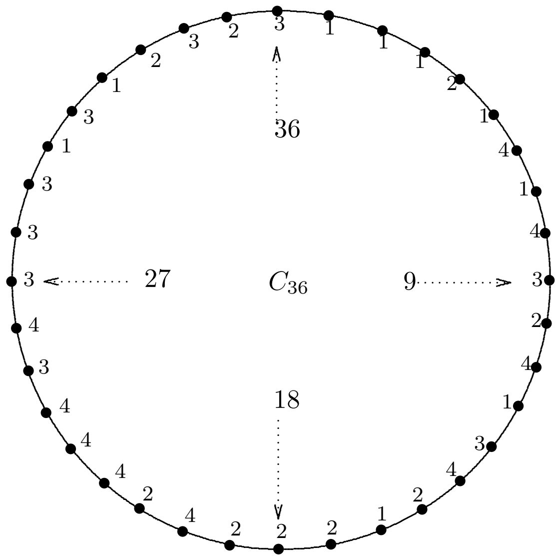

Figure 7. A minimum recognizable 4-coloring of C_{36}.

In general, we have the following lower bound for $\mathrm{rn}(C_n)$.

Proposition 2.1. *Let $k \geq 3$ be an integer. Then $rn(C_n) \geq k$ for all integers n such that*

$$\frac{(k-1)^3+(k-1)^2-2(k-1)+2}{2} \leq n \leq \frac{k^3+k^2}{2} \qquad \text{if } k \text{ is odd,}$$

$$\frac{(k-1)^3+(k-1)^2+2}{2} \leq n \leq \frac{k^3+k^2-2k}{2} \qquad \text{if } k \text{ is even.}$$

We conjecture that the lower bound for $\mathrm{rn}(C_n)$ in Proposition 2.1 is, in fact, an equality throughout.

Conjecture 2.2. Let $k \geq 3$ be an integer. Then $\text{rn}(C_n) = k$ for all integers n such that

$$\frac{(k-1)^3+(k-1)^2-2(k-1)+2}{2} \leq n \leq \frac{k^3+k^2}{2} \qquad \text{if } k \text{ is odd,}$$
$$\frac{(k-1)^3+(k-1)^2+2}{2} \leq n \leq \frac{k^3+k^2-2k}{2} \qquad \text{if } k \text{ is even.}$$

Our initial example concerning seating five groups of 15 students each satisfying certain conditions and illustrated in Figure 1 shows that $\text{rn}(C_{75}) = 5$. Therefore, Conjecture 2.2 is true for $k = 5$ and $n = 75$.

We describe how the recognizable 5-coloring of C_{75} in Figure 1 was constructed. (The recognizable 4-coloring of C_{36} shown in Figure 7 was constructed by a similar approach). Let $S = \{1, 2, 3, 4, 5\}$. With the aid of a so-called deBruijn digraph, a cyclic sequence

$$s: \ a_1, a_2, \ldots, a_{125}, a_{126} = a_1, a_{127} = a_2$$

of length 125 can be constructed whose terms are the elements of S and having the property that the 3-term subsequences a_i, a_{i+1}, a_{i+2} $(1 \leq i \leq 125)$ are all 125 3-permutations of the elements of S. This deBruijn digraph D has order 25 and $V(D)$ consists of the 2-permutations of the elements of S. An arc e of D joins two vertices ab and cd, where $a, b, c, d \in S$, and is directed from ab to cd if and only if $b = c$, and e is labeled abd. For example, 12 is joined to 23 (resulting in an arc labeled 123), but 23 is not joined to 12; while 12 is joined to 21 (resulting in an arc labeled 121) and 21 is joined to 12 (resulting in an arc labeled 212). Also, 11 is joined to 11 by a directed loop labeled 111. The resulting digraph D is connected and every vertex of D has outdegree 5 and indegree 5. This implies that D is Eulerian and so contains an Eulerian circuit whose 125 arcs can therefore be listed cyclically as e_1, $e_2, \ldots, e_{125}$, $e_{126} = e_1$ so that the labels of these arcs are the 3-permutations of the elements of S and such that if abc is the label of e_i $(1 \leq i \leq 125)$, then bcd is the label of e_{i+1} for some $d \in S$.

In our case, we are interested in constructing a cyclic sequence

$$s': \ b_1, b_2, \ldots, b_{75}, b_{76} = b_1, b_{77} = b_2$$

of length 75 whose terms are the elements of S and having the property that the 3-term subsequences b_i, b_{i+1}, b_{i+2} $(1 \leq i \leq 75)$ are all 75 3-permutations abc of the elements of S such that exactly one of a, b, c and c, b, a occurs among the 3-term subsequences of s'. In order to construct such a sequence s', we seek a spanning Eulerian subdigraph D' of D such that for each 3-permutation abc of the elements of S, exactly one arc of D' is labeled abc or cba. For example, D' must contain (1) the directed loop labeled 111, (2) both of the arcs labeled 121 and 212, (3) exactly one of the arcs labeled 112 and 211, and (4) exactly one of the arcs labeled 123 and 321. The recognizable 5-coloring of C_{75} in Figure 1 was constructed by finding an Eulerian subdigraph D' of D and an Eulerian circuit C' of D'. Therefore, Conjecture 2.2 can be verified for given k and n satisfying one of the conditions in this conjecture if an appropriate Eulerian digraph of order k^2 and size n can be constructed.

3. Recognizable Colorings of Trees

First, we consider paths. As a result of Proposition 2.1 and the fact that it is possible to have two additional color codes for vertices of degree 2 in P_n than for C_n when k is even, we have the following lower bound for $\mathrm{rn}(P_n)$.

Proposition 3.1. *Let $k \geq 3$ be an integer. Then $\mathrm{rn}(P_n) \geq k$ for all integers n such that*

$$\begin{array}{ll} \frac{(k-1)^3+(k-1)^2-2(k-1)+10}{2} \leq n \leq \frac{k^3+k^2+4}{2} & \textit{if } k \textit{ is odd,} \\ \frac{(k-1)^3+(k-1)^2+6}{2} \leq n \leq \frac{k^3+k^2-2k+8}{2} & \textit{if } k \textit{ is even.} \end{array}$$

Indeed, we conjecture that the lower bound for $\mathrm{rn}(P_n)$ in Proposition 3.1 is an equality throughout.

Conjecture 3.2. Let $k \geq 3$ be an integer. Then $\mathrm{rn}(P_n) = k$ for all integers n such that

$$\begin{array}{ll} \frac{(k-1)^3+(k-1)^2-2(k-1)+10}{2} \leq n \leq \frac{k^3+k^2+4}{2} & \text{if } k \text{ is odd,} \\ \frac{(k-1)^3+(k-1)^2+6}{2} \leq n \leq \frac{k^3+k^2-2k+8}{2} & \text{if } k \text{ is even.} \end{array}$$

It is known that $\mathrm{rn}(P_n) = 2$ if $2 \leq n \leq 8$ and $\mathrm{rn}(P_n) = 3$ if $9 \leq n \leq 20$. For example, minimum recognizable colorings for P_8 and P_{20} are shown in Figure 8 along with the corresponding color codes of their vertices.

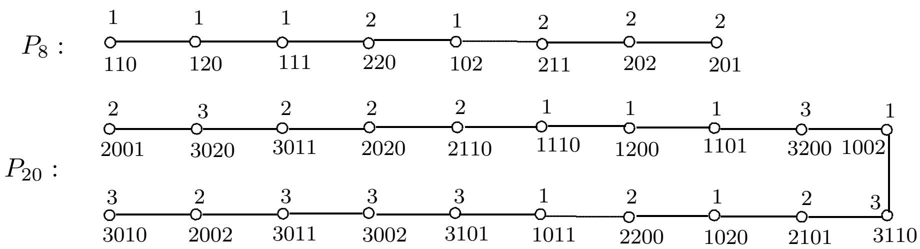

Figure 8. Minimum colorings for P_8 and P_{20}.

The largest integer n for which $\text{rn}(P_n) = 4$ is $n = 40$. A recognizable 4-coloring of P_{40} is shown in Figure 9 and so $\text{rn}(P_{40}) = 4$. The largest possible integer n for which $\text{rn}(P_n) = 5$ is $n = 77$. It can be shown that $\text{rn}(P_{77}) = 5$.

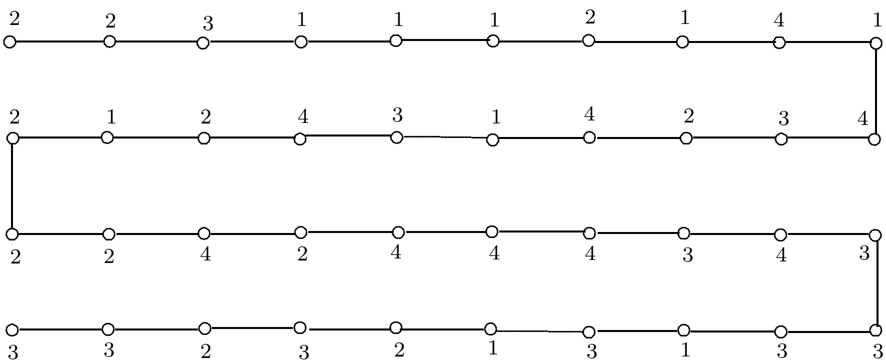

Figure 9. A minimum recognizable 4-coloring of P_{40}.

Now we consider trees more generally. Let T be a tree of order n having n_i vertices of degree i for $i \geq 1$. For each integer $n \geq 2$, let $D(n)$ be the maximum recognition number among all trees of order n and $d(n)$ the minimum recognition number among all trees of order n. That is, if $\mathcal{T}_n$ is the set of all trees of order n, then

$$\begin{aligned} D(n) &= \max\{\text{rn}(T) : T \in \mathcal{T}_n\}, \\ d(n) &= \min\{\text{rn}(T) : T \in \mathcal{T}_n\}. \end{aligned}$$

Therefore, $2 \leq d(n) \leq D(n) \leq n-1$. It is clear that $d(2) = D(2) = 2$. As we will see, the star $K_{1,n-1}$ of order $n \geq 3$ has recognition number $n-1$ and no tree of order $n \geq 3$ has recognition number n. Thus we have the following.

Observation 3.3. *For each integer $n \geq 3$, $D(n) = n-1$.*

It is known that if T is a tree of order n having n_i vertices of degree i for $i \geq 1$, then

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \dots \tag{1}$$

(see [9, p. 59], for example). By Corollary 1.5, if c is a recognizable k-coloring of a connected graph G of order at least 3, then G contains at most k^2 end-vertices and at most $\frac{k^3+k^2}{2}$ vertices of degree 2. It then follows by (1) that if T is a tree of order n with $\mathrm{rn}(T) = k$, then

$$n \leq k^2 + \frac{k^3+k^2}{2} + (k^2 - 2) = \frac{k^3 + 5k^2 - 4}{2}.$$

For example, if T is a tree of order n with $\mathrm{rn}(T) = 2$, then $n \leq 12$. The tree T shown in Figure 10 has order 12 with $\mathrm{rn}(T) = 2$. Observe that T has $2^2 = 4$ end-vertices, $\frac{2^3+2^2}{2} = 6$ vertices of degree 2, and $2^2 - 2 = 2$ vertices of degree 3. In fact, if $2 \leq n \leq 12$, then $d(n) = 2$.

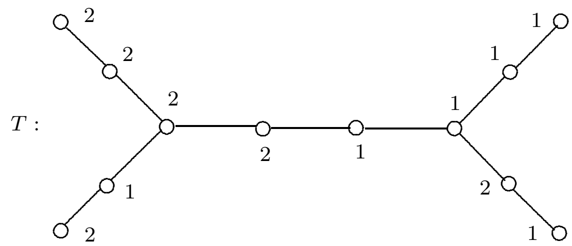

Figure 10. A tree of order 12 with $\mathrm{rn}(T) = 2$.

If T is a tree of order n with $rn(T) = 3$, then $n \leq 34$. The tree T of Figure 11 has order 34 and $rn(T) = 3$. This tree T contains 9 vertices of degree 1, 18 vertices of degree 2, and 7 vertices of degree 3. Therefore, $d(34) = 3$.

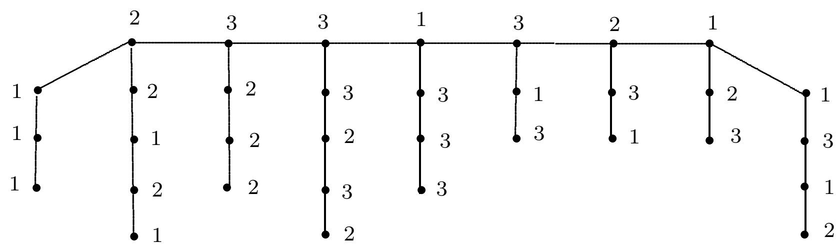

Figure 11. A tree T of order 34 with $rn(T) = 3$.

Also, if T is a tree of order n with $rn(T) = 4$, then $n \leq 70$. The tree T of Figure 12 has order 70 and $rn(T) = 4$. This tree T contains 16 vertices of degree 1, 40 vertices of degree 2, and 14 vertices of degree 3. This shows that $d(70) = 4$.

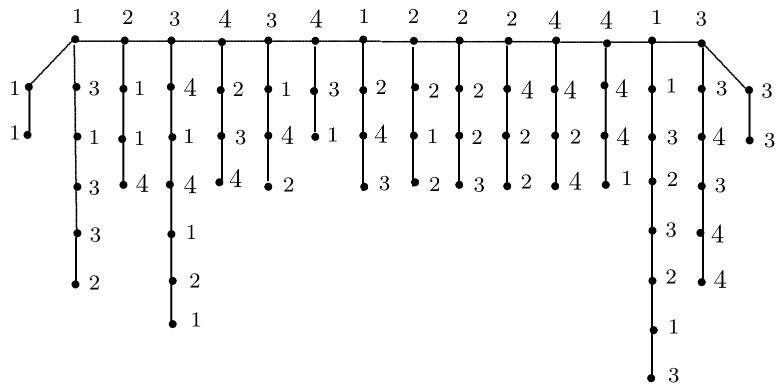

Figure 12. A tree T of order 70 with $rn(T) = 4$.

In general, we have the following conjecture.

Conjecture 3.4. For each integer $n \geq 3$, the minimum recognition number among all trees of order n is the unique integer k such that

$$\frac{(k-1)^3 + 5(k-1)^2 - 2}{2} \leq n \leq \frac{k^3 + 5k^2 - 4}{2}.$$

It is easy to see, however, that the minimum recognition number among all trees of order n is bounded below by the integer k described in Conjecture 3.4.

4. Recognizable Colorings of Complete Multipartite Graphs

In this section we determine the recognition numbers of all complete multipartite graphs. Let G be a complete k-partite graph for some positive integer k. If every partite set of G has a vertices for some positive integer a, then we write $G = K_{a(k)}$, where then $K_{a(1)} = \overline{K}_a$.

Theorem 4.1. *Let k and a be positive integers. Then the recognition number of the complete k-partite graph $K_{a(k)}$ is the unique positive integer ℓ for which $\binom{\ell-1}{a} < k \leq \binom{\ell}{a}$.*

Proof. Suppose that $G = K_{a(k)}$ has partite sets $U_1, U_2, \ldots, U_k$, where $|U_i| = a$ for $1 \leq i \leq k$. We first show that $\mathrm{rn}(G) \geq \ell$. Assume, to the contrary, that $\mathrm{rn}(G) \leq \ell - 1$. Then there exists a recognizable coloring c of G using $\ell - 1$ or fewer colors. Let $\mathcal{S} = \{1, 2, \ldots, \ell - 1\}$. For each integer i with $1 \leq i \leq k$, let

$$\mathcal{C}_i = \{c(x) : \ x \in U_i\}$$

be the set of the colors of the vertices of U_i. Then $\mathcal{C}_i$ is an a-element subset of S for $1 \leq i \leq k$. Since S has exactly $\binom{\ell-1}{a}$ distinct a-element subsets and $k > \binom{\ell-1}{a}$, it follows that there exist two partite sets U_s and U_t, where $1 \leq s \neq t \leq k$, such that $\mathcal{C}_s = \mathcal{C}_t$, that is,

$$\{c(x) : x \in U_s\} = \{c(y) : y \in U_t\}.$$

Thus, there exist $x \in U_s$ and $y \in U_t$ such that $c(x) = c(y)$. However then, $\mathrm{code}(x) = \mathrm{code}(y)$, which contracts the fact that c is a recognizable coloring of G. Therefore, $\mathrm{rn}(G) \geq \ell$.

Next, we show that $\mathrm{rn}(G) \leq \ell$. Let $\mathcal{L} = \{1, 2, \ldots, \ell\}$ and let

$$\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{\binom{\ell}{a}}$$

be the $\binom{\ell}{a}$ distinct a-element subsets of L. Since $k \leq \binom{\ell}{a}$, we can define a coloring c' of G that assigns the a distinct colors of $\mathcal{L}_i$ to the vertices of U_i for $1 \leq i \leq k$. Since $k > \binom{\ell-1}{a}$, there must be at least one vertex of G that is assigned color j for each color j with $1 \leq j \leq \ell$. Thus c' is an ℓ-coloring of G. It remains to show that c' is a recognizable coloring. Let u and v be two vertices of G such that $c'(u) = c'(v)$. Since the a vertices in each partite set

U_i are colored differently by c' for $1 \le i \le k$, it follows that u and v belong two different partite sets of G. We may assume, without loss of generality, that $u \in U_1$ and $v \in U_2$. Suppose that

$$\begin{aligned} \mathcal{C}'_1 &= \{c'(x) : x \in U_1\} = \{s_1, s_2, \ldots, s_a\}, \\ \mathcal{C}'_2 &= \{c'(y) : y \in U_2\} = \{t_1, t_2, \ldots, t_a\}. \end{aligned}$$

Since $\mathcal{C}'_1 \neq \mathcal{C}'_2$, either there is an element in $\mathcal{C}'_1$ that is not in $\mathcal{C}'_2$ or an element in $\mathcal{C}'_2$ that is not in $\mathcal{C}'_1$, say the former. We may assume, without loss of generality, that $s_1 \notin \mathcal{C}'_2$. Let w be the vertex colored s_1 in U_1. (Note that it is possible that $w = u$). Observe that

(1) u and v are both adjacent to every vertex in $V(G) - (U_1 \cup U_2)$, and so u and v are both adjacent to every vertex colored s_1 in $V(G) - (U_1 \cup U_2)$,

(2) v is adjacent to every vertex in U_1 but u is adjacent no vertex in U_1, and so v is adjacent to the only vertex colored s_1 in $U_1 \cup U_2$, namely the vertex w, while u is not adjacent to w.

Thus v is adjacent to every vertex colored s_1 in G, while u is adjacent to every vertex colored s_1 in G except w. Therefore, v is adjacent to exactly one more vertex colored s_1 in G than u is, and so the (s_1+1)st-coordinate in code(v) does not equal the (s_1+1)st-coordinate in code(u). Thus code$(u) \neq$ code(v). Hence c' is a recognizable ℓ-coloring of G and so $\mathrm{rn}(G) \le \ell$. Therefore, $\mathrm{rn}(G) = \ell$. ■

In particular, if $a = 2$, then, by solving $\ell^2 - \ell - 2t = 0$ for ℓ, we obtain $\ell = \frac{1+\sqrt{1+8t}}{2}$ and so

$$\mathrm{rn}(K_{2(t)}) = \left\lceil \frac{1+\sqrt{1+8t}}{2} \right\rceil .$$

If a complete multipartite graph G contains t_i partite sets of cardinality n_i for every integer i with $1 \le i \le k$, then we write $G = K_{n_1(t_1), n_2(t_2), \ldots, n_k(t_k)}$.

Corollary 4.2. *Let* $G = K_{n_1(t_1), n_2(t_2), \ldots, n_k(t_k)}$, *where* $n_1, n_2, \ldots, n_k$ *are* k *distinct positive integers. Then*

$$\mathrm{rn}(G) = \max\{\mathrm{rn}(K_{n_i(t_i)}) : \ 1 \le i \le k\}.$$

Proof. Let $\ell_i = \mathrm{rn}(K_{n_i(t_i)})$ for $1 \le i \le k$. Assume, without loss of generality, that
$$\ell_1 = \max\{\mathrm{rn}(K_{n_i(t_i)}) :\ 1 \le i \le k\}.$$

We first show that $\mathrm{rn}(G) \le \ell_1$. For each integer i with $1 \le i \le k$, let c_i be a recognizable ℓ_i-coloring of the subgraph $K_{n_i(t_i)}$ in G. We can now define a recognizable ℓ_1-coloring c of G by defining

$$c(x) = c_i(x) \text{ if } x \in V(K_{n_i(t_i)}) \text{ for } 1 \le i \le k.$$

Thus $\mathrm{rn}(G) \le \ell_1$. Next, we show that $\mathrm{rn}(G) \ge \ell_1$. Assume, to the contrary, that $\mathrm{rn}(G) = \ell \le \ell_1 - 1$. Let c' be a recognizable ℓ-coloring of G. Then c' induces a coloring c'_1 of the subgraph $K_{n_1(t_1)}$ in G such that $c'_1(x) = c(x)$ for all $x \in V(K_{n_1(t_1)})$. Since c'_1 uses at most ℓ colors and $\mathrm{rn}(K_{n_1(t_1)}) = \ell_1 > \ell$, it follows that c'_1 is not a recognizable coloring of $K_{n_1(t_1)}$, and so there exist two vertices u and v in $K_{n_1(t_1)}$ such that u and v have the same code with respect to c'_1. Since u and v are both adjacent to every vertex in $V(G) - V(K_{n_1(t_1)})$, it follows that u and v have the same code in G with respect to c', which is a contradiction. ■

In particular, if $t_1 = t_2 = \cdots = t_k = 1$, then $K_{n_i(t_i)} = K_{n_i(1)} = \overline{K}_{n_i}$ for $1 \le i \le k$. Since $\mathrm{rn}(\overline{K}_{n_i}) = n_i$ for $1 \le i \le k$, it follows that

$$\mathrm{rn}(K_{n_1,n_2,\ldots,n_k}) = \max\{n_i :\ 1 \le i \le k\},$$

where $n_1, n_2, \ldots, n_k$ are k distinct positive integers.

In the special case of complete bipartite graphs, we have the following.

Corollary 4.3. *For integers s and t with $1 \le s \le t$,*

$$\mathrm{rn}(K_{s,t}) = \begin{cases} t & \text{if } s < t, \\ t+1 & \text{if } s = t. \end{cases}$$

5. Realization Results on Recognition Numbers

In this section we first characterize those connected graphs of order n having recognition number n or $n-1$. We have already noted that the complete graph K_n of order n has recognition number n. In fact, it is the only connected graph of order n with this property.

Proposition 5.1. *If G is a nontrivial connected graph of order n, then*

$$\mathrm{rn}(G) = n \text{ if and only if } G = K_n.$$

Proof. As a consequence of Observation 1.3, $\mathrm{rn}(K_n) = n$ for each integer $n \geq 2$. Next, we show that K_n is the only connected graph of order n with recognition number n. Suppose that G is a connected graph of order n that is not complete. Thus G contains three vertices x, y, and z such that $xy \notin E(G)$ and $yz \in E(G)$. Define a coloring c that assigns color 1 to x and z, color 2 to y, and distinct colors from the set $\{3, 4, \ldots, n-1\}$ to the remaining $n-3$ vertices of G. Since c is a recognizable $(n-1)$-coloring of G, it follows that $\mathrm{rn}(G) \leq n-1$. ■

We now characterize those connected graphs of order $n \geq 4$ with recognition number $n-1$.

Theorem 5.2. *Let G be a connected graph of order $n \geq 4$. Then*

$$\mathrm{rn}(G) = n-1 \text{ if and only if } G = K_{1,n-1} \text{ or } G = C_4.$$

Proof. By Corollary 4.3, $\mathrm{rn}(K_{1,n-1}) = n-1$ and $\mathrm{rn}(C_4) = 3$. For the converse, let G be a connected graph of order $n \geq 3$ such that $G \neq K_{1,n-1}$ but $\mathrm{rn}(G) = n-1$. Thus $G \neq K_n$ by Proposition 5.1. Hence $n \geq 4$. Suppose first that $n = 4$. By Observation 5.3, G contains at least three vertices of the same degree. Since $G \neq K_{1,3}$, it follows that $G = C_4$. We may now assume that $n \geq 5$. Since G contains at least $n-1$ vertices of the same degree, G contains exactly $n-1$ vertices of the same degree or G is regular.

Case 1. *The graph G contains exactly $n-1$ vertices of the same degree.* Let $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$, where $v_1, v_2, \ldots, v_{n-1}$ have the same degree. Since $G \neq K_{1,n-1}$, the degree of each of the vertices $v_1, v_2, \ldots, v_{n-1}$ is at least 2. Furthermore, since $G \neq K_n$, it follows that G contains two nonadjacent vertices x and y.

Subcase 1.1. $x, y \in V(G) - \{v_0\}$. Since $\deg y \geq 2$, there exists $z \neq v_0$ such that $zy \in E(G)$. Since $N[x] \neq N[z]$, we can define a coloring c that assigns color 1 to v_0, x, z, color 2 to y, and assigns distinct colors of the set $\{3, 4, \ldots, n-2\}$ to the remaining $n-4$ vertices of G. Since $\deg v_0 \neq \deg x$ and $\deg v_0 \neq \deg z$, it follows by Observation 1.2 that $\mathrm{code}(v_0) \neq \mathrm{code}(x)$ and $\mathrm{code}(v_0) \neq \mathrm{code}(z)$. Furthermore, the third coordinate in $\mathrm{code}(x)$

is 0, while the third coordinate in code(z) is 1. Thus code(x) $\neq$ code(y). Therefore, c is a recognizable $(n-2)$-coloring of G and so $\mathrm{rn}(G) \leq n-2$, which is a contradiction.

Subcase 1.2. *One of x and y is v_0, say $x = v_0$.* Since G is connected, there exists a vertex w such that $wv_0 \in E(G)$. Thus $N[w] \neq N[y]$. We can define a recognizable $(n-2)$-coloring that assigns color 1 to v_0, w, y, and assigns distinct colors of the set $\{3, 4, \ldots, n-2\}$ to the remaining $n-4$ vertices of G. Thus $\mathrm{rn}(G) \leq n-2$, a contradiction.

Case 2. *The graph G is regular.* Since $\mathrm{rn}(C_n) \leq n-2$ for $n \geq 5$, it follows that G is r-regular for some integer $r \geq 3$. Since $G \neq K_n$, there exists two nonadjacent vertices u and v in G. Let x and y be distinct vertices such that $ux, vy \in E(G)$. If $uy \notin E(G)$ or $vx \notin E(G)$, then $N[u] \neq N[v]$ and $N[x] \neq N[y]$. Hence we can define a recognizable $(n-2)$-coloring that assigns color 1 to u and v, color 2 to x and y, and assigns the $n-4$ colors of the set $\{3, 4, \ldots, n-2\}$ to the remaining $n-4$ vertices of G. Thus $\mathrm{rn}(G) \leq n-2$. Therefore, we may assume that $uy \in E(G)$ and $vx \in E(G)$. This implies that if $w \in V(G) - \{u, v\}$, then either w is adjacent to both u and v or w is adjacent to neither of u and v.

Subcase 2.1. *Every vertex in $V(G) - \{u, v\}$ is adjacent to both u and v.* Then G is $(n-2)$-regular, where $n-2 \geq 3$. This implies that n is even and G is the graph obtained from K_n by removing a 1-factor F. Let $s_i t_i$ be an edge in F for $i = 1, 2, 3$ (and so $s_i t_i \notin E(G)$ $i = 1, 2, 3$). We can define a coloring c that assigns color 1 to t_1, s_2, t_3, color 2 to s_1, color 3 to t_2, color 4 to s_3, and assigns distinct colors of the set $\{5, 6, \ldots, n-2\}$ to the remaining $n-6$ vertices of G. Observe that $\mathrm{code}(t_1) = 12011\cdots$, $\mathrm{code}(s_2) = 12101\cdots$, and $\mathrm{code}(t_3) = 12110\cdots$. Thus c is a recognizable $(n-2)$-coloring of G and so $\mathrm{rn}(G) \leq n-2$, which is a contradiction.

Subcase 2.2. *There exists $w \in V(G) - \{u, v\}$ such that w is adjacent to neither u nor v.* Let s and t be distinct vertices that are adjacent to w in G. Then $N[u] \neq N[s]$ and $N[v] \neq N[t]$. Hence we can define a coloring c that assigns color 1 to u and s, color 2 to v and t, color 3 to w, and assigns the distinct $n-5$ colors from the set $\{4, 5, \ldots, n-2\}$ to the remaining $n-5$ vertices of G. Observe (1) that the fourth coordinate in code(u) is 0, (2) the fourth coordinate in code(s) is 1, (3) the fourth coordinate in code(v) is 0, and (4) the fourth coordinate in code(t) is 1. Thus code(u) $\neq$ code(s)

and code$(v) \neq$ code(t). Thus c is a recognizable $(n-2)$-coloring of G and so $\mathrm{rn}(G) \leq n-2$, which is a contradiction. ■

We have seen that if G is a nontrivial connected graph of order n have recognition number k, then $2 \leq k \leq n$. Next we show that every pair k, n of integers with $2 \leq k \leq n$ is realizable as the recognition number and order of some connected graph. The following observation will be useful in the proof of Theorem 5.4.

Observation 5.3. *If G is a nontrivial connected graph such that the maximum number of vertices of the same degree is k, then* $\mathrm{rn}(G) \leq k$.

Theorem 5.4. *For each pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph of order n having recognition number k.*

Proof. For $k = 2$, let G be the unique connected graph of order n containing exactly two vertices with equal degree. It then follows by Observation 5.3 that $\mathrm{rn}(G) = 2$. For $k = n$, let $G = K_n$ and $\mathrm{rn}(K_n) = n$ by Proposition 5.1. If $k > n-k$, then let $G = K_{n-k,k}$ and $\mathrm{rn}(K_{n-k,k}) = k$ by Proposition 4.3. Thus, we may assume that $3 \leq k \leq n-k$. We consider two cases, according to whether $k \geq 4$ or $k = 3$.

Case 1. $k \geq 4$. Then $3 < k \leq n-k \leq n-4$. Then $n-k \geq 4$. Let F be the unique connected graph of order $n-k$ containing exactly two vertices with equal degree. Then the degrees of the vertices of F are $1, 2, \ldots, \left\lfloor \frac{n-k}{2} \right\rfloor, \left\lfloor \frac{n-k}{2} \right\rfloor, n-k-1$. Let $V(F) = \{u_1, u_2, \ldots, u_{n-k}\}$, where $\deg_F u_1 = 1$ and $\deg_F u_{n-k} = n-k-1$. Since $n-k \geq 4$, it follows that $2 \leq \left\lfloor \frac{n-k}{2} \right\rfloor < n-k-1$ and so F has a unique end-vertex, namely u_1. The graph G is now constructed from F by adding k new vertices $v_1, v_2, \ldots, v_k$ and joining each vertex v_i $(1 \leq i \leq k)$ to u_{n-k}. Then the order of G is n. It remains to show that $\mathrm{rn}(G) = k$. Since the k vertices $v_1, v_2, \ldots, v_k$ have the same closed neighborhood, it follows by Observation 1.3 that $\mathrm{rn}(G) \geq k$. On the other hand, the maximum number of vertices of the same degree is k. It then follows by Observation 5.3 that $\mathrm{rn}(G) \leq k$. Therefore, $\deg(G) = k$.

Case 2. $k = 3$. Since $3 = k \leq n-k = n-3$, it follows that $n \geq 6$. Let F be the unique connected graph of order $n-2$ containing exactly two vertices with equal degree. Then the degrees of the vertices of F are $1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor, n-3$. Let $V(F) = \{u_1, u_2, \ldots, u_{n-2}\}$, where

$\deg_F u_1 = 1$ and $\deg_F u_{n-2} = n-3$. Since $n-2 \geq 4$, it follows that $\left\lfloor \frac{n-2}{2} \right\rfloor \geq 2$ and so u_1 is the unique end-vertex in F. Let v be the vertex adjacent to u_1 in F. Now the graph G is obtained from F by adding two new vertices v_1 and v_2 and joining each of v_1 and v_2 to v. Then the order of G is n. It remains to show that $\mathrm{rn}(G) = 3$. Since the three end-vertices u_1, v_1, v_2 have the same neighborhood, $\mathrm{rn}(G) \geq 3$. On the other hand, the maximum number of vertices of the same degree is 3. It then follows by Observation 5.3 that $\mathrm{rn}(G) \leq 3$. Therefore, $\deg(G) = 3$. ■

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