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Figure 1


Figure 3. $F^{1}$ (see [3, Figure 4])


Figure 2. (see [3, Figure 2])

Figure 1 presents a renewed, in fact improved, logo '3in1' GRAPHS. Both of the Figures 1 and 2 present an oriented graph, $D_{9}$, on 9 vertices provided that all crossings therein stand for vertices. In fact, labelling o $\leftarrow 0$ and oo $\leftarrow 7$ in '3in1' above is easily extendable to an isomorphism of the two digraphs. Refer therefore to labels of points in Figure 2. Only the points 0,7 , and 8 can be the crossing points there. Assume that the two figures present a digraph $D_{i}$ if exactly one of the three crossings is either a vertex and then $i=7$ or a crossing point and then $i=8, i$ being the order of $D_{i}$. Important new observation is that the group $\operatorname{Aut}\left(D_{9}\right)=\mathcal{C}_{6}$, the cyclic group generated by the permutation, say $\varphi=\left(\begin{array}{lll}0 & 7 & 8\end{array}\right)\left(\begin{array}{ll}1 & 3 \\ 5 & 2\end{array} 4\right.$ ). Then $\varphi^{3}=\left(\begin{array}{ll}1 & 2\end{array}\right)(34)(56)$ which when restricted to the vertex set $V\left(D_{i}\right)$ is in $\operatorname{Aut}\left(D_{i}\right), i=7,8,9$. In fact, $\operatorname{Aut}\left(D_{i}\right)=\mathcal{C}_{2}$ if $i=7,8$. Therefore digraphs $D_{7}$ and $D_{8}$ are independent of which points among $0,7,8$ are chosen to be vertices. That is why in only one picture the above logo presents three digraphs, one each of order 7,8 , or 9 depending on whether among points 0,7 , and 8 exactly any two points are crossing points, or exactly any point, or no point is a crossing point.

Thus there are $3^{2}$ ways to view Figure 2 as a '3in1'. The three digraphs are

- non-Hamiltonian,
- homogeneously bi-traceable (i.e., traceable to and from any vertex),
- 2-diregular (with id $=\mathrm{od}=2$ at each vertex) whence arc-minimal,
- oriented graphs. Moreover, the three numbers 7, 8, and 9 are the three smallest possible orders among such digraphs.

Each of the following digraphs $D^{i, \alpha}$ is another such an oriented graph, of order $i+2 \alpha+1$, for any positive integer $\alpha$. If $i=7,8,9$, then $D^{i, \alpha}$ can be seen to be one of infinitely many triples '3in1', see [3].

Let $D^{i, 0}$ stand for $D_{i}$ in case points $0,1, \ldots, i-1$ in Figure 2 make up the vertex set of $D_{i}, i=7,8,9$. For positive integer $\alpha$, let
$D^{i, \alpha}=D^{i, 0}-\{0\} \cup F^{\alpha}$ where $F^{\alpha}\left(\operatorname{cf} F^{1}\right.$ in Figure 3) is the arc-disjoint union of a 3-1 path and a 4-2 path, namely
$3,2^{\alpha}, 2^{\alpha-1}, \ldots, 2^{0}, 1^{\alpha}, 1^{\alpha-1}, \ldots, 1^{0}, 1, \quad 4,1^{\alpha}, 2^{\alpha}, 1^{\alpha-1}, 2^{\alpha-1}, \ldots, 1^{0}, 2^{0}, 2$, with all $2 \alpha+2$ inner vertices
$1^{0}, 2^{0}, 1^{1}, 2^{1}, \ldots, 1^{\alpha}, 2^{\alpha}$ being different from vertices of $D^{i, 0}$.
A digraph is called bi-detour homogeneous if at any vertex a detour starts and another detour terminates, detour being the name of the longest path. Assume that deleting two arcs 1-4 and 1-6 from $D^{i, \alpha}$ gives both the
$\operatorname{digraph} T^{i, \alpha}=D^{i, \alpha}-\{(1,4),(1,6)\}$ and the digraphical triple $\left(T^{i, \alpha},\{4,6\}, 1\right)$ with distinguished vertices: $1($ with od $=0)$ and 4,6 (both with id $=1$ ) but only in case $\alpha \geq 0$ and $i=7,8$ only (that is, $i \neq 9$ ). It has been proved recently in [1, Thm 4.3] that, for any positive integer $q$, compositions of any $q+1$ triples $T^{i, \alpha}$ give rise to bi-detour homogeneous oriented graphs of any order $n \geq 7 q+7$, of the smallest possible size $2 n$, and with detour of order $n-q(<n)$. The resulting order $n$ is the sum of orders of the involved triples $T^{i, \alpha}, i=7,8$.

Thus '3in1' has evolved into 'many-in-one', hasn't it?
Remarks. See [4] for some more information on '3in1'. Both Figures 1 and 2 above provide corrections of the misprinted Figures in [4]. The planar digraph $D_{9}$ was independently found by Hahn and Zamfirescu [2, Figure 5].

## References

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