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AN APPROXIMATION ALGORITHM FOR THE TOTAL COVERING PROBLEM

Pooya Hatami

Department of Mathematical Sciences
Sharif University of Technology
Tehran, Iran

e-mail: p_hatami@ce.sharif.edu

Abstract

We introduce a 2-factor approximation algorithm for the minimum total covering number problem.

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1. Introduction

A *vertex cover* of an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that if $e = uv \in E$, then $\{u, v\} \cap S \neq \emptyset$. A set $D \subseteq V \cup E$ is called a *total cover* if every element of $(V \cup E) \setminus D$ is adjacent or incident to an element in D.

The notion of total covering is first defined in [1], and then studied in many papers [2, 5, 12, 13]. Many variations of the covering problems including vertex covers, total covers, dominating sets, *et cetera* have been studied previously (see [6]).

The minimum total cover problem was first shown to be NP-hard in general graphs by Majumdar [10], where he also gives a linear-time algorithm for trees. Hedetniemi *et al.* [7] showed that the problem is NP-hard for bipartite and chordal graphs. Manlove [11] demonstrates NP-hardness for planar bipartite graphs of maximum degree 4.

Trivially for every graph a vertex cover together with all isolated vertices constitute a total cover. It is well-known that a maximal matching can be used to find a vertex cover of size at most twice the minimum vertex cover: If M is a maximal matching of the graph G, the set S of all $2|M|$ vertices involved in M constitute a vertex cover of G. Moreover a vertex cover of G has at least $|M|$ elements, because every vertex is involved in at most one matching edge. Thus taking the vertices which are involved in a maximal matching gives a 2-approximation algorithm for the minimum vertex cover problem.

It is widely believed that it is NP-hard to approximate the vertex cover problem to within any factor smaller than 2, and recently Khot and Regev [9] proved that the *Unique Games Conjecture* would imply this. So far, the best known lower bound is a recent result of Dinur and Safra [3] which shows that it is NP-hard to approximate this problem to within any factor smaller than $10\sqrt{5} - 21 \approx 1.36067$.

The approximability of the problem of finding a minimum total cover does not seem to have received explicit attention in the literature previously. However given a graph $G = (V, E)$, the relationship $\alpha_2(G) = \gamma(T(G))$ holds, where $\alpha_2(G)$ denotes the minimum size of a total cover in G, $\gamma(G)$ denotes the minimum size of a dominating set in G, and $T(G)$ denotes the total graph of G (this is the graph with vertex set $V \cup E$, and two vertices are adjacent in $T(G)$ if and only if the corresponding elements are adjacent or incident as vertices or edges of G). It follows from the correspondence that the minimum total cover problem is approximable within a factor of $1 + \log n$, where $n = |V|$ [8]. Also, if $\Delta(G) \leq k$, then $\Delta(T(G)) \leq 2k + 1$. It follows that, in a graph of maximum degree k, the problem of finding a minimum total cover is approximable within a factor of $H_{2(k+1)} - \frac{1}{2}$ [4], where $H_i = \sum_{j=1}^{i} \frac{1}{j}$ is the ith Harmonic number.

We introduce a simple and elementary algorithm which finds a total cover of size at most twice the size of an optimal total covering. Note that, for $k \geq 3$, $H_{2(k+1)} - \frac{1}{2} \geq 2$, implying that the above derived results would be improved upon this 2-approximation algorithm.

2. The Approximation Algorithm

In this section we introduce an approximation algorithm for computing the minimum total cover number of a graph.

After that straightforward 2-approximation algorithm for the minimum vertex cover problem, it is tempting to try the same algorithm for the total cover problem. It is easy to see that if we modify this algorithm to include all isolated vertices too, we obtain an approximation algorithm for the total cover problem. The following example shows that the algorithm is not a $(4-\epsilon)$-approximation: Consider the graph illustrated in Figure 1 for even n. The maximum matching of this graph is of size n while the set S which consists of v and all edges of the form $e = u_i u_{i+1}$ is a total cover of size $\frac{n}{2}+1$ of the graph. Lemma 1 will immediately conclude that the mentioned algorithm has factor 4.

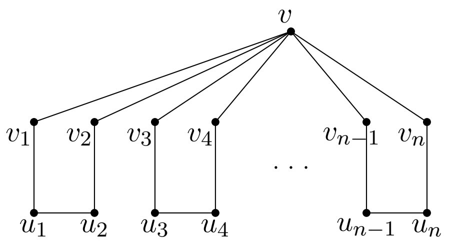

Figure 1. A hard example for maximal matching algorithm.

Next we introduce a 2-approximation algorithm for this problem. Consider a graph $G = (V, E)$ with t isolated vertices. Let M be a maximum matching in G of size m. Let k be the number of vertices that **(i)** are not involved in M, and **(ii)** are adjacent to both endpoints of an edge in M; we call these *bad* vertices. Note that since M is of maximum size, if a bad vertex w is adjacent to both endpoints of $e = uv$, then neither u nor v is adjacent to any other vertex outside M. We will find a total cover S of size $m + k + t$ in G through the following algorithm:

1. Obviously every isolated vertex must be in S. Remove all these vertices from G.
2. Select a matching edge $e = v_1v_2$ where both v_1 and v_2 are adjacent to a bad vertex v. Add v and e to S and remove v, v_1, v_2 from G. Repeat this until all bad vertices are removed.
3. After the first two steps the size of S is $2k + t$. Now we have a graph G_1 with a maximum matching M_1 of size $m - k$ without any bad vertices. Next we apply the following step:
   - Pick the edges $e = uv \in M_1$ in an arbitrary order, and note that at most one of u and v is adjacent to some vertices in $G_1 \setminus M_1$:
     - If one of u and v is adjacent to some vertices $z \in G_1 \setminus M_1$ which are not covered by S, then add that vertex to S.
     - Otherwise add e to S.

It is clear that S is of size $m+k+t$. We now show that S covers every edge between vertices covered by M_1 (It is clear that S covers all other elements of G). Suppose that $e_1 = u_1v_1$ and $e_2 = u_2v_2$ are matching edges in M_1 and the edge $e = u_1u_2$ is not covered. Then none of e_1, e_2, u_1, u_2 are in S, and so both v_1, v_2 are in S. Suppose that among e_1 and e_2, the edge e_1 was picked first. So there is a vertex $w_1 \in G_1 \setminus M_1$ adjacent to v_1 and there is a vertex $w_2 \in G_1 \setminus M_1$ which is adjacent to v_2 but not to v_1. The path w_1,v_1,u_1,u_2,v_2,w_2 is an augmenting path for M_1 and this contradicts the fact that M is a maximum matching of G.

Lemma 1. *The minimum total cover of G has at least $\frac{m+k}{2} + t$ elements.*

Proof. Call every triangle consisting of a bad vertex v and a matching edge whose both endpoints are adjacent to v a *bad* triangle. There are k bad vertices, and no two bad vertices can share a common matching edge, thus there exist at least k bad triangles.

Suppose that S is a total cover in G. Let $A \subseteq S$ be a *maximal* set of *edges* which covers $2|A|$ edges of M, consisting of the edges each covering precisely two edges of M. Let $B = S \setminus A$. Every bad triangle has at least one edge which is not covered by A. Since no edge is incident to two bad triangles with distinct vertices, no element (that is, neither a vertex nor an edge) can cover two edges from two disjoint bad triangles. Since the number of the bad triangles is at least k, we have $|B| \geq k+t$, as B has to cover the isolated vertices too.

Since B covers at most $|B|-t$ edges of M, $S = A \cup B$ covers at most $2|A| + |B| - t$ edges of M. Thus $2|A| + |B| - t \geq m$. From this inequality and $|B| \geq k+t$ we get $2(|A| + |B|) \geq m + k + 2t$ which implies that $|S| = |A| + |B| \geq \frac{m+k}{2} + t$. ∎

Theorem 1. *The minimum total cover problem admits a 2-approximation algorithm.*

Proof. Immediately from Lemma 1. ∎

Consider the graph illustrated in Figure 1. Our algorithm finds a total cover of size $n+1$. The graph in Figure 1 has a total cover of size $\frac{n}{2}+1$. The result of our algorithm is $2-o(1)$ times the size of the minimum total cover of the graph. So our algorithm is not a $(2-\epsilon)$-approximation, for any ϵ.

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