Discussiones Mathematicae
Graph Theory 28 (2008) 23–34

COMPETITION HYPERGRAPHS OF DIGRAPHS WITH CERTAIN PROPERTIES II HAMILTONICITY

Martin Sonntag

Faculty of Mathematics and Computer Science
TU Bergakademie Freiberg
Prüferstraße 1, D–09596 Freiberg, Germany

e-mail: sonntag@mathe.tu-freiberg.de

and

Hanns-Martin Teichert

Institute of Mathematics
University of Lübeck
Wallstraße 40, D–23560 Lübeck, Germany

e-mail: teichert@math.uni-luebeck.de

Abstract

If $D = (V, A)$ is a digraph, its *competition hypergraph* $C\mathcal{H}(D)$ has vertex set V and $e \subseteq V$ is an edge of $C\mathcal{H}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = N_D^-(v) = \{w \in V | (w, v) \in A\}$. We give characterizations of $C\mathcal{H}(D)$ in case of hamiltonian digraphs D and, more general, of digraphs D having a τ-cycle factor. The results are closely related to the corresponding investigations for competition graphs in Fraughnaugh *et al.* [4] and Guichard [6].

Keywords: hypergraph, competition graph, hamiltonian digraph.

2000 Mathematics Subject Classification: 05C65, 05C20, 05C45.

1. Introduction and Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges. Loops are allowed only in digraphs; per definition they do not appear in competition graphs or competition hypergraphs.

In 1968 Cohen [2] introduced the *competition graph* $C(D)$ associated with a digraph $D = (V, A)$ representing a food web of an ecosystem. $C(D) = (V, E)$ is the graph with the same vertex set as D (corresponding to the species) and

$$E = \{\{u, w\} | u \neq w \wedge \exists v \in V : (u, v) \in A \wedge (w, v) \in A\},$$

i.e., $\{u, w\} \in E$ if and only if u and w compete for a common prey $v \in V$.

Surveys of the large literature around competition graphs can be found in Roberts [11], Kim [8] and Lundgren [9].

In our paper [13] it is shown that in many cases competition hypergraphs yield a more detailed description of the predation relations among the species in $D = (V, A)$ than competition graphs. If $D = (V, A)$ is a digraph its *competition hypergraph* $C\mathcal{H}(D) = (V, \mathcal{E})$ has the vertex set V and $e \subseteq V$ is an edge of $C\mathcal{H}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V \mid (w, v) \in A\}$. In this case we say $v \in V = V(D)$ *corresponds to* $e \in \mathcal{E}$ and vice versa.

In standard terminology concerning digraphs we follow Bang-Jensen and Gutin [1]. With $d_D^-(v), d_D^+(v), N_D^-(v)$ and $N_D^+(v)$ we denote the *in-degree, out–degree, in-neighbourhood* and *out-neighbourhood* of a vertex v in a digraph D, respectively.

For a graph G, let us call a collection $\{C_1, \ldots, C_p\}$ an *edge clique cover* of G, if each $C_i \subseteq V(G)$ generates a clique in G (not necessarily maximal) or $C_i = \emptyset$, and every edge of G is contained in at least one of these cliques. Obviously, the edges of $C\mathcal{H}(D)$ correspond to certain cliques in $C(D)$, and this proves to be very useful in the following.

If $M = (m_{ij})$ is the adjacency matrix of digraph D, then the competition graph $C(D)$ is the row graph $RG(M)$ (see Lundgren and Maybee [10]; Greenberg, Lundgren and Maybee [5]). To find a similar characterization for competition hypergraphs, we defined in [13] the *row hypergraph* $R\mathcal{H}(M)$. The vertices of this hypergraph correspond to the rows of M, i.e., to the vertices $v_1, v_2, \ldots, v_n$ of D, and the edges correspond to certain columns;

in detail:

$$\mathcal{E}(R\mathcal{H}(M)) =$$
$$= \Big\{\{v_{i_1}, \ldots, v_{i_k}\} \mid k \geq 2 \wedge \exists\, j \in \{1, \ldots, n\} : m_{ij} = 1 \Leftrightarrow i \in \{i_1, \ldots, i_k\}\Big\}.$$

This notion yields immediately the following result.

Lemma 1 ([13]). *Let D be a digraph with adjacency matrix M. Then the competition hypergraph $C\mathcal{H}(D)$ is the row hypergraph $R\mathcal{H}(M)$.*

Note that any permutation of rows or columns in M does not change the row hypergraph $R\mathcal{H}(M)$ (up to isomorphism). Conversely, for a competition hypergraph $\mathcal{H}$ with n vertices and t edges we call each $(n \times n)$-matrix M with entries 0 or 1 a *competition matrix of* $\mathcal{H}$ if $\mathcal{H} \cong R\mathcal{H}(M)$. Such a competition matrix is said to be *standardized* if $e_j \in \mathcal{E}(\mathcal{H})$ corresponds to column j of M for $j = 1, \ldots, t$ and all entries are 0 in columns $t+1, \ldots, n$.

Obviously, every competition matrix M of a competition hypergraph $\mathcal{H}$ can be transformed into a standardized one by permuting columns and replacing entries 1 by 0 in columns, which contain only a single entry 1 (both operations do not influence $R\mathcal{H}(M)$).

Results for competition graphs of hamiltonian digraphs are given in Fraughnaugh *et al.* [4].

Theorem 2 ([4]). *A graph G with n vertices is a competition graph of a hamiltonian digraph without loops if and only if G has an edge clique cover $\{C_1, C_2, \ldots, C_n\}$ with a system of distinct representatives $\{v_n, v_1, \ldots, v_{n-1}\}$ such that $v_n \in C_1, v_i \in C_{i+1}$ $(i = 1, \ldots, n-1)$ and*

$$\forall i \in \{1, \ldots, n\} : v_i \notin C_i. \tag{1}$$

In the same paper [4] it is shown that condition (1) may be omitted in Theorem 2 if D may have loops, Guichard [6] had success in combining both results if G has $n \geq 3$ vertices.

Theorem 3 ([6]). *A graph G with $n \geq 3$ vertices is a competition graph of a hamiltonian digraph without loops if and only if G has an edge clique cover $\{C_1, \ldots, C_n\}$ with a system of distinct representatives.*

In the following we provide some results concerning competition hypergraphs, discuss relations between the investigations for competition graphs

and competition hypergraphs of hamiltonian digraphs and prove characterizations for competition hypergraphs of hamiltonian and related digraphs. An important point of view is the fact that these characterizations contain conditions only depending on the edge set of the competition hypergraph.

2. Tools

A graph G with n vertices is the competition graph of a digraph which may have loops if and only if there is an edge clique cover of G containing at most n cliques (cf. Dutton and Brigham [3]). Moreover, if additionaly $G \neq K_2$ is fulfilled, G is even the competition graph of a digraph without loops (cf. Roberts and Steif [12]). Hence the conditions in Theorem 2 and Theorem 3 provide that G is the competition graph of a digraph which may have loops and a digraph without loops, respectively. This is one reason that the additional condition (1) of Theorem 2 may be omitted in Theorem 3.

For hypergraphs the following results are known.

Theorem 4 ([13]). *A hypergraph $\mathcal{H}$ with n vertices is a competition hypergraph of a digraph which may have loops if and only if $|\mathcal{E}(\mathcal{H})| \leq n$.*

Because of the numerous possibilities for edge cardinalities in hypergraphs, the result for digraphs without loops becomes more complicated. For $t \in \mathbb{N}$ we define

$$\mathcal{M}_k = \{M_k \subseteq \{1, \ldots, t\} |\ |M_k| = k\} \quad \text{for} \quad k = 1, \ldots, t.$$

Theorem 5 ([13]). *Let $\mathcal{H}$ be a hypergraph with n vertices and $\mathcal{E}(\mathcal{H}) = \{e_1, \ldots, e_t\}$. Then $\mathcal{H}$ is a competition hypergraph of a digraph without loops if and only if*

$$\forall\, k \in \{1, \ldots, t\}\ \forall\, M_k \in \mathcal{M}_k : \Big| \bigcap_{j \in M_k} e_j \Big| \leq n - k. \tag{2}$$

In the following we will need several times

Hall's Theorem ([7]). *Let $A_1, \ldots, A_t$ be arbitrary sets. Then $A_1, \ldots, A_t$ have a system of distinct representatives if and only if*

$$\forall\, k \in \{1, \ldots, t\}\ \forall\, M_k \in \mathcal{M}_k : \Big| \bigcup_{j \in M_k} A_j \Big| \geq k. \tag{3}$$

3. Results

There are two interesting points of view for the investigations of competition hypergraphs of hamiltonian digraphs:

(a) As mentioned in the introduction the $t \leq n$ edges of the competition hypergraph $C\mathcal{H}(D)$ correspond to certain cliques of a suitable edge clique cover $\{C_1, \ldots, C_n\}$ of the competition graph $C(D)$. In Theorem 2 and 3 there are conditions for all these n cliques $C_1, \ldots, C_n$. In case of hypergraphs it would be desirable to formulate conditions only for the $t \leq n$ edges, and this will be possible.

(b) Considering Theorems 2 and 3 the question arises whether in case of loopless digraphs a condition corresponding to (1) is needed or not?

Our results will show that, unfortunately, the answer to the question (b) is yes. However, if we do not postulate such a condition we can prove a weaker result; this motivates the following definition. According to Bang-Jensen and Gutin [1] a system $\{\vec{c}_1, \ldots, \vec{c}_\tau\}$ of oriented cycles in a digraph D is called a *τ-cycle factor* if every vertex of D is contained in exactly one cycle $\vec{c}_j \in \{\vec{c}_1, \ldots, \vec{c}_\tau\}$. Clearly, for $\tau = 1$ the digraph D is hamiltonian. The following result characterizes competition hypergraphs of digraphs D having a τ-cycle factor. A class of examples given later will show that sometimes $\tau \geq 2$ is unavoidable.

Theorem 6. *Let $\mathcal{H}$ be a hypergraph with n vertices and $\mathcal{E}(\mathcal{H}) = \{e_1, \ldots, e_t\}$. Then the following conditions are equivalent.*

(i) *$\mathcal{H}$ is the competition hypergraph of a loopless digraph D having a τ-cycle factor.*

(ii) *It holds*

$$(4) \quad \forall\, k \in \{1, \ldots, t\}\ \forall\, M_k \in \mathcal{M}_k : \Big| \bigcup_{j \in M_k} e_j \Big| \geq k \wedge \Big| \bigcap_{j \in M_k} e_j \Big| \leq n - k.$$

(iii) *$\mathcal{H}$ is the competition hypergraph of a loopless digraph and $\mathcal{E}(\mathcal{H})$ has a system of distinct representatives.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathcal{H}$ is the competition hypergraph of a digraph D having a τ-cycle factor $\{\vec{c}_1, \ldots, \vec{c}_\tau\}$ and the adjacency matrix $M = (m_{ij})$ for some vertex labelling, such that $V = \{v_1, \ldots, v_n\}$. For each $j \in \{1, \ldots, n\}$ let v_j^- be the unique vertex with $(v_j^-, v_j) \in A(\vec{c}_\mu)$ for some $\mu \in \{1, \ldots, \tau\}$.

Then $S' = \{v_1^-, \ldots, v_n^-\}$ is a system of distinct representatives for the *column sets* $c_j = \{v_i \mid i \in \{1, \ldots, n\} \wedge m_{ij} = 1\}, j = 1, \ldots, n$. By Lemma 1 we have $\mathcal{H} = C\mathcal{H}(D) = R\mathcal{H}(M)$. The edges $\mathcal{E}(\mathcal{H}) = \{e_1, \ldots, e_t\}$ correspond to those columns of M containing at least two entries 1, thus

$$\forall j \in \{1, \ldots, t\} \; \exists \; i_j \in \{1, \ldots, n\} : e_j = c_{i_j}.$$

Hence there is a system of distinct representatives $S'' \subseteq S'$ for $\{e_1, \ldots, e_t\}$. Therefore, using Hall's Theorem, we obtain the first part of (4) and the second one is true by Theorem 5.

(ii) $\Rightarrow$ (i). Suppose (4) is true. Using Theorem 5 it follows from the second part of (4) that $\mathcal{H}$ is the competition hypergraph of a digraph without loops; in the following we construct such a digraph having additionally a τ-cycle factor. Choosing $k = t$ in (4) we obtain $|\mathcal{E}(\mathcal{H})| = t \leq n$. Let $M^1 = (m_{ij}^1)$ be the standardized competition matrix of $\mathcal{H}$ for some labelling of the vertices of $\mathcal{H}$, i.e., $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$, and it holds

$$\forall j \in \{1, \ldots, t\} : e_j = \{v_i \mid i \in \{1, \ldots, n\} \;\wedge\; m_{ij}^1 = 1\}.$$

Note that the columns $t+1, \ldots, n$ of M^1 contain only the entry 0. The matrix M^1 will be transformed three times in the following; observe that the resulting matrices M^2, M^3, M^4 are competition matrices of the same hypergraph $\mathcal{H}$. Because of Hall's Theorem and the first part of (4) there is a system of distinct representatives $S = \{\bar{v}_1, \ldots, \bar{v}_t\}$ for $\{e_1, \ldots, e_t\}$. For the last $n - t$ column sets it holds

$$\forall j \in \{t+1, \ldots, n\} : c_j^1 = \{v_i \mid i \in \{1, \ldots, n\} \;\wedge\; m_{ij}^1 = 1\} = \emptyset\,. \tag{5}$$

Hence the system S can be enlarged to a system of distinct representatives $\bar{S} = \{\bar{v}_1, \ldots, \bar{v}_t, \bar{v}_{t+1} \ldots, \bar{v}_n\}$ for the column sets $\{c_1^2 = e_1, \ldots, c_t^2 = e_t, c_{t+1}^2, \ldots, c_n^2\}$, i.e., the matrix $M^2 = (m_{ij}^2)$ arises from M^1 by setting $m_{ij}^2 = 1$ if $v_i = \bar{v}_j$ for $j = t+1, \ldots, n$.

Next we show that M^2 and $\bar{S}$ can be transformed into a matrix $M^3 = (m_{ij}^3)$ and an enlargement $\widetilde{S} = \{\widetilde{v}_1, \ldots, \widetilde{v}_n\}$ of S, respectively, having the additional property

$$\forall i \in \{1, \ldots, n\} \; \exists \; j \in \{1, \ldots, n\} : m_{ij}^3 = 0\,. \tag{6}$$

We distinguish three cases:

(a) For $t = n$ the second part of (4) yields (for $k = t = n$) : $|\bigcap_{j=1}^{n} e_j| = 0$, therefore (6) is true, i.e., $\widetilde{S} = \bar{S}$.

(b) For $t \leq n-2$ it follows with (5) that at least two columns of M^2 contain $(n-1)$ times the entry 0 and once the entry 1, appearing in different rows. Hence $\widetilde{S} = \bar{S}$ and (6) is true.

(c) For $t = n-1$ we choose $k = t = n-1$ in the second part of (4) and obtain

$$\Big|\bigcap_{j=1}^{n-1} e_j\Big| \leq n-(n-1) = 1\,. \tag{7}$$

Hence there is at most one row $(m_{p1}^2, \ldots, m_{pn}^2) = (1, \ldots, 1)$. In this case change the representatives $\bar{v}_1 = v_q$ and $\bar{v}_n = v_p$ of c_1^2 and c_n^2, respectively, i.e., $\widetilde{v}_1 = v_p, \widetilde{v}_n = v_q$. By (7) this yields $(m_{p1}^3, \ldots, m_{pn}^3) = (1, \ldots, 1, 0)$ and $(m_{q1}^3, \ldots, m_{qn}^3) = (1, \ldots, 0, \ldots, 1)$.

Thus (6) is fulfilled.

We consider the obtained competition matrix M^3 of $\mathcal{H}$ with column sets $c_j^3 = \{v_i \,|\, i \in \{1, \ldots, n\} \wedge m_{ij}^3 = 1\}$, $j = 1, \ldots, n$, where $c_j^3 = e_j$ for $j = 1, \ldots, t$. Further we define $\bar{c}_j^3 = \{v_i \,|\, i \in \{1, \ldots, n\} \wedge m_{ij}^3 = 0\}$. The second part of (4) can be written as

$$\forall\, k \in \{1, \ldots, t\}\ \forall\, M_k \in \mathcal{M}_k : \Big|\bigcap_{j \in M_k} c_j^3\Big| \leq n-k\,. \tag{8}$$

Because of $|c_{t+1}^3| = \cdots = |c_n^3| = 1$ it follows

$$\forall\, \widetilde{C} \subseteq \{c_1^3, \ldots, c_n^3\} : \widetilde{C} \cap \{c_{t+1}^3, \ldots, c_n^3\} \neq \emptyset \Rightarrow \Big|\bigcap \widetilde{C}\Big| \leq 1\,. \tag{9}$$

Together with (6) we obtain from (9) that (8) can be generalized to

$$\forall\, k \in \{1, \ldots, n\}\ \forall\, M_k \in \mathcal{M}_k : \Big|\bigcap_{j \in M_k} c_j^3\Big| \leq n-k$$

and this is equivalent to

$$\forall\, k \in \{1, \ldots, n\}\ \forall\, M_k \in \mathcal{M}_k : \Big|\bigcup_{j \in M_k} \bar{c}_j^3\Big| \geq k\,. \tag{10}$$

Using Hall's Theorem the existence of a system of distinct representatives for $\{\bar{c}_1^3, \ldots, \bar{c}_n^3\}$ follows. Let $M^4 = (m_{ij}^4)$ arise from M^3 by permuting of the columns of M^3 such that $m_{jj}^4 = 0$, where v_j is the representative of $\bar{c}_j^4, j = 1, \ldots, n$. Then M^4 is the adjacency matrix of a digraph D without loops. Obviously, $\{v_1, \ldots, v_n\} = V(D)$ (in a certain ordering) is also a system of distinct representatives for $\{c_1^4, \ldots, c_n^4\}$ with the additional property $i \neq j$, if v_i represents c_j^4.

It remains to show the existence of a τ-cycle factor in D. For this purpose we consider the system of distinct representatives for $\{c_1^4, \ldots, c_n^4\}$ mentioned above and define a digraph $D' = (V', A')$ by $V' := \{1, \ldots, n\}$ and $A' := \{(i,j) | i,j \in \{1, \ldots, n\} \wedge v_i \text{ is the representative for } c_j^4\}$.

Then A' has the properties:

- $|A'| = n$;
- $\forall\, i,j \in \{1, \ldots, n\} : (i,j) \in A' \Rightarrow i \neq j$;
- $\{i \,|\, \exists\, j \in \{1, \ldots, n\} : (i,j) \in A'\} = \{1, \ldots, n\}$
  $= \{j \,|\, \exists\, i \in \{1, \ldots, n\} : (i,j) \in A'\}$.

Consequently, the adjacency matrix of D' (which can be obtained from M^4 by replacing certain entries 1 by 0, i.e., D' is isomorphic to a subdigraph of D) contains exactly one entry 1 per column and one entry 1 per row.

Therefore, in D' every vertex $i \in V'$ has $d_{D'}^-(i) = d_{D'}^+(i) = 1$. Hence, all components of D' are directed cycles corresponding to a τ-cycle factor $\{\vec{c}_1, \ldots, \vec{c}_\tau\}$ of D.

(ii) $\Leftrightarrow$ (iii). The equivalence of both conditions follows immediately from Hall's Theorem and Theorem 5. ■

As mentioned before Theorem 6, in the following we give a class of nonhamiltonian digraphs $\widetilde{D}_{2n}, n \geq 3$, the competition hypergraphs $\widetilde{\mathcal{H}}_{2n} := C\mathcal{H}(\widetilde{D}_{2n})$ of which fulfill condition (iii) of Theorem 6 but being not competition hypergraphs of any hamiltonian digraph without loops (cf. Lemma 7 below). For $V(\widetilde{D}_{2n}) = \{v_1 = x_1, \ldots, v_n = x_n, v_{n+1} = y_1, \ldots, v_{2n} = y_n\}$ we define the arcs by

$$\begin{aligned} A(\widetilde{D}_{2n}) = &\{(x_i, x_j) \,|\, i \neq j \,\wedge\, i,j \in \{1, \ldots, n\}\} \\ \cup\, &\{(y_i, y_j) \,|\, i \neq j \,\wedge\, i,j \in \{1, \ldots, n\}\} \\ \cup\, &\{(x_i, y_j) \,|\, i,j \in \{1, \ldots, n\}\}. \end{aligned} \tag{11}$$

The adjacency matrix $\widetilde{M} = (\widetilde{m}_{ij})$ of such a digraph is given by

$$\widetilde{m}_{ij} = \begin{cases} 0 & \text{for } i = j \ \vee \ (i \geq n \ \wedge \ j \leq n), \\ 1 & \text{otherwise}. \end{cases}$$

For example $n = 3$ yields

$$\widetilde{M} = \left(\begin{array}{ccc|ccc} 0 & \boxed{1} & 1 & 1 & 1 & 1 \\ 1 & 0 & \boxed{1} & 1 & 1 & 1 \\ \boxed{1} & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 1 & 0 & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \end{array}\right)$$

and $\widetilde{D}_6, \widetilde{\mathcal{H}}_6 := C\mathcal{H}(\widetilde{D}_6)$ given in Figure 1.

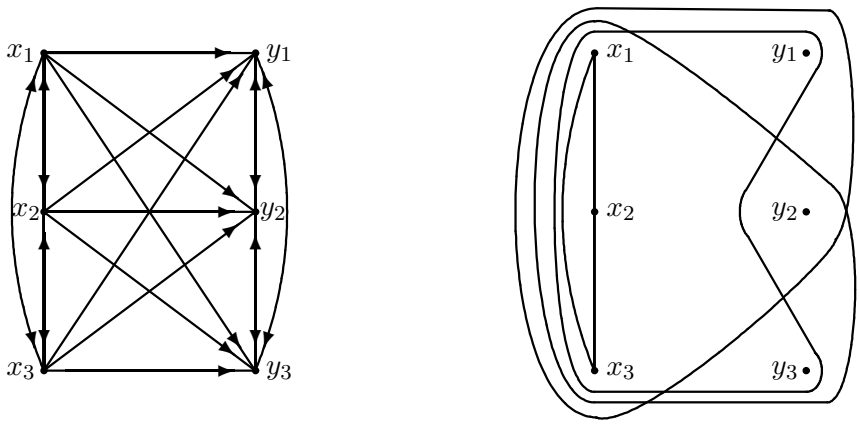

Figure 1. The digraph $\widetilde{D}_6$ and its competition hypergraph.

Clearly, $\widetilde{D}_{2n}$ is not hamiltonian, has a 2-cycle-factor and $\widetilde{\mathcal{H}}_{2n}$ fulfills (iii) of Theorem 6 (the representatives of the edges of $\widetilde{\mathcal{H}}_{2n}$ can be chosen corresponding to the framed elements in $\widetilde{M}$).

Lemma 7. *Let $n \geq 3$. Then there is no hamiltonian digraph D without loops such that $\widetilde{\mathcal{H}}_{2n} = \mathcal{CH}(\widetilde{D}_{2n}) = \mathcal{CH}(D)$.*

Proof. From (11) we obtain $\mathcal{E}(\widetilde{\mathcal{H}}_{2n}) = \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$, where
$\forall\, j \in \{1, \ldots, n\} : e_j := \{x_1, \ldots, x_n\} \setminus \{x_j\}$ and
$\forall\, j \in \{1, \ldots, n\} : e_{n+j} := \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \setminus \{y_j\}$.

For every loopless digraph D with $\mathcal{CH}(D) = \widetilde{\mathcal{H}}_{2n}$ it follows:

(a) $\forall\, j \in \{1, \ldots, n\} : y_j \in V(D)$ corresponds to $e_{n+j} \in \mathcal{E}(\widetilde{\mathcal{H}}_{2n})$ (since y_j is the only vertex not belonging to e_{n+j}); therefore for $j \in \{1, \ldots, n\}$: $\{(x_i, y_j) \,|\, i \in \{1, \ldots, n\}\} \cup \{(y_i, y_j) \,|\, i \neq j \wedge i \in \{1, \ldots, n\}\} \subseteq A(D)$.

(b) $\forall\, j \in \{1, \ldots, n\} : x_j \in V(D)$ corresponds to $e_j \in \mathcal{E}(\widetilde{\mathcal{H}}_{2n})$ (because of (a)) $y_1, \ldots, y_n$ cannot correspond to e_j and x_j is the only vertex of $\{x_1, \ldots, x_n\}$ not belonging to e_j); therefore for $j \in \{1, \ldots, n\}$: $\{(x_i, x_j) \,|\, i \neq j \wedge i \in \{1, \ldots, n\}\} \subseteq A(D)$.

Formula (11), (a) and (b) imply $A(\widetilde{D}_{2n}) \subseteq A(D)$. Since every vertex of D corresponds to a hyperedge of $\widetilde{\mathcal{H}}_{2n}$, no additional arcs can appear in D (because this would result in a change of $\widetilde{\mathcal{H}}_{2n}$). Consequently, $D = \widetilde{D}_{2n}$ is the only loopless digraph with the competition hypergraph $\widetilde{\mathcal{H}}_{2n}$ and the proof is complete. ■

Note that the competition graph $C(\widetilde{D}_{2n}) \cong K_{2n}$, which is of course the competition graph of a hamiltonian digraph without loops.

Finally we characterize the competition hypergraphs of hamiltonian digraphs without loops; for this purpose the condition (12) below, which corresponds to (1) in Theorem 2, is necessary.

Theorem 8. *A hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ and $\mathcal{E}(\mathcal{H}) = \{e_1, \ldots, e_t\}$ is the competition hypergraph of a hamiltonian digraph without loops if and only if there is a subset $\{j_1, \ldots, j_t\} \subseteq \{1, \ldots, n\}$ of pairwise distinct indices such that $\{v_{j_1-1}, \ldots, v_{j_t-1}\}$ (indices taken mod n) is a system of distinct representatives of $\{e_1, \ldots, e_t\}$ and furthermore*

$$\forall\, i \in \{1, \ldots, t\} : v_{j_i} \notin e_i\,. \tag{12}$$

Proof. Suppose $\mathcal{H}$ is the competition hypergraph of a loopless digraph D having the hamiltonian cycle $(v_1, v_2, \ldots, v_n, v_1)$. For $\mathcal{E}(\mathcal{H}) = \{e_1, \ldots, e_t\}$ it follows

$$\forall\, i \in \{1, \ldots, t\}\, \exists\, j_i \in \{1, \ldots, n\} : e_i = N_D^-(v_{j_i}).$$

Clearly, $v_{j_i} \notin e_i$ (because D has no loops) and the hamiltonian cycle yields $v_{j_i-1} \in e_i$. Conversely, suppose $v_{j_i-1} \in e_i$ and $v_{j_i} \notin e_i$ for $i = 1, \ldots, t$. Consider the digraph D with $V(D) = V(\mathcal{H}), N_D^-(v_{j_i}) = e_i$ for $i = 1, \ldots, t$ and $(v_{k-1}, v_k) \in A(D)$ for $k \notin \{j_1, \ldots, j_t\}$. Then D is a loopless digraph with the hamiltonian cycle $(v_1, v_2, \ldots, v_n, v_1)$ and we have $\mathcal{H} = C\mathcal{H}(D)$. ■

As an immediate consequence we obtain a characterization for the case of hamiltonian digraphs with loops allowed.

Corollary 9. *A hypergraph $\mathcal{H}$ is the competition hypergraph of a hamiltonian digraph (which may have loops) if and only if $\mathcal{E}(\mathcal{H})$ has a system of distinct representatives.*

Acknowledgement

The authors would like to thank one the referees for his valuable suggestions, especially to the proof of Theorem 6 and to Lemma 7.

References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications (Springer, London, 2001).

[2] J.E. Cohen, Interval graphs and food webs: a finding and a problem (Rand Corporation Document 17696-PR, Santa Monica, CA, 1968).

[3] R.D. Dutton and R.C. Brigham, *A characterization of competition graphs*, Discrete Appl. Math. **6** (1983) 315–317.

[4] K.F. Fraughnaugh, J.R. Lundgren, S.K. Merz, J.S. Maybee and N.J. Pullman, *Competition graphs of strongly connected and hamiltonian digraphs*, SIAM J. Discrete Math. **8** (1995) 179–185.

[5] H.J. Greenberg, J.R. Lundgren and J.S. Maybee, *Inverting graphs of rectangular matrices*, Discrete Appl. Math. **8** (1984) 255–265.

[6] D.R. Guichard, *Competition graphs of hamiltonian digraphs*, SIAM J. Discrete Math. **11** (1998) 128–134.

[7] P. Hall, *On representation of subsets*, J. London Math. Soc. **10** (1935) 26–30.

[8] S.R. Kim, *The competition number and its variants*, in: J. Gimbel, J.W. Kennedy and L.V. Quintas (eds.), Quo vadis, graph theory?, Ann. of Discrete Math. **55** (1993) 313–326.

[9] J.R. Lundgren, *Food webs, competition graphs, competition-common enemy graphs and niche graphs*, in: F. Roberts (ed.), Applications of combinatorics and graph theory to the biological and social sciences, IMA **17** (Springer, New York, 1989) 221–243.

[10] J.R. Lundgren and J.S. Maybee, *A characterization of graphs of competition number m*, Discrete Appl. Math. **6** (1983) 319–322.

[11] F.S. Roberts, *Competition graphs and phylogeny graphs*, in: L. Lovász (ed.), Graph theory and combinatorial biology; Proc. Int. Colloqu. Balatonlelle (Hungary) 1996, Bolyai Soc. Math. Studies **7** (Budapest, 1999) 333–362.

[12] F.S. Roberts and J.E. Steif, *A characterization of competition graphs of arbitrary digraphs*, Discrete Appl. Math. **6** (1983) 323–326.

[13] M. Sonntag and H.-M. Teichert, *Competition hypergraphs*, Discrete Appl. Math. **143** (2004) 324–329.

Received 14 January 2005
Revised 24 September 2007
Accepted 31 December 2007