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FACTORING DIRECTED GRAPHS WITH RESPECT TO THE CARDINAL PRODUCT IN POLYNOMIAL TIME

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Abstract

By a result of McKenzie [4] finite directed graphs that satisfy certain connectivity and thinness conditions have the unique prime factorization property with respect to the cardinal product. We show that this property still holds under weaker connectivity and stronger thinness conditions. Furthermore, for such graphs the factorization can be determined in polynomial time.

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1. INTRODUCTION

Factorizations of graphs with respect to the cardinal product were first studied in the context of relational structures by McKenzie [4]. For finite directed

and undirected graphs McKenzie's results imply unique prime factorization under certain connectivity and thinness conditions.

His results do not lead to factorization algorithms. For the strong product, which is a special case of the cardinal product, this task was first solved by Feigenbaum and Schäffer. In [1] they presented a polynomial algorithm for the prime factorization of connected graphs with respect to the strong product. Their procedure consists of three parts: First the problem of factorizing a graph G is reduced to the factorization of a thin graph G/R. This follows the ideas of McKenzie [4]. Then G/R is factored. This is the main and most difficult part. It is effected by construction of the so-called Cartesian skeleton H and the prime factor decomposition of H with respect to the Cartesian product. Finally the factorization of G/R is extended to the original graph G.

A variant of this algorithm was proposed by Imrich [2] for the prime factorization of undirected nonbipartite connected graphs with respect to the cardinal product.

In the case of directed graphs the second part of the decomposition procedure, the factorization of thin graphs, can easily be adapted to the factorization of $R^+|R^-$-connected R^+-thin graphs. This is the topic of the present paper.

As in the case of the strong and the cardinal product of undirected graphs, the proof of the correctness of the algorithm also shows that the prime factorization is unique. This is important, because the class of $R^+|R^-$-connected R^+-thin graphs is not identical with the class of $R^+|R^-$- and $R^-|R^+$-connected thin graphs, for which McKenzie showed unique prime factorization. (McKenzie's connectivity condition is stronger, but his thinness condition weaker than ours.)

Thus, the present results also slightly extend the class of undirected graphs that are known to have unique prime factorizations with respect to the cardinal product. To our knowledge this is the only such extension since 1971.

The reduction to R-thin graphs, as introduced by McKenzie [4], has different properties than the reduction to R^+-thin graphs, which is the case that is of relevance for us. A fortiori this also holds for the third part of the factorization procedure, that is, the extension of the factorization to the original graph. In this case the factorization may also become non-unique. A comprehensive treatment of these parts is planned in a forthcoming paper.

2. The Cardinal Product

In this section we define the cardinal product of directed graphs, and basic connectivity conditions.

By a *directed graph* $G = (V, A)$ we mean set V together with a set A of ordered pairs $\langle x, y \rangle$ of vertices of G. We allow that both $\langle x, y \rangle$ and $\langle y, x \rangle$ are in A and do not require x, y to be distinct. Thus, A is a subset of the Cartesian product $V \times V$.

V is the *vertex set* of G and A the *set of arcs* of G. The vertex x is the *origin* and y the *terminus* of $\langle x, y \rangle$. In the case when $x = y$ we speak of a *loop*. In analogy to the undirected case we call a graph G with $A(G) = V(G) \times V(G)$ complete. If it has n vertices it will be denoted by K_n^d to distinguish it from the ordinary complete graph K_n (where any two distinct vertices are connected by an undirected edge.)

We say $A(G)$ is *reflexive* if A contains all loops $\langle x, x \rangle$, where $x \in V(G)$. It is *symmetric* if $\langle x, y \rangle \in A(G)$ if and only if $\langle y, x \rangle \in A(G)$. By abuse of language one also says that G is reflexive, respectively symmetric. Symmetric directed graphs correspond to undirected graphs by identification of pairs of edges with opposite directions.

The *out-neighborhood* $N^+(x)$ of a vertex x, compare Figure 1, is defined as the set

$$\{ y \in V \mid \langle x, y \rangle \in A \}.$$

Analogously one defines the *in-neighborhood* $N^-(x)$. Clearly a directed graph is uniquely defined by its vertex set and the out-neighborhoods of the vertices.
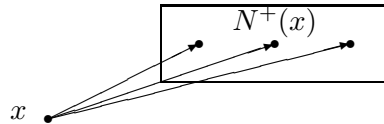

Figure 1. $N^+(x)$

The *cardinal product* $G_1 \times G_2$ of two directed graphs G_1, G_2 is defined on the Cartesian product $V(G_1) \times V(G_2)$ of the vertex sets of the factors, and the out-neighborhood of a vertex $x = (x_1, x_2)$ is the Cartesian product of the out-neighborhoods of x_1 in G_1 and x_2 in G_2:

$$N^+_{G_1 \times G_2}(x_1, x_2) = N^+_{G_1}(x_1) \times N^+_{G_2}(x_2).$$

The cardinal product is commutative, associative, and the single vertex with a loop, that is K_1^d, is a unit. The cardinal product of reflexive symmetric graphs corresponds to the strong product of undirected graphs.

A graph G is prime with respect to the cardinal product if $G = G_1 \times G_2$ implies that G_1 or G_2 are equal to K_1^d.

A graph G is $R^+|R^-$-connected if for all x, $y \in V(G)$ an $n \in \mathbb{N}$ and a sequence $(x_i)_{0 \le i \le n}$ can be found such that $x_0 = x$, $x_n = y$ and

$$N^+(x_i) \cap N^+(x_{i+1}) \neq \emptyset \quad \text{for} \quad 0 \le i < n.$$

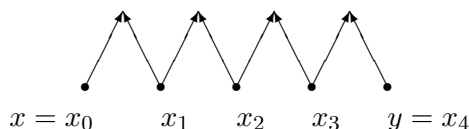$R^-|R^+$-*connectedness* is defined analogously.

Figure 2. An $x - y$ sequence in an $R^+|R^-$-connected graph.

Lemma 2.1. *Let $G = G_1 \times G_2 \times \cdots \times G_k$. If G is $R^+|R^-$-connected or $R^-|R^+$-connected, then this property is inherited by all factors.*

Proof. Clear. ■

The converse also holds, but as we do not need the result in the sequel, we leave the proof to the reader.

3. Prime Factorizations

Clearly every finite graph must have at least one prime factor decomposition with respect to the cardinal product. It need not be unique, for examples cf. [3]. However, it is unique if certain connectivity conditions are met.

Theorem 3.1 (McKenzie [4])**.** *Let G be an $R^+|R^-$- and $R^-|R^+$-connected finite graph. Then G has a unique representation as a cardinal product of prime graphs, up to isomorphisms and the order of the factors.*

Feigenbaum and Schäffer [1] showed that this factorization of a graph G can be found in polynomial time if $A(G)$ is reflexive and symmetric. Imrich [2] extended this result to graphs that are not reflexive. Of course the connectivity conditions still have to be met. We formulate this as a theorem.

Theorem 3.2 (Feigenbaum and Schäffer [1], Imrich [2]). *Let $G = (V, A)$ be an $R^+|R^-$- and $R^-|R^+$-connected finite graph, where A is symmetric, that is, where $\langle x, y \rangle \in A$ if and only if $\langle y, x \rangle \in A$. Then the prime factor decomposition of G with respect to the cardinal product can be found in polynomial time.*

In the proof of the above theorems the graphs under consideration where reduced to so-called thin graphs first and then factored. Following McKenzie we say two vertices of G are in the relation R ($\approx$ in his terminology) if both their out-neighborhoods and their in-neighborhoods are the same. A graph G is then called *thin* if no two vertices of G are in the relation R.

R is an equivalence relation on the set of vertices of G. As usual we define the *quotient graph G/R* as follows: the vertex set of G/R is the set of all equivalence classes $\{\overline{x} \mid x \in V(G)\}$ of $V(G)$ with respect to R, and $\langle \overline{x}, \overline{y} \rangle \in A(G/R)$ if there are vertices $a \in \overline{x}$, $b \in \overline{y}$ with $\langle a, b \rangle \in A(G)$. The following lemma holds:

Lemma 3.3 (McKenzie [4]). *Let G be a directed graph. Then*

(i) *G/R is thin.*
(ii) *If $G = G_1 \times G_2$ is $R^+|R^-$- and $R^-|R^+$-connected, then $G/R = G_1/R \times G_2/R$.*

Although McKenzie's result does not require the graphs to be symmetric, the algorithms in [1, 2] rely on it. In our case $N^+(x)$ can be different from $N^-(x)$, and we could not directly adapt the algorithm to this case. However, an analysis of the factorization algorithms in [1, 2] and the key lemmas on which they are based shows that they remain valid if no two vertices of G have the same out-neighborhood (or if no two vertices have the same in-neighborhood). This motivates the following definitions:

Two vertices of G are in the relation R^+ if their N^+-neighborhoods are the same. Clearly R^+ is an equivalence relation. R^- is defined analogously. A graph is then called R^+-*thin*, respectively R^--*thin*, if all equivalence classes of the relation R^+, respectively R^-, consist of just one element.

The graph in Figure 3 illustrates some of these properties. It is neither thin, nor R^+- or R^--thin, but G/R is a single arrow and both R^+- and R^--thin.
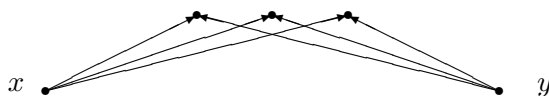
Figure 3. A graph that is not R^+-thin.

Theorem 3.4. *Let $G = (V, A)$ be an $R^+|R^-$- and $R^-|R^+$-connected finite graph. If G/R is R^+-thin (or R^--thin), then the prime factor decomposition of G with respect to the cardinal product can be found in polynomial time.*

Outline of the proof. This theorem generalizes Lemma 8 of [2]. The connectivity conditions are the same, symmetry is not required, but a stronger thinness condition.

An inspection of the proof in [2] reveals, that the factorization of G/R goes through if all neighborhoods are replaced by out-neighborhoods and if no two different vertices have the same out-neighborhoods. (Or if all neighborhoods are replaced by in-neighborhoods and if no two different vertices have the same in-neighborhoods.)

The proof in [2] continues with the definition of a graph H on the vertex set of G/R, which has the property that every decomposition $G_1 \times G_2$ of G with respect to the direct product induces a decomposition $H_1 \square H_2$ of H, so that $V(H_i) = V(G_i)$ $(i \in \{1, 2\})$. A graph with this property is called Cartesian skeleton of G. The main part of the proof consists in showing that it can be computed in polynomial time. It can be done with Algorithm 1 of [2], which works for out-neighborhoods (our case) just as for neighborhoods (the case treated in [2]). For its correctness in the case of directed graphs one has to prove analoga of Lemma 2 and 3 of [2]. The details are not difficult but time consuming and are presented in all technicalities in the dissertation of the second author.

Using the Cartesian skeleton one finds the vertex sets of all possible divisors of G/R and subsequently its PFD as in Lemma 8 of [2]. The factorization of G is then obtained as in [2]. ∎

For a comprehensive presentation of the algorithm in [2] we also refer to [3, p. 167–177].

The class of graphs for which we can find the prime factor decomposition with respect to the cardinal product in polynomial time by application of this theorem is still smaller than the class of graphs that have the unique

prime factorization property by McKenzie. In the next section we show that
the algorithm also can be used to show uniqueness of the decomposition in
some cases not covered by Theorem 3.1 and to find it in polynomial time.
These considerations also lead to graphs with tractable non-unique prime
factorizations.

4.   FACTORING $R^+|R^-$-CONNECTED R^+-THIN GRAPHS

Lemma 3.3 and the importance of $R^+|R^-$-connectedness and R^+-thinness
in Theorem 3.4 suggest to repeat the previous investigations with R^+, re-
spectively R^-, in the role of R. In analogy to Lemma 3.3 we obtain:

Lemma 4.1. *Let G be the cardinal product of two nontrivial directed graphs
G_1 and G_2. If all N^+-neighborhoods of the vertices of G are nonempty, then*

$$G/R^+ = G_1/R^+ \times G_2/R^+.$$

Proof. Two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in the relation R^+
if and only if $N^+(x) = N^+(y)$. This is equivalent to $N^+(x_1) \times N^+(x_2) =
N^+(y_1) \times N^+(y_2)$. Since $N^+(x)$ and $N^+(y)$ are both nonempty this is pos-
sible if and only if $N^+(x_1) = N^+(y_1)$ and $N^+(x_2) = N^+(y_2)$, that is, if
$x_1 R^+ y_1$ and $x_2 R^+ y_2$.   ∎

Note that $R^+|R^-$-connectivity implies that the N^+-neighborhoods are
nonempty.

Corollary 4.2. *Let G be the cardinal product of two nontrivial directed
graphs G_1 and G_2. If all N^+-neighborhoods of the vertices of G are non-
empty, then the following statements are equivalent:*

 (i) *G is R^+-thin.*
 (ii) *G_1 and G_2 are R^+-thin.*

Clearly Lemma 4.1 and Corollary 4.2 remain valid if N^+ is replaced by N^-,
and R^+ by R^-.

The problem with the analogy is that G/R is always thin, but G/R^+
need not be R^+-thin, see Figure 4. It may thus be necessary to quotient by
R^+ several times until one reaches an R^+-thin graph. For $R^+|R^-$-connected
R^+-thin graphs (or $R^-|R^+$-connected R^--thin graphs) we then have the
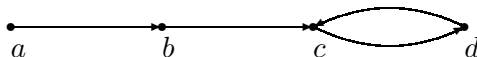following theorem.

Figure 4. Example of a graph G where G/R^+ is not R^+-thin.

Theorem 4.3. *Let $G = (V, A)$ be an $R^+|R^-$-connected R^+-thin graph, or an $R^-|R^+$-connected R^--thin graph. Then the prime factor decomposition of G with respect to the cardinal product is unique and can be found in polynomial time.*

Again the proof is completely analogous to the one in [2]. However, the conditions are quite different from those of McKenzie: Our thinness condition is stronger, since R^+- (or R^-)-thin graphs are thin, but thin graphs need not be R^+- (or R^-)-thin. On the other hand, our connectivity condition is weaker, because we only need either $R^+|R^-$- or $R^-|R^+$-connectedness, and not both.

This somewhat enlarges the class of known graphs with unique prime factorization with respect to the cardinal product.

To complete the analogy between R^+ and R, we then have to find factorizations of $R^+|R^-$-connected graphs G from factorizations of G/R^+; similarly for $R^-|R^+$-connected graphs.

This problem is quite different from the extension of a factorization of G/R to G. To see this, consider the subgraphs induced by the vertices in an equivalence class with respect to R compared to those that are induced by the vertices in an equivalence class of R^+. In the first case the graphs will have no arcs at all or the graph is a K_n^d. In the second case the graphs have no arcs at all again, or they consist of a K_s^d together with vertices whose out-neighborhoods are the vertices of K_s^d (and whose in-neighborhoods are empty). We call such graphs $R_{s,r}^+$, where r is their total number of vertices.

In the first case the prime factors of K_n^d are all $K_{n'}^d$, where n' is a prime divisor of n, and the prime factorization is unique. In the second case the prime factors are of the form $R_{s',r'}^+$, with $s' \leq r'$ and $s'|s$, $r'|r$. The prime factorization need not be unique either, for example

$$R_{2,2}^+ \times R_{1,3}^+ \text{ and } R_{2,3}^+ \times R_{1,2}^+$$

are two different prime factorizations of $R_{2,6}^+$.

We plan to treat the prime factorizations of $R^+|R^-$- and $R^-|R^+$-connected graphs in more detail in a forthcoming publication.

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