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A PROOF OF THE CROSSING NUMBER OF $K_{3,n}$ IN A SURFACE

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Abstract

In this note we give a simple proof of a result of Richter and Siran by basic counting method, which says that the crossing number of $K_{3,n}$ in a surface with Euler genus ε is

$$\left\lfloor \frac{n}{2\varepsilon+2} \right\rfloor \left\{ n - (\varepsilon+1)\Big(1 + \left\lfloor \frac{n}{2\varepsilon+2} \right\rfloor\Big) \right\}.$$

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1. Introduction

In [1], Guy and Jenkyns showed that the crossing number of $K_{3,n}$ in the torus is $\lfloor (n-3)^2/12 \rfloor$. In [2], Richter and Siran generalized their result and showed the following:

Theorem 1.1. *If the surface Σ has Euler genus ε, then the crossing number of $K_{3,n}$ in Σ is given by*

$$cr_\Sigma(K_{3,n}) = \left\lfloor \frac{n}{2\varepsilon+2} \right\rfloor \left\{ n - \Big(\varepsilon+1)(1 + \left\lfloor \frac{n}{2\varepsilon+2} \right\rfloor\Big) \right\}. \tag{1}$$

(The *Euler genus* of a surface Σ is $2h$ if Σ is the sphere with h handles and k if Σ is the sphere with k crosscaps.) In this note, we give a simple proof of Theorem 1.1 by using basic counting method. In the following, we will denote the right hand side of (1) by $f(\varepsilon, n)$.

2. Proof of Theorem 1.1

To prove that $cr_\Sigma(K_{3,n}) \le f(\varepsilon, n)$, one can refer to [2] for the drawings. To complete the proof, it suffices to show that

$$cr_\Sigma(K_{3,n}) \ge f(\varepsilon, n). \tag{2}$$

We will prove (2) by induction. For $n \le 2\varepsilon + 2$, from [3] and [4], we know that $K_{3,n}$ can be embedded in Σ. Therefore, $cr_\Sigma(K_{3,n}) = 0 = f(\varepsilon, n)$, which shows that (2) is true for $n \le 2\varepsilon + 2$.

Therefore we may assume that $n > 2\varepsilon + 2$. Let $n = (2\varepsilon + 2)q + r$ where $0 \le r \le 2\varepsilon + 1$. Then

$$f(\varepsilon, n) = (\varepsilon + 1)(q^2 - q) + qr. \tag{3}$$

Note that, in a crossing-free drawing of a (connected) subgraph of $K_{3,n}$ in Σ, every face has even degree. Let t_j be the number of regions with j bounding arcs; and F, E, V be the number of faces, arcs, vertices, respectively. Then $t_j = 0$ if j is odd, $F = t_4 + t_6 + t_8 + ...$, and $2E = 4t_4 + 6t_6 + 8t_8 + ...$, and by the Euler's formula for Σ,

$$V \ge 2 - \varepsilon + E - F, \tag{4}$$

$$V \ge 2 - \varepsilon + t_4 + 2t_6 + 3t_8 + \ldots \ge 2 - \varepsilon + F. \tag{5}$$

Suppose we have an optimal drawing of $K_{3,n}$ in Σ, i.e., one with $cr_\Sigma(K_{3,n})$ crossings, and that by removing $cr_\Sigma(K_{3,n})$ edges, a crossing-free drawing is produced. Then (4) and (5) give $E - V = (3n - cr_\Sigma(K_{3,n})) - (3 + n) \le F + \varepsilon - 2 \le V + 2\varepsilon - 4 = 3 + n + 2\varepsilon - 4$, so

$$cr_\Sigma(K_{3,n}) \ge n - 2 - 2\varepsilon. \tag{6}$$

If $q = 1$, then $n = (2\varepsilon + 2) + r$. Then by (3) and (6), we have

$$cr_\Sigma(K_{3,(2\varepsilon+2)+r}) \ge r = f(\varepsilon, (2\varepsilon + 2) + r).$$

This implies that (2) holds for $q = 1$.

Therefore we may assume that $q \ge 2$. Since $K_{3,n}$ contains n different $K_{3,n-1}$ and each of $K_{3,n-1}$ contains at least $f(\varepsilon, n - 1)$ crossings by induction hypothesis. Note that a crossing in a drawing of $K_{3,n}$ appears in $n - 2$

different drawings of $K_{3,n-1}$. Hence

$$cr_\Sigma(K_{3,n}) \geq \frac{n}{n-2}\, cr_\Sigma(K_{3,n-1}) = \frac{n}{n-2}\, f(\varepsilon, n-1). \tag{7}$$

From (3) and (7), we have

$$cr_\Sigma(K_{3,n}) \geq \begin{cases} (\varepsilon+1)(q^2-q) + qr - 1 + \dfrac{qr+r-2}{n-2}, & \text{if } 1 \leq r \leq 2\varepsilon+1; \\ (\varepsilon+1)(q^2-q), & \text{if } r = 0. \end{cases} \tag{8}$$

Note that $q \geq 2$ and $1 \leq r \leq 2\varepsilon+1$ imply that $\frac{qr+r-2}{n-2} > 0$. Hence (3), (8) and the fact that the crossing number is an integer imply that (2) holds for $q \geq 2$. This completes the proof of Theorem 1.1.

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