

COMPETITION HYPERGRAPHS OF DIGRAPHS
WITH CERTAIN PROPERTIES I
STRONG CONNECTEDNESS

MARTIN SONNTAG

Faculty of Mathematics and Computer Science
TU Bergakademie Freiberg
Prüferstraße 1, D-09596 Freiberg, Germany
e-mail: sonntag@mathe.tu-freiberg.de

AND

HANNS-MARTIN TEICHERT

Institute of Mathematics
University of Lübeck
Wallstraße 40, D-23560 Lübeck, Germany
e-mail: teichert@math.uni-luebeck.de

Abstract

If $D = (V, A)$ is a digraph, its *competition hypergraph* $\mathcal{CH}(D)$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V \mid (w, v) \in A\}$. We tackle the problem to minimize the number of strong components in D without changing the competition hypergraph $\mathcal{CH}(D)$. The results are closely related to the corresponding investigations for competition graphs in Fraughnaugh *et al.* [3].

Keywords: hypergraph, competition graph, strong component.

2000 Mathematics Subject Classification: 05C65, 05C20, 05C40.

1. INTRODUCTION AND DEFINITIONS

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, graphs $G = (V(G), E(G))$ and digraphs $D = (V(D), A(D))$ considered here may have isolated vertices but no multiple edges and no loops.

In 1968 Cohen [2] introduced the *competition graph* $C(D)$ associated with a digraph $D = (V, A)$ representing a food web of an ecosystem. $C(D) = (V, E)$ is the graph with the same vertex set as D (corresponding to the species) and

$$E = \{\{u, v\} \mid u \neq v \wedge \exists w \in V : (u, w) \in A \wedge (v, w) \in A\},$$

i.e., $\{u, v\} \in E$ iff u and v compete for a common prey $w \in V$.

Surveys of the large literature around competition graphs can be found in Roberts [6], Kim [4] and Lundgren [5].

In [8] it is shown that in many cases competition hypergraphs yield a more detailed description of the predation relations among the species in $D = (V, A)$ than competition graphs. If $D = (V, A)$ is a digraph its *competition hypergraph* $\mathcal{CH}(D) = (V, \mathcal{E})$ has the vertex set V and $e \subseteq V$ is an edge of $\mathcal{CH}(D)$ iff $|e| \geq 2$ and there is a vertex $v \in V$, such that $e = \{w \in V \mid (w, v) \in A\}$. In this case we say $v \in V = V(D)$ *corresponds to* $e \in \mathcal{E}$ and vice versa.

In our investigations, we make intensive use of the fact that in the digraphs under consideration no vertex is a hunter of itself. Moreover, it is obvious that loops play no role for the connectedness of digraphs.

In standard terminology concerning digraphs we follow Bang-Jensen and Gutin [1]. With $d_D^-(v), d_D^+(v), N_D^-(v)$ and $N_D^+(v)$ we denote the *in-degree*, *out-degree*, *in-neighbourhood* and *out-neighbourhood* of a vertex v in a digraph D , respectively. A set of t isolated vertices is denoted as I_t , and $i(G)$ is the number of isolated vertices in G , where G is a graph or a hypergraph. For a subset \hat{V} of vertices let $D[\hat{V}]$ be the subdigraph of D generated by \hat{V} . For a graph G , let us use $\hat{m}(G)$ to denote the *edge clique cover number of G* , i.e., the smallest number of cliques covering all the edges of G .

2. RESULTS

Competition graphs of strongly connected digraphs are investigated in Fraughnaugh *et al.* [3]. The most interesting result is the following characterization.

Theorem 1 ([3]). *A graph G with $n \geq 3$ vertices is the competition graph of a strongly connected digraph if and only if $\widehat{m}(G) + i(G) \leq n$.*

Consider the edge clique cover of $C(D)$ where each clique is formed by the hunters of a prey $v \in V(D)$. Then these cliques correspond to the edges of the competition hypergraph $CH(D)$ (cf. Sonntag and Teichert [8]). However, Theorem 1 cannot be generalized to competition hypergraphs; only one direction can be shown.

Lemma 2. *If \mathcal{H} is a competition hypergraph of a strongly connected digraph with n vertices, then $|\mathcal{E}(\mathcal{H})| + i(\mathcal{H}) \leq n$.*

Proof. Let $\mathcal{H} = CH(D)$ where D is strongly connected. Then each edge in \mathcal{H} corresponds to $N_D^-(v)$ for some $v \in V(D)$ with $|N_D^-(v)| \geq 2$, hence $|\mathcal{E}(\mathcal{H})| \leq n$. Because D is strongly connected, for each isolated vertex \widehat{v} in \mathcal{H} there is a vertex $w \neq \widehat{v}$ with $(\widehat{v}, w) \in A(D)$ and $N_D^-(w) = \{\widehat{v}\}$. Therefore no edge of \mathcal{H} corresponds to $N_D^-(w)$ and we obtain $|\mathcal{E}(\mathcal{H})| + i(\mathcal{H}) \leq n$. ■

In the following we give an example of an infinite family of competition hypergraphs $CH(D)$ which fulfill the inequality in Lemma 2 but are not competition hypergraphs of a strongly connected digraph D . After that we tackle the problem to minimize the number of strong components in D without changing the competition hypergraph.

Following Bang-Jensen and Gutin [1] for a digraph D with strong components D_1, \dots, D_k we define the *strong component digraph* $SC(D)$ as follows:

$$V(SC(D)) = \{w_1, \dots, w_k\},$$

$$A(SC(D)) = \{(w_i, w_j) \mid i \neq j \wedge \exists x \in V(D_i) \exists y \in V(D_j) : (x, y) \in A(D)\}.$$

Because $SC(D)$ is acyclic, it is possible to arrange the strong components D_1, \dots, D_k of D in an *acyclic ordering* (cf. [1]), i.e., they are denoted such that

$$\forall i, j \in \{1, \dots, k\} \forall x \in V(D_i) \forall y \in V(D_j) : i \neq j \wedge (x, y) \in A(D) \Rightarrow i < j.$$

In the first instance we restrict our investigations to digraphs having no trivial strong components, i.e., $\forall i \in \{1, 2, \dots, k\} : |V(D_i)| > 1$. We denote such digraphs as *digraphs with (nontrivial) strong components* D_1, \dots, D_k .

At the end of the paper we will discuss some problems which can be caused by trivial strong components.

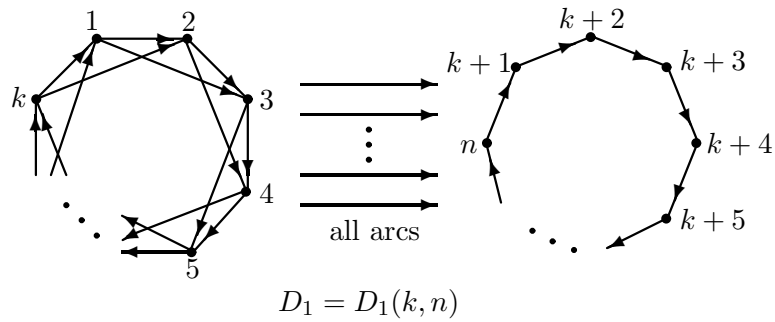
A graph G with n vertices is a competition graph of some digraph D , if and only if $G \neq K_2$ and $\hat{m}(G) \leq n$ (cf. Roberts and Steif [7]). From Theorem 1 it follows that every competition graph G without isolated vertices is even a competition graph of a strongly connected digraph.

A characterization of hypergraphs which are competition hypergraphs $\mathcal{CH}(D)$ of digraphs D (without loops) is given in Sonntag and Teichert [8]. The question arises whether or not — analogously to graphs — every such hypergraph $\mathcal{CH}(D)$ without isolated vertices is also a competition hypergraph of a strongly connected digraph \tilde{D} . The answer is no, as the following class of examples will show. Later in this section we discuss characteristic structures appearing in the digraphs D of these examples, and this will be the starting point for our further investigations.

Example. Let $n \geq 5$ und $k \in \{3, 4, \dots, n-2\}$. Then $D_1 = D_1(k, n)$ has the vertices $V(D_1) = \{1, \dots, n\}$ and the arcs

$$\begin{aligned} A(D_1) = & \{(i, i+1) \mid i \in \{1, \dots, k-1\}\} \cup \{(k, 1)\} \cup \\ & \{(i, i+2) \mid i \in \{1, \dots, k-2\}\} \cup \{(k-1, 1), (k, 2)\} \cup \\ & \{(i, i+1) \mid i \in \{k+1, \dots, n-1\}\} \cup \{(n, k+1)\} \cup \\ & \{(i, j) \mid i \in \{1, \dots, k\} \wedge j \in \{k+1, \dots, n\}\}. \end{aligned}$$

The digraph D_1 and its competition hypergraph $\mathcal{H}_1 = \mathcal{CH}(D_1)$ are shown in Figure 1.



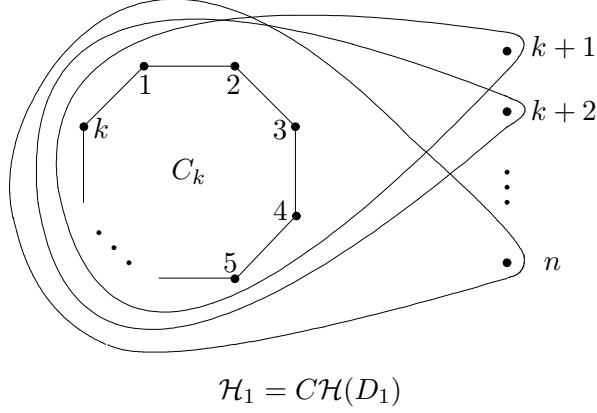


Figure 1. The digraph $D_1 = D_1(k, n)$ and its competition hypergraph $\mathcal{H}_1 = \text{CH}(D_1)$.

Clearly, D_1 is not strongly connected; consequently we obtain

Lemma 3. *Let $n \geq 5, k \in \{3, 4, \dots, n-2\}$ and $D_1 = D_1(k, n)$. Then there is no strongly connected digraph \tilde{D}_1 with $\mathcal{H}_1 = \text{CH}(D_1) = \text{CH}(\tilde{D}_1)$.*

Proof. Let $\tilde{D} = (V, A)$ be a digraph with $\mathcal{H}_1 = \text{CH}(\tilde{D})$ and $V = V_1 \cup V_2$, where $V_1 = \{1, \dots, k\}, V_2 = \{k+1, \dots, n\}$. Then \mathcal{H}_1 has k edges with 2 vertices (α -edges) and $(n-k)$ edges with $(k+1)$ vertices (β -edges). Because each β -edge \tilde{e} contains V_1 , for the vertex \tilde{v} corresponding to \tilde{e} it holds $\tilde{v} \in V_2$. Hence the existence of $(n-k)$ β -edges implies that each $v \in V_2$ corresponds to one of these β -edges, and it follows $\{(i, j) \mid i \in V_1 \wedge j \in V_2\} \subseteq A$.

The considerations above imply that $\hat{v} \in V_1$ for each vertex \hat{v} corresponding to an α -edge. Now assume there is an arc $(v_2, v_1) \in A$ with $v_i \in V_i$; $i = 1, 2$. Since $|N_{\tilde{D}}^-(v_1)| \geq 2$ we obtain $N_{\tilde{D}}^-(v_1) \in \mathcal{E}(\mathcal{H}_1)$, but $N_{\tilde{D}}^-(v_1)$ is neither an α -edge (because of $v_2 \in V_2$ and $v_2 \in N_{\tilde{D}}^-(v_1)$) nor a β -edge (because of $v_1 \in V_1$ and $v_1 \notin N_{\tilde{D}}^-(v_1)$), a contradiction. Hence there are no arcs from V_2 to V_1 in \tilde{D} , i.e., \tilde{D} is not strongly connected. ■

Thus, for each $n \geq 5$, there exists a connected competition hypergraph $\text{CH}(D)$ (of a digraph D with n vertices) being not the competition hypergraph of any strongly connected digraph D' .

The question arises, for what reasons there is no strongly connected digraph \tilde{D}_1 with $\mathcal{H}_1 = \text{CH}(D_1) = \text{CH}(\tilde{D}_1)$; in other words: why does

$D_1 = D_1(k, n)$ have no strongly connected "competition equivalent" digraph \tilde{D}_1 ?

As we will see the three reasons are

- the existence of all arcs from the "left" strong component $D_1[\{1, 2, \dots, k\}]$ to the "right" strong component $D_1[\{k+1, k+2, \dots, n\}]$ of $D_1 = (V, E)$;
- different vertices $v \neq v'$ have different sets of predecessors $N^-(v) \neq N^-(v')$ and
- every vertex $v \in V$ has at least two predecessors.

These three properties can even be used to characterize the digraphs having no competition equivalent strong digraphs.

Definition. A digraph $D = (V, A)$ with (nontrivial) strong components D_1, \dots, D_k (in acyclic ordering) is an *mcce-digraph* iff $k = 1$ (i.e., D is strongly connected) or $k > 1$ and

- (a) $\forall i, j \in \{1, 2, \dots, k\} \forall v \in V(D_i) \forall v' \in V(D_j) : i < j \Rightarrow (v, v') \in A$;
- (b) $\forall v \in V : |N^-(v)| \geq 2$;
- (c) $\forall v, v' \in V : v \neq v' \Rightarrow N^-(v) \neq N^-(v')$.

The abbreviation **mcce**-digraph comes from **m**aximal **c**onected with respect to **c**ompetition **e**quivalence.

The main results of this section are the following two theorems, which will be proved in Section 4.

Theorem 4. *For every digraph $D = (V, A)$ (with nontrivial strong components) there exists an mcce-digraph D' with $\mathcal{CH}(D) = \mathcal{CH}(D')$.*

In the following section we will give a constructive proof of Theorem 4 using an algorithm (Algorithm MCCE). Algorithm MCCE will be able to construct a competition equivalent mcce-digraph D' to a given digraph D such that the connectedness of D' is "best possible" in the sense of

Theorem 5. *A competition hypergraph $\mathcal{CH}(D)$ of a digraph $D = (V, A)$ (with nontrivial strong components) is the competition hypergraph of a strongly connected digraph iff every competition equivalent mcce-digraph D' of D is strongly connected.*

3. ALGORITHM

In Algorithm MCCE we will need three basic operations closely related to the defining properties of mcce-digraphs. For this end let $D = (V, A)$ be a digraph with the (nontrivial) strong components D_1, \dots, D_k (in acyclic ordering). Operations A, B and C modify D and generate a new digraph $D' = (V, A')$ as described below.

Operation A: Interchange of in-neighbourhoods.

Let $i, j \in \{1, 2, \dots, k\}$ with $i \neq j$ and $v \in V(D_i)$, $v' \in V(D_j)$ be two non-adjacent vertices. We obtain D' from D by interchanging the in-neighbourhoods of v and v' , i.e.,:

$$N_{D'}^-(v) := N_D^-(v') \quad \text{and} \quad N_{D'}^-(v') := N_D^-(v).$$

Operation B: Vertices of in-degree 1.

Let $i, j \in \{1, 2, \dots, k\}$, $i < j$, $v \in V(D_i)$ with $|N_D^-(v)| = 1$ and $\forall v_i \in V(D_i) \forall v_j \in V(D_j) : (v_i, v_j) \in A(D)$. Delete the incoming arc of v , add an arc (v', v) for an arbitrarily chosen $v' \in V(D_j)$, i.e., the only difference between D and D' is that in D' the vertex v has $N_{D'}^-(v) = \{v'\}$.

Operation C: Separation of in-neighbourhoods.

Let $v_1, v_2, \dots, v_s \in V$ with $N_D^-(v_1) = N_D^-(v_2) = \dots = N_D^-(v_s)$. Delete the incoming arcs of v_2, v_3, \dots, v_s and add the arcs $(v_1, v_2), (v_2, v_3), \dots, (v_{s-1}, v_s)$, i.e., in D' we have $N_{D'}^-(v_1) = N_D^-(v_1)$, $N_{D'}^-(v_2) = \{v_1\}$, $N_{D'}^-(v_3) = \{v_2\}, \dots$, $N_{D'}^-(v_s) = \{v_{s-1}\}$.

Now we discuss some important properties of the described operations.

Lemma 6. *Let $D = (V, A)$ be a digraph having only nontrivial strong components. Let $D' = (V, A')$ be the digraph constructed from D*

- (A) *by applying Operation A to $v \in V(D_i)$ and $v' \in V(D_j)$ or*
- (B) *by applying Operation B to $v \in V(D_i)$ and $v' \in V(D_j)$, where $i < j$, $|N_D^-(v)| = 1$ and $\forall v_i \in V(D_i) \forall v_j \in V(D_j) : (v_i, v_j) \in A$ hold.*

Then we obtain:

- (1) $V(D_i) \cup V(D_j)$ *is contained in a strong component D'_l of D' .*
- (2) *If $D_i = D_{i_1}, D_{i_2}, \dots, D_{i_t} = D_j$ induce a path in $SC(D)$, then $V(D_{i_1}) \cup V(D_{i_2}) \cup \dots \cup V(D_{i_t})$ is contained in a strong component D'_l of D' .*
- (3) $CH(D) = CH(D')$.

Proof. (A): First, we consider Operation A.

(1) For $u, u' \in V(D_i) \cup V(D_j)$ we have to demonstrate the existence of a (u, u') -path $w_{D'}^{u, u'}$ in D' .

The existence of $w_{D'}^{u, u'}$ is evident if $u, u' \in V(D_i)$ or $u, u' \in V(D_j)$ or $i < j \wedge u \in V(D_i) \wedge u' \in V(D_j)$ and in D there is a (u, u') -path $w_D^{u, u'}$ which does not contain incoming arcs of v and v' . In this case we choose $w_{D'}^{u, u'} = w_D^{u, u'}$.

Now let $u, u' \in V(D_i)$ and $w_D^{u, u'} = (u, \dots, v^-, v, \dots, u')$ with $v^- \in N_D^-(v)$. In D' we have $v^- \in N_{D'}^-(v')$ and we find a $v'^- \in N_{D'}^-(v') = N_{D'}^-(v)$ such that there is a (v', v'^-) -path $w_{D'}^{v', v'^-} := w_D^{v', v'^-}$, where w_D^{v', v'^-} obviously does not contain an incoming arc of v' . We modify $w_D^{u, u'}$ and obtain a (u, u') -path $w_{D'}^{u, u'}$ in D' in the following way:

$$w_{D'}^{u, u'} = (u, \dots, v^-, w_{D'}^{v', v'^-}, v, \dots, u').$$

The case $u, u' \in V(D_j)$ and $w_D^{u, u'} = (u, \dots, v'^-, v', \dots, u')$ with $v'^- \in N_D^-(v')$ can be considered analogously.

If $u \in V(D_i) \wedge u' \in V(D_j)$, then there is a $v^- \in N_D^-(v) \cap V(D_i) \subseteq N_{D'}^-(v')$ such that in $D'[V(D_i)]$ (as well as in $D[V(D_i)]$) there exists a (u, v^-) -path $w_{D'}^{u, v^-} = w_D^{u, v^-}$ not containing any incoming arc of v (and, obviously, of v'). Moreover, in $D'[V(D_j)]$ (as well as in $D[V(D_j)]$) there is a (v', u') -path $w_{D'}^{v', u'} = w_D^{v', u'}$, not containing any incoming arc of v' (and, obviously, of v). Consequently, we have $w_{D'}^{u, u'} = (w_{D'}^{u, v^-}, w_{D'}^{v', u'})$. The case $u \in V(D_j) \wedge u' \in V(D_i)$ follows analogously.

(2) is an immediate conclusion of (1).

(3) is obvious, because the interchange of in-neighbourhoods does not influence the set $\{N_D^-(v) | v \in V \wedge |N_D^-(v)| > 1\} = \mathcal{E}(CH(D))$, i.e., $CH(D)$ remains unchanged.

(B): Now, we investigate Operation B.

(1) Again, for $u, u' \in V(D_i) \cup V(D_j)$ we need a (u, u') -path $w_{D'}^{u, u'}$ in D' . If $u \in V(D_i) \wedge u' \in V(D_j)$, because of $(u, u') \in A$ we have the path $w_{D'}^{u, u'} = (u, u')$. In the cases $u, u' \in V(D_j)$ or $u, u' \in V(D_i) \wedge v \notin V(w_D^{u, u'}) \setminus \{u\}$, we choose $w_{D'}^{u, u'} = w_D^{u, u'}$. Let $u, u' \in V(D_i) \wedge v \in V(w_D^{u, u'}) \setminus \{u\}$ and $w_D^{u, u'} = (u, \dots, v^-, v, \dots, u')$. Owing to $(v^-, v'), (v', v) \in A'$ we obtain the wanted path by $w_{D'}^{u, u'} = (u, \dots, v^-, v', v, \dots, u')$, where the vertex v' was inserted in $w_D^{u, u'}$ between v^- and v .

Now $u \in V(D_j) \wedge u' \in V(D_i)$. Since D_j and D_i are strong components of D , there is a (u, v') -path $w_D^{u, v'}$ in D_j and a (v, u') -path $w_D^{v, u'}$ in D_i . It follows easily that both paths are also paths in D' , consequently we can choose $w_{D'}^{u, u'} = (w_D^{u, v'}, w_D^{v, u'})$.

Again, (2) is a direct conclusion from (1).

(3) Operation B does not change $\mathcal{CH}(D)$, because we manipulate only a vertex v having in-degree 1 in D as well as in D' and therefore v does not correspond to any edge in the competition hypergraph. ■

Proposition 7. *For every digraph $D = (V, A)$ (with nontrivial strong components) there is a digraph $D' = (V, A')$ with the nontrivial strong components D'_1, \dots, D'_k (in acyclic ordering) and $\mathcal{CH}(D) = \mathcal{CH}(D')$, such that $SC(D')$ is a transitive tournament and*

$$\forall i, j \in \{1, 2, \dots, k\} \forall v \in V(D'_i) \forall v' \in V(D'_j) : i < j \Rightarrow (v, v') \in A'.$$

Proof. Starting with D , the iterated application of Operation A to pairs (u, u') of non-adjacent vertices $u \in V(D_i)$ and $u' \in V(D_j)$, where $i \neq j$, leads to D' . ■

Lemma 8. *Let $D = (V, A)$ be a digraph having only nontrivial strong components and $D' = (V, A')$ be the digraph constructed from D by applying Operation C to $v_1, v_2, \dots, v_s \in V$ with $N_D^-(v_1) = N_D^-(v_2) = \dots = N_D^-(v_s)$.*

Then we obtain:

- (1) *There is a strong component D_j of D such that $v_1, v_2, \dots, v_s \in V(D_j)$.*
- (2) *$D'[V(D_j)]$ is a strong component of D' .*
- (3) *$\mathcal{CH}(D) = \mathcal{CH}(D')$.*

Proof. (1) Assume, $v_x \in V(D_{x'})$ and $v_y \in V(D_{y'})$, where $x, y \in \{1, 2, \dots, s\}$, $x \neq y$, $x', y' \in \{1, 2, \dots, k\}$ and $x' \neq y'$. Without loss of generality let $x' < y'$, i.e., because of the acyclic ordering of the strong components of D there is no arc from $D_{y'}$ to $D_{x'}$. Since $D_{y'}$ is strongly connected, v_y must have a predecessor v_y^- in $D_{y'}$. Due to $N_D^-(v_1) = N_D^-(v_2) = \dots = N_D^-(v_s)$ the vertex $v_y^- \in V(D_{y'})$ is also a predecessor of $v_x \in V(D_{x'})$, i.e., (v_y^-, v_x) is an arc from $D_{y'}$ to $D_{x'}$, a contradiction.

(2) Let $u, u' \in V(D_j)$ and $u \neq u'$. If in D there is a (u, u') -path $w_D^{u, u'}$ with $\{v_1, v_2, \dots, v_s\} \cap V(w_D^{u, u'}) = \emptyset$, then $w_D^{u, u'}$ is also a (u, u') -path $w_{D'}^{u, u'}$ in D' .

Otherwise, let $v_x, v_y \in \{v_1, v_2, \dots, v_s\}$ such that in $w_D^{u,u'}$ v_x is the first and v_y is the last of these vertices appearing in $w_D^{u,u'} = (u = u_0, u_1, \dots, u_{p-1}, u_p = v_x, \dots, u_q = v_y, u_{q+1}, \dots, u_t = u')$. In D' we substitute $u_p = v_x, \dots, u_q = v_y$ in $w_D^{u,u'}$ by v_1, v_2, \dots, v_y and obtain a (u, u') -path $w_{D'}^{u,u'} = (u = u_0, u_1, \dots, u_{p-1}, v_1, v_2, \dots, v_y, u_{q+1}, \dots, u_t = u')$.

(3) Operation C does not change $\mathcal{CH}(D)$, since in $\mathcal{CH}(D)$ the vertices $\{v_1, v_2, \dots, v_s\}$ correspond to one and the same hyperedge $e = N_D^-(v_1)$. In D' the vertex v_1 corresponds to this hyperedge e and the remaining vertices $\{v_2, v_3, \dots, v_s\}$ have in-degree 1, i.e., they do not correspond to any edge of the competition hypergraph $\mathcal{CH}(D')$. ■

Now we give

Algorithm MCCE.

Let $D = (V, A)$ be a digraph with (nontrivial) strong components D_1, \dots, D_k (in acyclic ordering).

1. **Apply** Operation A as long as possible and
obtain a digraph $D' = (V, A')$ with k' strong components.
2. **Let** $k := k'$, $D := D'$ and D_1, \dots, D_k be the (new) strong components of $D = D'$ (in acyclic ordering).
3. **If** D is strongly connected (i.e., $k = 1$),
then goto 6.
4. **If** $\exists v : |N_D^-(v)| = 1$ (obviously, in this case $v \in V(D_1)$ must be valid),
then choose $v' \in V(D_k)$,
apply Operation B to v and v' ,
obtain a strongly connected digraph $D := D' = (V, A')$ and
goto 6.
5. **If** $\exists i \exists u, v \in V(D_i) : N_D^-(u) = N_D^-(v)$,
then apply Operation C to u and v and
obtain a digraph $D := D' = (V, A')$, where $|N_{D'}^-(v)| = 1$;
if $i < k$,
then choose $v' \in V(D_k)$,
apply Operation B to v and v' ,
obtain a digraph $D' = (V, A')$ with strong components
 $D'_1, \dots, D'_{k'}$ (in acyclic ordering; note that
 $D'_{k'} = D'[V(D_i) \cup V(D_{i+1}) \cup \dots \cup V(D_k)]$) and

let $k := k'$, $D := D'$ and D_1, \dots, D_k be the (new) strong components of $D = D'$ (in acyclic ordering);
if $k > 1$ (note that in this case because of $v \in V(D_k)$ and $|N_D^-(v)| = 1$ no vertex $u \in V(D_1)$ is a predecessor of v),
then choose $u \in V(D_1)$,
apply Operation A to u and v and
obtain a strongly connected digraph $D := D' = (V, A')$.

6. **Stop.**

4. PROOFS AND CONCLUDING REMARKS

In this section we prove Theorems 4 and 5. Obviously, to show Theorem 4 it suffices to verify Algorithm MCCE.

Proof of Theorem 4. At first we verify the feasibility of Algorithm MCCE. Obviously, it suffices to investigate steps 4 and 5.

In step 4 we have $k > 1$ and Operation A cannot be applied to D . Consequently the validity of condition (a) of the definition of an mcce-digraph follows. In particular this means that there are all possible arcs from vertices of D_1 to vertices of D_k . Therefore, for a vertex $v \in V(D_1)$ with in-degree 1 and an arbitrary vertex $v' \in V(D_k)$ Operation B can be applied.

Besides we remark that after steps 1–3 the strong component digraph $SC(D)$ is a transitive tournament (cf. Proposition 7). Hence, D_1, D_2, \dots, D_k induce a path in $SC(D)$ and Operation B evidently provides a strongly connected digraph D' (cf. Lemma 6(2)).

In step 5 it is trivial that Operation C is feasible. Since in case $i < k$ there are all arcs from $V(D_i)$ to $V(D_k)$, Operation B can be applied (note that Operation C before influenced only incoming arcs of D_i or arcs inside the strong component D_i) to $v \in V(D_i)$ (with in-degree 1) and an arbitrary vertex $v' \in V(D_k)$.

Again, since D_i, D_{i+1}, \dots, D_k induce a path in $SC(D)$, Lemma 6(2) provides that Operation B results in a strongly connected subdigraph $D'[V(D_i) \cup V(D_{i+1}) \cup \dots \cup V(D_k)]$. Because there are no arcs in D' from $V(D_i) \cup V(D_{i+1}) \cup \dots \cup V(D_k)$ to $V(D_1) \cup V(D_2) \cup \dots \cup V(D_{i-1})$ we obtain $D'_{k'} = D'[V(D_i) \cup V(D_{i+1}) \cup \dots \cup V(D_k)]$.

After updating k , D and D_1, D_2, \dots, D_k the vertex v (with in-degree 1) is now in the strong component D_k . Therefore the only predecessor of v is in D_k and in case $k > 1$ Operation A can be applied to $u \in V(D_1)$ and v .

Consequently, Algorithm MCCE is feasible.

Now we verify that Algorithm MCCE results in an mcce-digraph having the same competition hypergraph as the initial digraph.

Starting from the initial digraph $D = (V, A)$, Algorithm MCCE uses Operations A, B and C to construct a new digraph. Owing to Lemma 6 and Lemma 8 this procedure does not change the competition hypergraph $\mathcal{CH}(D)$.

Let D be the new digraph constructed in Algorithm MCCE. Steps 3, 4 and 5 lead to a strongly connected digraph D , i.e., in these cases D is an mcce-digraph.

Now let D result from steps 1 and 2, where $k > 1$. Since Operation A cannot be applied any longer, in D there are no non-adjacent vertices $v \in V(D_i)$ and $v' \in V(D_j)$ with $i \neq j$. Therefore condition (a) of the definition of an mcce-digraph holds.

Because D was computed in steps 1 and 2, neither the premise of step 4 nor the premise of step 5 can be fulfilled. But this is equivalent to property (b) and property (c) of an mcce-digraph, respectively. ■

Proof of Theorem 5. It suffices to show that if the competition hypergraph $\mathcal{CH}(D)$ of the digraph $D = (V, A)$ (with nontrivial strong components) is the competition hypergraph of a strongly connected digraph, then every competition equivalent mcce-digraph D' of D is strongly connected. Let $\tilde{D} = (V, \tilde{A})$ be strongly connected and competition equivalent to D .

Assume, $D' = (V, A')$ is a competition equivalent mcce-digraph of the digraph $D = (V, A)$ and D_1, \dots, D_k are the strong components of D' in acyclic ordering, where $k > 1$.

Then, obviously, D' and \tilde{D} are competition equivalent and we set $\mathcal{H} = (V, \mathcal{E}) := \mathcal{CH}(D') = \mathcal{CH}(\tilde{D})$.

Since \tilde{D} is strongly connected, we obtain

$$(1) \quad \exists v' \in V(D_k) \exists v \in U := \bigcup_{i=1}^{k-1} V(D_i) : (v', v) \in \tilde{A}.$$

Property (b) in the definition of mcce-digraph implies $\forall w \in V : N_{D'}^-(w) \in \mathcal{E}$ and property (c) yields $\forall w, w' \in V : w \neq w' \Rightarrow N_{D'}^-(w) \neq N_{D'}^-(w')$. I.e., every vertex $w \in V$ corresponds to a hyperedge $N_{D'}^-(w) \in \mathcal{E}$ and all these hyperedges are pairwise distinct. Because of $\mathcal{E}(\mathcal{CH}(D')) = \mathcal{E}(\mathcal{CH}(\tilde{D})) = \mathcal{E}$ the same holds for \tilde{D} , i.e., for the hyperedges $N_{\tilde{D}}^-(w) \in \mathcal{E}$.

Since in D' there is no arc from $V(D_k)$ to $U = \bigcup_{i=1}^{k-1} V(D_i)$ and we have no loops, it follows $\forall u \in U : N_{D'}^-(u) \subset U$. On the other hand, the acyclic ordering of the strong components D_1, \dots, D_k of D' and property (a) in the definition of mcce-digraph provide

$$(2) \quad \forall e \in \mathcal{E}(\mathcal{CH}(D')) = \mathcal{E} : e \cap V(D_k) \neq \emptyset \Rightarrow U \subset e.$$

Owing to the competition equivalence of D' and \tilde{D} for every vertex $w \in V(\tilde{D}) = V$ there is a vertex $w^* \in V(D') = V$ such that $N_{\tilde{D}}^-(w) = N_{D'}^-(w^*)$, and vice versa.

Let us consider the vertex v from (1) and let $v^* \in V$ with $N_{\tilde{D}}^-(v) = N_{D'}^-(v^*)$. Because there are no loops in \tilde{D} we have $v \notin N_{\tilde{D}}^-(v) = N_{D'}^-(v^*)$, and consequently $U \not\subset N_{\tilde{D}}^-(v)$. Property (2) implies $N_{\tilde{D}}^-(v) \cap V(D_k) = \emptyset$. This contradicts (1). ■

Up to now, we excluded trivial strong components from our considerations. One reason is that algorithm MCCE could handle such components only under special assumptions.

The following example shows an infinite family of digraphs $D_2(k, n)$ having one trivial strong component, where Operation A fails, if we try to apply it to the trivial strong component.

Example. Let $n \geq 6$ and $k \in \{3, 4, \dots, n-3\}$. Then $D_2 = D_2(k, n)$ has the vertices $V(D_2) = \{1, \dots, n\}$ and the arcs

$$\begin{aligned} A(D_2) = & \{(i, i+1) \mid i \in \{1, \dots, k-1\}\} \cup \{(k, 1)\} \cup \\ & \{(i, i+2) \mid i \in \{1, \dots, k-2\}\} \cup \{(k-1, 1), (k, 2)\} \cup \\ & \{(i, i+1) \mid i \in \{k+2, \dots, n-1\}\} \cup \{(n, k+2)\} \cup \\ & \{(i, j) \mid i \in \{1, \dots, k\} \wedge j \in \{k+2, \dots, n\}\} \cup \\ & \{(i, k+1) \mid i \in \{1, \dots, k\}\} \cup \{(k+1, j) \mid j \in \{k+3, \dots, n\}\}. \end{aligned}$$

Note that $(k+1, k+2) \notin A(D_2)$. The digraph $D_2 = D_2(k, n)$ and its competition hypergraph $\mathcal{H}_2 = \mathcal{CH}(D_2)$ are shown in Figure 2.

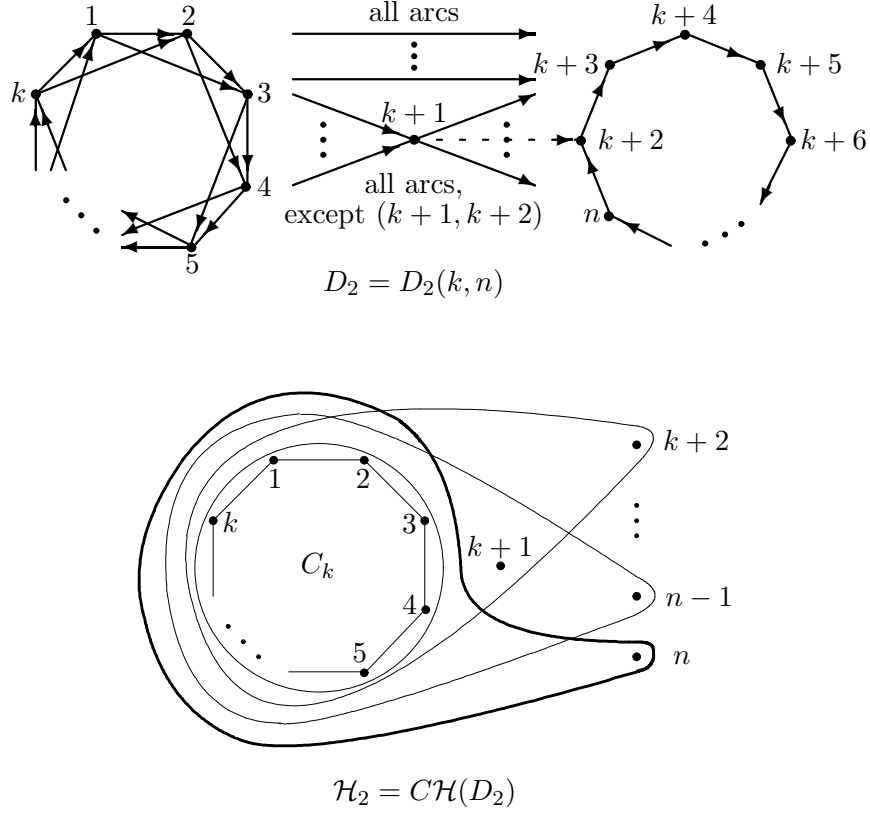


Figure 2. The digraph $D_2 = D_2(k, n)$ and its competition hypergraph $\mathcal{H}_2 = \mathcal{CH}(D_2)$.

The digraph $D_2(k, n)$ is not strongly connected and we obtain

Lemma 9. *Let $n \geq 6, k \in \{3, 4, \dots, n-3\}$ and $D_2 = D_2(k, n)$. Then every digraph \tilde{D} being competition equivalent to D_2 has at least three strong components.*

Proof. Let $\tilde{D} = (V, A)$ be a digraph with $\mathcal{H} := \mathcal{CH}(\tilde{D}) = \mathcal{CH}(D_2)$ and $V = V_1 \cup V_2 \cup \{k+1\}$ with $V_1 = \{1, \dots, k\}, V_2 = \{k+2, \dots, n\}$. There are four types of edges in \mathcal{H} :

- $(n - k - 2)$ α -edges of the form $V_1 \cup \{k + 1, k + t\}, t \in \{2, \dots, n - k - 1\}$,
- 1 β -edge $V_1 \cup \{n\}$ (thick lined edge),
- 1 γ -edge V_1 ,
- k δ -edges of the form $\{k, 1\}$ or $\{i, i + 1\}, i \in \{1, \dots, k - 1\}$.

Because $V_1 \cup \{k + 1\}$ is contained in each α -edge, the corresponding vertices to these α -edges belong to $\{k + 2, \dots, n\}$, i.e., exactly one of these vertices, say $(k + 2)$, is still available as the corresponding vertex to another edge. Either $(k + 2)$ corresponds to the β -edge, then $(k + 1)$ corresponds to the γ -edge (case 1) or vice versa (case 2). Hence the vertices $1, \dots, k$ correspond to the δ -edges.

In case 1 we obtain that there is no arc from V_2 to $V_1 \cup \{k + 1\}$ and no arc from $k + 1$ to V_1 . Consequently, each pair of vertices x, y from different vertex sets out of $V_1, \{k + 1\}$ and V_2 has the property that x and y have to be in different strong components of \tilde{D} . Therefore, \tilde{D} consists of at least three strong components. Changing the roles of $k + 1$ and $k + 2$, case 2 can be considered analogously. ■

By Lemma 9 it follows that Operation A has to fail if we try to apply this operation to the non-adjacent vertices $k + 1, k + 2 \in V(D_2)$. In detail we see that if we would change the sets of predecessors $N_{D_2}^-(k + 1) = \{1, \dots, k\}$ and $N_{D_2}^-(k + 2) = \{1, \dots, k, n\}$ (cf. Operation A), we would obtain a digraph D'_2 with the same competition hypergraph $\mathcal{CH}(D_2) = \mathcal{CH}(D'_2)$, but the vertices $k + 1$ and $k + 2$ would still belong to different strong components. So in D'_2 we have $N_{D'_2}^-(k + 2) = \{1, \dots, k\}$, i.e., there is no path from any vertex of $\{k + 1, k + 3, k + 4, \dots, n\}$ to $k + 2$. Moreover, it is obvious that there exists no path from $\{k + 1, k + 2, \dots, n\}$ to $\{1, \dots, k\}$.

Note that Operations A, B, C can be also applied to digraphs with trivial strong components if we assume that all vertices explicitly mentioned in these operations are contained in nontrivial components; results analogous to Lemma 6 and Lemma 8 can be verified.

In special cases trivial strong components of the digraph $D = (V, A)$ can be integrated into other strong components without changing the competition hypergraph:

Remark 10. Let $D = (V, A)$ be a connected digraph with the strong components D_1, \dots, D_k (in acyclic ordering) and let D_l be a trivial component, e.g. $V(D_l) = \{v\}$. If one of the following conditions is fulfilled, then there is a digraph \tilde{D} competition equivalent to D with the strong components $D'_1, \dots, D'_{k'}$ (in acyclic ordering), such that $v \in V(D'_{l'})$ with $|V(D'_{l'})| > 1$ and $l' \in \{1, \dots, k'\}$:

- (a) $\exists u \in V - \{v\} \exists w_D^{u,v} : N_D^-(u) = \emptyset \wedge w_D^{u,v}$ is a (u, v) -path in D ;
- (b) $|N_D^-(v)| \leq 1 \wedge N_D^+(v) \neq \emptyset$;
- (c) $\exists i \in \{1, \dots, l-1\} \exists j \in \{l+1, \dots, k\} \exists x \in V(D_i) \exists y \in V(D_j) \exists w_D^{x,y} : D_i, D_j$ nontrivial $\wedge w_D^{x,y}$ is an (x, y) -path in D containing $v \wedge$
 $\left(\begin{array}{l} \text{(i)} \ (\exists x_i \in V(D_i) \exists y_j \in V(D_j) : (x_i, y_j) \notin A) \vee \\ \text{(ii)} \ (\exists x_i \in V(D_i) : |N_D^-(x_i)| \leq 1) \vee \\ \text{(iii)} \ (\exists x_i, y_i \in V(D_i) : N_D^-(x_i) = N_D^-(y_i)) \end{array} \right).$

Proof. *Case (a).* If we add a new arc (v, u) to $A(D)$, we obtain a competition equivalent digraph \tilde{D} , where all strong components of D containing a vertex $x \in V(w_D^{u,v})$ are included in one strong component $D'_{l'}$ of \tilde{D} .

Case (b). We delete the incoming arc of v and add an arc (u, v) , where $u \in N_D^+(v)$. Then the vertices v and u are in one strong component $D'_{l'}$ of \tilde{D} . Note that in this case \tilde{D} may be even disconnected; this can be avoided by using (a) instead of (b) (if possible) or by the application of Operation A after deleting the incoming arc of v and adding (u, v) .

Case (c). Since D_i and D_j are nontrivial strong components connected by $w_D^{x,y}$, where v is contained in $w_D^{x,y}$, we can proceed as follows (similarly to Algorithm MCCE):

If (i) is fulfilled Operation A (cf. Lemma 6(2)) yields the desired result.

If (i) is not valid but (ii) is true, then Operation B can be used (cf. Lemma 6(2)) and we are done.

If neither (i) nor (ii) is fulfilled, it remains to consider (iii). We apply Operation C (cf. Lemma 8) and obtain $z \in \{x_i, y_i\} \subseteq V(D_i)$ with indegree 1. Now Operation B (cf. Lemma 6(2)) completes the proof. ■

It seems to be very difficult to generalize Algorithm MCCE and Theorems 4 and 5 such that trivial strong components can be included without stint.

One reason is the more complicated structure of the digraphs under consideration. On the other hand, Operations A, B and C do not work for a lot of configurations, where trivial strong components occur. It seems to be hopeless to search for modifications of Operations A, B and C or for new operations in order to obtain a complete description of the competition hypergraphs of strongly connected digraphs in analogy to Theorem 5.

Acknowledgement

The authors would like to thank one of the referees for his helpful suggestions.

REFERENCES

- [1] J. Bang-Jensen and G. Gutin, *Digraphs: theory, algorithms and applications* (Springer, London, 2001).
- [2] J.E. Cohen, *Interval graphs and food webs: a finding and a problem* (Rand Corporation Document 17696-PR, Santa Monica, CA, 1968).
- [3] K.F. Fraughnaugh, J.R. Lundgren, S.K. Merz, J.S. Maybee and N.J. Pullman, *Competition graphs of strongly connected and hamiltonian digraphs*, SIAM J. Discrete Math. **8** (1995) 179–185.
- [4] S.R. Kim, *The competition number and its variants*, in: J. Gimbel, J.W. Kennedy and L.V. Quintas (eds.), *Quo vadis, graph theory?*, Ann. of Discrete Math. **55** (1993) 313–326.
- [5] J.R. Lundgren, *Food webs, competition graphs, competition-common enemy graphs and niche graphs*, in: F. Roberts (ed.), *Applications of combinatorics and graph theory to the biological and social sciences*, IMA **17** (Springer, New York, 1989) 221–243.
- [6] F.S. Roberts, *Competition graphs and phylogeny graphs*, in: L. Lovász (ed.), *Graph theory and combinatorial biology*; Proc. Int. Colloqu. Balatonlelle (Hungary) 1996, Bolyai Soc. Math. Studies **7** (Budapest, 1999) 333–362.
- [7] F.S. Roberts and J.E. Steif, *A characterization of competition graphs of arbitrary digraphs*, Discrete Appl. Math. **6** (1983) 323–326.
- [8] M. Sonntag, H.-M. Teichert, *Competition hypergraphs*, Discrete Appl. Math. **143** (2004) 324–329.

Received 14 January 2005
 Revised 24 September 2007
 Accepted 31 December 2007