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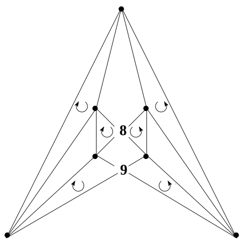

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'3in1' ENHANCED: THREE SQUARED WAYS TO '3in1' GRAPHS

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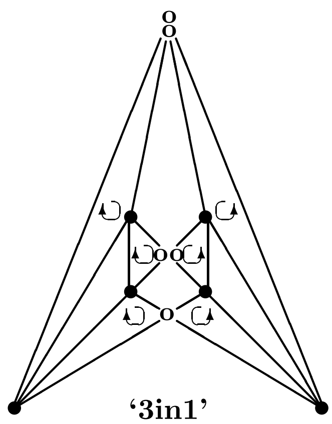

Figure 1

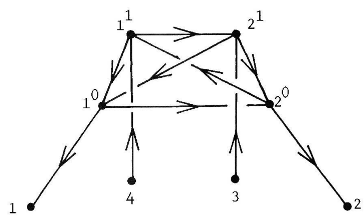

Figure 3. F^1 (see [3, Figure 4])

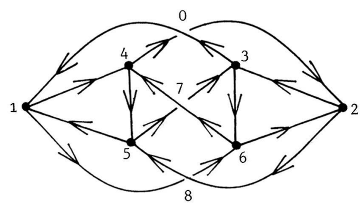

Figure 2. (see [3, Figure 2])

Figure 1 presents a renewed, in fact improved, logo '3in1' GRAPHS. Both of the Figures 1 and 2 present an oriented graph, D_9, on 9 vertices provided that all crossings therein stand for vertices. In fact, labelling o $\leftarrow$ 0 and oo $\leftarrow$ 7 in '3in1' above is easily extendable to an isomorphism of the two digraphs. Refer therefore to labels of points in Figure 2. Only the points 0, 7, and 8 can be the crossing points there. Assume that the two figures present a digraph D_i if exactly one of the three crossings is either a vertex and then $i = 7$ or a crossing point and then $i = 8$, i being the order of D_i. Important new observation is that the group $\mathrm{Aut}(D_9) = \mathcal{C}_6$, the cyclic group generated by the permutation, say $\varphi = (0\ 7\ 8)(1\ 3\ 5\ 2\ 4\ 6)$. Then $\varphi^3 = (1\ 2)(3\ 4)(5\ 6)$ which when restricted to the vertex set $V(D_i)$ is in $\mathrm{Aut}(D_i)$, $i = 7, 8, 9$. In fact, $\mathrm{Aut}(D_i) = \mathcal{C}_2$ if $i = 7, 8$. Therefore digraphs D_7 and D_8 are independent of which points among 0, 7, 8 are chosen to be vertices. That is why in only one picture the above logo presents three digraphs, one each of order 7, 8, or 9 depending on whether among points 0, 7, and 8 exactly any two points are crossing points, or exactly any point, or no point is a crossing point.

Thus there are 3^2 ways to view Figure 2 as a '3in1'. The three digraphs are

- non-Hamiltonian,
- homogeneously bi-traceable (i.e., traceable to and from any vertex),
- 2-diregular (with id = od = 2 at each vertex) whence arc-minimal,
- oriented graphs. Moreover, the three numbers 7, 8, and 9 are the three smallest possible orders among such digraphs.

Each of the following digraphs $D^{i,\alpha}$ is another such an oriented graph, of order $i + 2\alpha + 1$, for any positive integer α. If $i = 7, 8, 9$, then $D^{i,\alpha}$ can be seen to be one of infinitely many triples '3in1', see [3].

Let $D^{i,0}$ stand for D_i in case points $0, 1, \ldots, i-1$ in Figure 2 make up the vertex set of D_i, $i = 7, 8, 9$. For positive integer α, let

$D^{i,\alpha} = D^{i,0} - \{0\} \cup F^\alpha$ where F^α (cf F^1 in Figure 3) is the arc-disjoint union of a 3–1 path and a 4–2 path, namely

$3, 2^\alpha, 2^{\alpha-1}, \ldots, 2^0, 1^\alpha, 1^{\alpha-1}, \ldots, 1^0, 1, \quad 4, 1^\alpha, 2^\alpha, 1^{\alpha-1}, 2^{\alpha-1}, \ldots, 1^0, 2^0, 2,$ with all $2\alpha + 2$ inner vertices

$1^0, 2^0, 1^1, 2^1, \ldots, 1^\alpha, 2^\alpha$ being different from vertices of $D^{i,0}$.

A digraph is called *bi-detour homogeneous* if at any vertex a detour starts and another detour terminates, *detour* being the name of the longest path. Assume that deleting two arcs 1–4 and 1–6 from $D^{i,\alpha}$ gives both the digraph $T^{i,\alpha} = D^{i,\alpha} - \{(1,4),(1,6)\}$ and the digraphical triple $(T^{i,\alpha}, \{4,6\}, 1)$ with distinguished vertices: 1 (with od = 0) and 4, 6 (both with id = 1) but only in case $\alpha \geq 0$ and $i = 7, 8$ only (that is, $i \neq 9$). It has been proved recently in [1, Thm 4.3] that, for any positive integer q, compositions of any $q+1$ triples $T^{i,\alpha}$ give rise to bi-detour homogeneous oriented graphs of any order $n \geq 7q+7$, of the smallest possible size $2n$, and with detour of order $n-q$ ($< n$). The resulting order n is the sum of orders of the involved triples $T^{i,\alpha}$, $i = 7, 8$.

Thus '3in1' has evolved into 'many-in-one', hasn't it?

Remarks. See [4] for some more information on '3in1'. Both Figures 1 and 2 above provide corrections of the misprinted Figures in [4]. The planar digraph D_9 was independently found by Hahn and Zamfirescu [2, Figure 5].

References

[1] S.A. van Aardt, F.E.S. Bullock, J. Górska and Z. Skupień, *On detour homogeneous digraphs*, Discrete Math., to appear.

[2] S. Hahn and T. Zamfirescu, *Bihomogeneously traceable oriented graphs*, Rend. Sem. Mat. Univ. Politecn. Torino **39** (1981) 137–145.

[3] Z. Skupień, *On homogeneously traceable nonhamiltonian digraphs and oriented graphs*, in: G. Chartrand, Y. Alavi, D.L. Goldsmith, L. Lesniak-Foster and D.R. Lick, eds., The Theory and Applications of Graphs (T.A.G.) (Proc. Fourth Int. Conf. on T.A.G., held in Kalamazoo, MI., 1980) (Wiley, 1981) 517–527.

[4] Z. Skupień, *Logo* '3in1', in: Z. Skupień and R. Kalinowski, guest eds., Discuss. Math. Graph Theory **19** (1999) 116.

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