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A SOKOBAN-TYPE GAME AND ARC DELETION WITHIN IRREGULAR DIGRAPHS OF ALL SIZES

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Abstract

Digraphs in which ordered pairs of out- and in-degrees of vertices are mutually distinct are called *irregular*, see Gargano *et al.* [3]. Our investigations focus on the problem: what are possible sizes of irregular digraphs (oriented graphs) for a given order n? We show that those sizes in both cases make up integer intervals. The extremal sizes (the endpoints of these intervals) are found in [1, 5]. In this paper we construct, with help of Sokoban-type game, n-vertex irregular oriented graphs (irregular digraphs) of all intermediate sizes.

Keywords: irregular digraph, all sizes.

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1. Introduction

Let $G = (V(G), E(G))$ be a *digraph* (a directed graph without loops and without multiple arcs), where $V(G)$ and $E(G)$ denote the vertex set and the arc set, respectively. The cardinality of $V(G)$ $(E(G))$ is called the *order* (the *size*) of G. A digraph without 2-cycles is called an *oriented graph.* By $\mathrm{od}_G(u)$ and $\mathrm{id}_G(u)$ we denote the *out-degree* and the *in-degree* of the vertex u, respectively. The ordered pair $(\mathrm{od}_G(u), \mathrm{id}_G(u))$ will be called the *degree pair* of u in G. The set of all degree pairs of vertices of a given digraph G will be called the *degree set* of G.

A digraph G is called *irregular* if different vertices have distinct degree pairs, i.e., for each $u, v \in V(G)$ the following implication holds

$$u \neq v \Rightarrow (\mathrm{od}_G(u),\, \mathrm{id}_G(u)) \neq (\mathrm{od}_G(v),\, \mathrm{id}_G(v)).$$

The class of digraphs with this property of irregularity was introduced in the paper [3]. Properties of oriented graphs (digraphs) of above class were studied in papers [1, 2, 3, 4] and [5]. In particular, the minimum and the maximum size of these graphs have been found in papers [1] and [5].

It is clear that if G is n-vertex irregular oriented graph and D_G is the degree set of G, then

$$D_G \subset \{\underbrace{(0,0)}_{B_0}, \underbrace{(1,0),(0,1)}_{B_1}, \underbrace{(2,0),(1,1),(0,2)}_{B_2}, \dots, \underbrace{(n-1,0),(n-2,1),\dots,(1,n-2),(0,n-1)}_{B_{n-1}}\},$$

where $B_i = \{(a,b) : a+b = i,\ a, b \text{ are non-negative integers}\}$, $i = 0, \dots, n-1$.

Let $\epsilon_n^{\min}(or)$ and $\epsilon_n^{\max}(or)$ denote the minimum and the maximum size, respectively, for irregular oriented graphs of a given order n. Analogously, by $\epsilon_n^{\min}(di)$ $(\epsilon_n^{\max}(di))$ we denote the minimum (maximum) size for irregular digraphs. An irregular oriented graph is called *largest* (*smallest*) if it has the maximum (minimum) size among all irregular oriented graphs of given order. In similar way we define a *largest* and a *smallest* digraph.

The results of papers [1, 3] and [5] permit to obtain in simple way the following properties.

Property 1.

(a) *The transitive tournament T_n $(n \geq 1)$ on n vertices is the unique n-vertex largest irregular oriented graph, and B_{n-1} is the degree set of this tournament.*

(b) *The complement $\overline{G}$ of an irregular digraph G is an irregular digraph, too.*

(c) $\epsilon_n^{\min}(di) = \epsilon_n^{\min}(or)$ *and* $\epsilon_n^{\max}(di) = (n-1)n - \epsilon_n^{\min}(or)$.

(d) $\epsilon_n^{\max}(di) > \epsilon_n^{\max}(or) > \epsilon_n^{\min}(or)$ *for* $n > 2$.

It is easy to note that for given integer $n > 1$ there exist the unique numbers t and m such that

$$n = 1 + 2 + \cdots + t + m, \quad \text{where} \quad 1 \leq m \leq t + 1. \tag{1}$$

For more information on t and m see [4, 5]. In the paper [1] parameters t and m have used to the characterization of the family of all degree sets of smallest irregular oriented graphs with given order n. In particular, n-element sets $(n > 3)$ of the following form (2):

$$\left.\begin{array}{l} \bigcup_{i=0}^{t-2} B_i \cup K, \ \text{where} \\ K = \begin{cases} B_{t-1} \cup B_t(m) & \text{for } t \text{ or } m \text{ even}, \\ \left(B_{t-1} \setminus \{(\frac{1}{2}(t-1), \frac{1}{2}(t-1))\}\right) \cup B_t(m+1) & \text{for } t \text{ and } m \text{ odd}, \end{cases} \\ \qquad \text{where } B_t(k) \subset B_t, \ |B_t(k)| = k, \ \text{and } (a,b) \in B_t(k) \Leftrightarrow (b,a) \in B_t(k) \end{array}\right\}$$

belong to this family.

From mentioned characterization immediately follows

Property 2. *Let $n > 3$, t and m be integers for which* (1) *holds. Each set of the form* (2) *is the degree set of some smallest irregular oriented graph of order n.*

A solution of the following problem is the main aim of our paper.

Problem. Let $n > 2$ be positive integer and let $\phi = \epsilon_n^{\max}(or) - \epsilon_n^{\min}(or)$. Construct a sequence $G_0, G_1, \ldots, G_\phi$ of irregular oriented graphs of order n such that:

(a) G_0 is largest, G_ϕ is smallest,

(b) $G_i = G_{i-1} - e_i$, for some $e_i \in E(G_{i-1})$, where $i = 1, 2, \ldots, \phi$.

Solve analogous problem for digraphs.

For $n = 3$ a solution of above problem for oriented graphs can be given by the sequence G_0, G_1, G_2, where

$$G_0 = T_3, \quad G_1 = G_0 - (v_1, v_3), \quad G_2 = G_1 - (v_1, v_2) \text{ and}$$

T_3 denotes the tournament with the vertex set $\{v_1, v_2, v_3\}$ and the arc set $\{(v_1, v_2), (v_2, v_3), (v_1, v_3)\}$, and for digraphs by the sequence D_0, D_1, D_2, D_3, D_4, where

$$D_0 = \overline{G_2}, \quad D_1 = D_0 - (v_1, v_2), \quad D_2 = D_1 - (v_1, v_3),$$
$$D_3 = D_2 - (v_3, v_1), \quad D_4 = D_3 - (v_3, v_2)$$

and G_2 is the last element of previous sequence.

For $n > 3$, constructing the sequence $G_0, G_1, \ldots, G_\phi$ we start from the largest irregular oriented graph and we proceed to obtain a smallest one whose the degree set has the form described in (2). The order of the deletion of arcs will be described by a winning strategy in some Sokoban-type game given in next section.

2. Some Sokoban-Type Game

Sokoban is a puzzle game that can be found at various sites on the Internet (for instance `http://www.cs.ualberta.ca/~ games/Sokoban/`, `http://www.geocities.com/sokoban_game/`) and through commercial vendors. If sources are correct, Sokoban is a classic game invented in Japan. The original game was written by Hiroyuki Imabayashi. In 1980 it won a computer game contest. Mr Hiroyuki Imabayashi is the president of the Thinking Rabbit Inc. The object of the game is to push boxes (or balls) into their correct positions with a minimal number of pushes and moves. The boxes can only be pushed, never pulled, and only one can be pushed at one time. Initial and correct positions of boxes are given at the beginning of the game. It sounds easy, but the levels difficulty changes from very easy to extremely difficult, some takes hours and even days to figure them out. The simplicity and elegance of the rules have made Sokoban one of the most popular logic games. Over the years many versions for all platforms have been made and new levels are created all the time.

Our game runs on a triangle board of hexagonal areas placed in the plane as it is presented in Figure 1a).

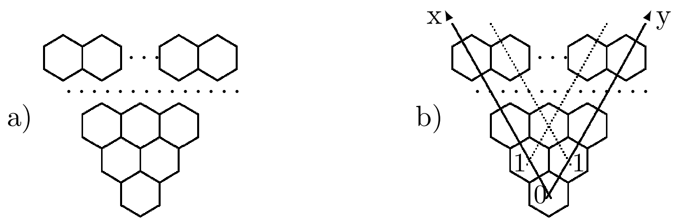

Figure 1

For a description of our game coordinate system with axes x and y (grid points are centers of hexagons, see Figure 1b)) will be used. A hexagon will be (i, j)-th if its center has coordinates (i, j). Suppose that the board has n hexagons on the top. By k-th *row*, $k = 1, 2, \ldots, n$, we mean the set all (i, j)-th hexagons for which $i + j = k - 1$. Obviously k-th row has k hexagons.

At the beginning of the game in n-th row there are n balls, say $b_1, b_2, \ldots, b_n$, allocated into $(n-1, 0)$-th, $(n-2, 1)$-th, $\ldots$, $(0, n-1)$-th hexagons, respectively. The aim of the game is to push all balls into hexagons of n-element set which has a given property $\mathcal{T}$. Each n-element set with property $\mathcal{T}$ will be called a *target* set. By a move in our game we mean the choice two balls, say b_i and b_j where $i < j$, and the push the ball b_i down on the right, the ball b_j down on the left, into one row below keeping to the board, i.e., if before the move chosen balls b_i and b_j occupy (p, q)-th and (r, s)-th hexagon, respectively, where $p > 0$ and $s > 0$, then after the move they are allocated into $(p-1, q)$-th and $(r, s-1)$-th hexagon, respectively. The locations of remaining balls are not changed. The move in which the balls b_i and b_j, $i < j$, are chosen will be denoted by (b_i, b_j). A player must keep the following rules.

Rule 1. Two balls b_i and b_j can be together chosen to the moves only once.

Rule 2. After each move, any two balls can never be in the same hexagon. It means, after each move, n balls occupy n different hexagons.

The player wins the game if all balls are pushed into hexagons of some target set. In other cases the player losses.

A sequence $S = ((l_1, r_1), (l_2, r_2), \ldots, (l_m, r_m))$ of moves will be called a *winning strategy* in our game if Rule 1 and Rule 2 are satisfied and the player doing these moves wins.

Example 1. Note that the game with three balls has no winning strategy if each target set has a single hexagon from each row, while $S_1 = ((b_1, b_3), (b_1, b_2))$ and $S_2 = ((b_1, b_3), (b_2, b_3))$ are winning strategies if some target set consists of $(0, 0)$-th, $(1, 0)$-th and $(0, 1)$-th hexagons.

Example 2. Let as consider the game with balls b_i, $1 \leq i \leq 5$, in which each target set contains all hexagons from 1-th and 2-th rows and two hexagons from 3-th row. Then $S = ((b_1, b_5), (b_1, b_4), (b_1, b_3), (b_1, b_2), (b_2, b_5), (b_2, b_3), (b_4, b_5))$ is a winning strategy while $S^* = ((b_1, b_5), (b_2, b_4), (b_3, b_4), (b_1, b_4), (b_2, b_5), (b_3, b_5))$ is not. Moreover S^* can not be lengthened to a winning strategy. In Figure 2 initial board, the board after the move (b_1, b_5), the board after the moves $(b_1, b_5), (b_1, b_4)$ and the board after all moves from S are presented.

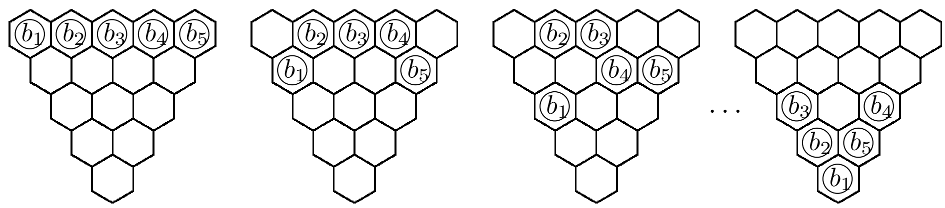

Figure 2

We will consider three types of our game in dependence of the location of target sets.

Type I. $n = p + q$, $p \neq q$. Each target set consists of all hexagons from p-th and q-th rows.

Type II. $n = p + q$, $p \geq q \geq 1$. Each target set consists of all hexagons from p-th row and q hexagons from $(p+1)$-th row or $p-1$ hexagons from p-th row and $q+1$ hexagons from $(p+1)$-th row.

Type III. $n > 3$, $n = 1 + 2 + \cdots + t + m$, $1 \leq m \leq t + 1$. Each target set consists of all hexagons from 1-th to t-th rows and m hexagons from $(t+1)$-th row or all hexagons from 1-th to $(t-1)$-th rows, $t-1$ hexagons from t-th row and $m+1$ hexagons from $(t+1)$-th row.

3. A Winning Strategy for our Game

First we give some notations. If $L = (l_1, l_2, \ldots, l_p)$ and $R = (r_1, r_2, \ldots, r_q)$ are sequences, then by (L, R) and LR we denote sequences defined as follows

$(L, R) = ((l_1, r_1), (l_2, r_2), \ldots, (l_m, r_m))$, where $m = \min\{p, q\}$.

$LR = (l_1, l_2, \ldots, l_p, r_1, r_2, \ldots, r_q)$ (it is known as the concatenation of L and R).

If L (or R) is the sequence of the length 0, then we assume that (L, R) is the sequence of the length 0 too, and $LR = R$ (or $LR = L$).

So, for positive integer m we have

$$L^m = \underbrace{LL \ldots L}_{m\ times} = \underbrace{(\underbrace{l_1, l_2, \ldots, l_p}_{L}, \underbrace{l_1, l_2, \ldots, l_p}_{L}, \ldots, \underbrace{l_1, l_2, \ldots, l_p}_{L})}_{m\ times}$$

and, in particular,

$$(a)^m = \underbrace{(a)(a) \ldots (a)}_{m\ times} = (\underbrace{a, a, \ldots, a}_{m\ times}).$$

Moreover, if $m = 0$, then we assume that L^m is the sequence of the length 0.

Let p, q be positive integers and let $(z_1, z_2, \ldots, z_{p+q})$ be a sequence. Put

$$\left.\begin{aligned} &L_1 = (z_p)^q (z_{p-1})^q \ldots (z_1)^q, \quad R_1 = (z_{p+q}, z_{p+q-1}, \ldots, z_{p+1})^p, \\ &r = \left\lfloor \tfrac{p+q}{2} \right\rfloor, \quad f = \left\lceil \tfrac{qr}{r+1} \right\rceil \\ &L_2 = (z_r)^q (z_{r-1})^q \ldots (z_1)^q, \quad R_2 = (z_{p+q}, z_{p+q-1}, \ldots, z_{r+1})^f. \end{aligned}\right\} \tag{3}$$

Theorem 1. *If $p < q$, then the sequence (L_1, R_1) is a winning strategy for Type* I *game with the balls $z_1, z_2, \dots, z_{p+q}$.*

Theorem 2. *If $p \geq q$ and $p + q$ is odd, then the sequence (L_2, R_2) is a winning strategy for Type* II *game with the balls $z_1, z_2, \dots, z_{p+q}$.*

***Proof of Theorems* 1 *and* 2.** It is not difficult to check that (L_1, R_1) and (L_2, R_2) are sequences of moves for which Rule 1 holds. To the proof that Rule 2 holds we note that:

- if two balls, say x and y, under of accomplishment moves from (L_j, R_j), are pushed into the same hexagon, say H, then x is from L_j and y is from R_j, $j = 1, 2$;
- the hexagon H is not from the top of the board and H is not from the lower row of each target set;
- there exists exactly one move in (L_j, R_j), say m_α, after which x occupies the hexagon H;
- if m_β is the first move in (L_j, R_j) after which y occupies H, then, by restrictions for the locations of x, y and H, we obtain $\beta - \alpha > 0$, it means that x is placed in H earlier than y.

Counting the number of moves in which a given ball appears it is easy to check that all balls are settled in hexagons which form the corresponding target set. ■

Now, we proceed to obtain the description of a winning strategy for Type II game in case when the number of balls on the top is even. Suppose that p and q are positive integers, $p \geq 2$, $p + q$ is even. Let $(z_1, z_2, \dots, z_{p+q})$ be a sequence. Put

$$\tag{4} \left.\begin{aligned} &k = \left\lfloor \tfrac{p}{2} \right\rfloor, \quad s = \left\lceil \tfrac{q}{2} \right\rceil, \quad f_1 = \left\lceil \tfrac{qk}{k+s} \right\rceil, \\ &L_3 = (z_k)^q (z_{k-1})^q \dots (z_1)^q, \\ &R_3 = (z_{k+2s+1}, z_{k+2s+2}, \dots, z_{p+q}, z_{k+1}, z_{k+2}, \dots, z_{k+s})^{f_1}. \end{aligned}\right\}$$

Let k_i be the number of pairs in (L_3, R_3) with z_{k+i}, $i = 1, 2, \dots, s$. Note that $p+q = 2(k+s)$ and $k_i = f_1$ or $k_i = f_1 - 1$. Let x be the index that ball among $z_{k+2s+1}, z_{k+2s+2}, \dots, z_{p+q}$ which was used as the last one in (L_3, R_3).

Put

$$\left.\begin{aligned}
&f_2 = \left\lceil \tfrac{k_1+k_2+\cdots+k_s}{k} \right\rceil,\\
&L_4 = (z_{k+2s})^{k_1}(z_{k+2s-1})^{k_2}\dots(z_{k+s+1})^{k_s},\\
&R_4 = \begin{cases} (z_{x+1}, z_{x+2}, \dots, z_{p+q})(z_{k+2s+1}, z_{k+2s+2}, \dots, z_{p+q})^{f_2} & \text{if} \quad x < p+q,\\ (z_{k+2s+1}, z_{k+2s+2}, \dots, z_{p+q})^{f_2} & \text{if} \quad x = p+q, \end{cases}
\end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned}
&f_3 = \left\lceil \tfrac{s(q-1)-(k_1+k_2+\cdots+k_s)}{s} \right\rceil,\\
&L_5 = (z_{k+s})^{q-1-k_s}(z_{k+s-1})^{q-1-k_{s-1}}, \dots, (z_{k+1})^{q-1-k_1},\\
&R_5 = (z_{k+s+1}, z_{k+s+2}, \dots, z_{k+2s})^{f_3}.
\end{aligned}\right\} \tag{6}$$

Theorem 3. *If $p \geq q$ and $p+q > 3$ is even, then the sequence (L_3, R_3) $(L_4, R_4)(L_5, R_5)$ is a winning strategy for Type* II *game with balls $z_1, z_2, \dots, z_{p+q}$.*

Proof. By similar argumentations as for (L_2, R_2) we can prove that Rule 1 and Rule 2 hold for (L_3, R_3). To prove that Rule 1 holds for (L_j, R_j), $j = 4, 5$, it is sufficient to note that $q - s - 1 \leq f_1 - 1 \leq k_i \leq f_1 \leq k$. These inequalities simply follow from definitions of k, s, f_1, k_i. Moreover, the inequality $f_1 \geq q - k_i$ warrants Rule 2 for (L_5, R_5).

Now we will show that Rule 2 holds for (L_4, R_4). Let $A = (z_{k+s+1}, z_{k+s+2}, \dots, z_{k+2s})$ and $B = (z_{k+2s+1}, z_{k+2s+2}, \dots, z_{p+q})$. Note that if two balls under of accomplishment moves from (L_4, R_4) are pushed into the same hexagon, then one of them belongs to A and second one belongs to B.

Let us consider the locations of balls of $A \cup B$ after all moves from (L_3, R_3). The balls of A are in hexagons on $(p+q)$-th row. The locations of balls from B depend on x. Namely, if $x = p+q$, then all balls of B are located into hexagons of $(p+q-f_1)$-th row, if $x < p+q$, then the balls $z_{k+2s+1}, z_{k+2s+2}, \dots, z_x$ are in $(p+q-f_1)$-th row and the balls $z_{x+1}, z_{x+2}, \dots, z_{p+q}$ are in $(p+q-f_1+1)$-th row.

Consider the case when $x = p+q$. Note that under accomplishment of moves from (L_4, R_4) the balls of A are pushed only into hexagons from $(p+q-i)$-th rows, where $i = 1, 2, \dots, k_i$, while balls of B are pushed in $(p+q-f_1-j)$-th row, where $j \geq 0$. So, if all k_i are equal to $f_1 - 1$, then Rule 2 holds for (L_4, R_4). Assume that $k_i = f_1$ for some $i \in \{1, 2, \dots, s\}$, then $k_1 = f_1$. Let us denote by H_1, $H_2, \dots, H_{f_1}$ hexagons in which there are allocated the balls z_{k+2s+1}, $z_{k+2s+2}, \dots, z_{k+2s+f_1}$, respectively, after all moves from (L_3, R_3). Among hexagons occupied by balls from B after all moves from (L_3, R_3) only hexagons H_1,$H_2, \dots, H_{f_1}$ can be occupied by balls from A after moves from (L_4, R_4). First f_1 moves from (L_4, R_4) push the ball z_{k+2s} into H_{f_1} while the ball z_{k+2s+f_1} is pushed lower. The balls z_{k+2s+1}, $z_{k+2s+2}, \dots, z_{k+2s+f_1-1}$ are pushed lower too, so the hexagons H_1,$H_2, \dots, H_{f_1-1}$ became empty. Thus, for all moves from (L_4, R_4) Rule 2 holds.

In case $x < p+q$ we prove that Rule 2 holds for all moves from (L_4, R_4) in similar way as above.

It is not difficult to check that considered strategy follows to a target set for Type II game. ■

Finally, we describe a winning strategy for Type III game with the balls $b_1, b_2, \dots, b_n$, $n > 3$. Let t and m be integers for which equality (1) is true. For $i = 1, 2, \dots, t$ put

$$S^i_{n-s_i} = (L_1, R_1), \tag{7}$$

where $s_i = 1 + 2 + \cdots + i$, $s_0 = 0$, L_1 and R_1 are sequences described in (3) for $p = i$, $q = n - s_i$ and $z_j = b_{s_{i-1}+j}$, $j = 1, 2, \dots, n - s_{i-1}$.

Let L_2, R_2, L_3, R_3, L_4, R_4 and L_5, R_5 be sequences described in (3), (4), (5) and (6), respectively, for $p = t$, $q = m$ and $z_j = b_{s_{t-1}+j}$, $j = 1, 2, \dots, t + m$. Put

$$S^* = \begin{cases} S^t_{t+1} & \text{if} \quad m = t + 1, \\ (L_2, R_2) & \text{if} \quad m \le t \text{ and } t + m \text{ is odd}, \\ (L_3, R_3)(L_4, R_4)(L_5, R_5) & \text{if} \quad m \le t \text{ and } t + m \text{ is even}. \end{cases} \tag{8}$$

Theorem 4. *The sequence $S = S^1_{n-1} S^2_{n-3} \dots S^{t-1}_{n-s_{t-1}} S^*$ is a winning strategy for Type* III *game with the balls $b_1, b_2, \dots, b_n$.*

Proof. Rule 1 holds, by definition of S. Let us consider a move g from S. We will show that after g balls $b_1, b_2, \dots, b_n$ occupy pairwise different hexagons. Let g be the move from $S^i_{n-s_i}$, $1 \le i \le t - 1$. Note that after the last move from $S^1_{n-1} S^2_{n-3} \dots S^{i-1}_{n-s_{i-1}}$ the balls $b_1, b_2, \dots, b_{s_{i-1}}$ are allocated into j-th rows, $j = 1, 2, \dots, i - 1$, while remaining balls — into $(n - s_{i-1})$-th row, where $n - s_{i-1} > i - 1$. Moves from $S^i_{n-s_i}$ are make with balls

$b_{s_{i-1}+1}, b_{s_{i-1}+2}, \ldots, b_n$ and they allocate these balls into hexagons of j-th rows, where $j > i-1$. Thus, after the move g, the balls $b_1, b_2, \ldots, b_n$ occupy pairwise different hexagons, by Theorem 1. A similar situation appears, if g is a move from S^*. Then, after the move g, the balls $b_1, b_2, \ldots, b_n$ occupy pairwise different hexagons, by Theorem 1, Theorem 2 or Theorem 3. ■

4. A Relationship Between the Game and Irregular Digraphs

Let $n > 3$ be positive integer and let T_n be n-vertex transitive tournament. Suppose that T_n has vertices $v_1, v_2, \ldots, v_n$ and that these vertices are numbered in such a way that

$$(v_i, v_j) \in E(T_n) \Leftrightarrow i < j.$$

Then $(\text{od}_{T_n}(v_i), \text{id}_{T_n}(v_i)) = (n-i, i-1)$, $i = 1, 2, \ldots, n$.

Let us consider Type III game with n balls $b_1, b_2, \ldots, b_n$. Notice that, the locations of hexagons of n-th row in the board form the set B_{n-1}. At the beginning of the game, the ball b_i, $i = 1, 2, \ldots, n$, occupies that hexagon whose the location is the degree pair of the vertex v_i in T_n. Let γ be the mapping $b_i \to v_i$ between $\{b_1, b_2, \ldots, b_n\}$ and $V(T_n)$. Then for each pair (b_i, b_j), where $i < j$, the pair $(\gamma(b_i), \gamma(b_j))$ is the arc (v_i, v_j) of T_n and after the move (b_i, b_j) locations of hexagons occupied by balls b_i and b_j are equal to the degree pairs of vertices v_i and v_j in $T_n - (v_i, v_j)$, respectively.

Let $S = ((b_{i_1}, b_{j_1}), (b_{i_2}, b_{j_2}), \ldots, (b_{i_\phi}, b_{j_\phi}))$ be the winning strategy for Type III game described in Theorem 4 and let C be the set of hexagons occupied by all balls after all moves of S. The move (b_{i_1}, b_{j_1}) corresponds to the deletion of the arc (v_{i_1}, v_{j_1}) from T_n. By Rule 1, the move (b_{i_2}, b_{j_2}) corresponds to the deletion of the arc (v_{i_2}, v_{j_2}) from $T_n - (v_{i_1}, v_{j_1})$, and so on. For $k = 1, 2, \ldots, \phi$ put

$$e_k = (v_{i_k}, v_{j_k})$$

and

$$G_0 = T_n \text{ and } G_k = G_{k-1} - e_k. \tag{9}$$

By Rule 2, each G_k is an irregular oriented graph. Because locations of hexagons of C form a set of pairs described in Property 2, then G_ϕ is an irregular oriented graph with the minimum size, so $\phi = \epsilon_n^{\max}(or) - \epsilon_n^{\min}(or)$ and

Theorem 5. *The sequence* $G_0, G_1, \ldots, G_\phi$*, described in* (9), *gives a solution of the Problem for oriented graphs.* ∎

Put

$$D_0 = \overline{G_\phi}, \quad D_k = \overline{G_{\phi-k}} \text{ for } k = 1, 2, \ldots, \phi. \tag{10}$$

Then, by Property 1, each D_k is an irregular digraph, D_0 is some largest one. Moreover, $D_\phi = \overline{T_n}$ and $D_k = D_{k-1} - e'_k$, where $e'_k = e_{\phi-k+1}$.

Let σ be the following correspondence between sets $\{b_1, b_2, \ldots, b_n\}$ and $V(\overline{T_n})$

$$b_i \to v_{n-i+1}, \ i = 1, 2, \ldots, n,$$

and let

$$e'_{\phi+k} = (\sigma(b_{i_k}), \sigma(b_{j_k})) \text{ for } k = 1, 2, \ldots, \phi$$

((b_{i_k}, b_{j_k}) is k-th move of the winning strategy S). Then each $e'_{\phi+k}$ is an arc in $\overline{T_n}$. For $k = 1, 2, \ldots, \phi$ put

$$D_{\phi+k} = D_{\phi+k-1} - e'_{\phi+k}. \tag{11}$$

Theorem 6. *The sequence* $D_0, D_1, \ldots, D_\phi, D_{\phi+1}, D_{\phi+2}, \ldots, D_{2\phi}$ *of digraphs described in* (10) *and* (11) *is a solution of the Problem for digraphs.* ∎

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