Discussiones Mathematicae
Graph Theory 27 (2007) 541–547

FRACTIONAL DOMINATION IN PRISMS

Matthew Walsh

Department of Mathematical Sciences
Indiana-Purdue University
Fort Wayne, Indiana 46805, USA

e-mail: walshm@ipfw.edu

Abstract

Mynhardt has conjectured that if G is a graph such that $\gamma(G) = \gamma(\pi G)$ for all generalized prisms πG then G is edgeless. The fractional analogue of this conjecture is established and proved by showing that, if G is a graph with edges, then $\gamma_f(G \times K_2) > \gamma_f(G)$.

Keywords: fractional domination, graph products, prisms of graphs.

2000 Mathematics Subject Classification: 05C69.

Throughout let us assume that graphs are finite and simple; our notation concurs with [3]. Let $G = (V, E)$ be a graph; the (*closed*) *neighbourhood* $N[v]$ of a vertex $v \in V$ consists of v itself and all vertices $u \in V$ such that $u \sim v$. A set $S \subseteq V$ is *independent* if no two members of S are adjacent; S is *dominating* if $\cup_{v \in S} N[v] = V$. The size of a smallest dominating set in G is denoted by $\gamma(G)$ and termed the *domination number* of G.

By generalizing "set" to "fuzzy set" in the definition of domination, one can define the concept of fractional domination. A function $f : V \to [0, 1]$ is a *fractional dominating function* precisely when $\sum_{u \in N[v]} f(u) \geq 1$ for all $v \in V$. If one defines the size of a fractional dominating function f by $|f| = \sum_{v \in V} f(v)$ then one can talk about the minimum size of a fractional dominating function of G; this is the *fractional domination number* of G and denoted by $\gamma_f(G)$. Since the characteristic function of a dominating set in G is clearly a fractional dominating function of G, $\gamma_f(G) \leq \gamma(G)$.

(Notation will sometimes be abused in the following standard fashions: if S is a set of vertices, then $f(S) = \sum_{v \in S} f(v)$. Thus, $|f| = f(V)$. In the particular case where the set in question is the closed neighbourhood $N[v]$ of the vertex v, the notation is further condensed to $f[v] = f(N[v])$.)

An *equitable partition* $P_1, \ldots, P_k$ of the vertices of a graph G is a partition with the properties that every induced graph $G[P_i]$ is regular, and every induced bipartite graph between two cells P_i, P_j is biregular. The following result can be found in [5].

Theorem 1. *If G is a graph that admits an equitable partition $\{P_i\}_{i=1}^k$, then there exists a minimum fractional dominating function of G that is constant on each cell P_i, $i = 1, \ldots, k$.*

Suppose that G is a graph and π a permutation on its vertex set V. The *generalized prism* πG is the graph with vertex set $V_\pi = V \times \{0, 1\}$, with $(u, i) \sim (v, j)$ when either $i = j$ and $u \sim v$ in G, or else $i \neq j$ and $v = \pi(u)$. When $\pi = 1$, the identity permutation, then the graph $1G = G \times K_2$ is called the *prism* of G.

The following result from [1] is easily shown.

Lemma 2. *For any graph G and any permutation π of its vertex set, $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$.*

A graph G for which $\gamma(G) = \gamma(\pi G)$ for any permutation π is a *universal γ-fixer*; if $2\gamma(G) = \gamma(\pi G)$ for all π, then G is a *universal γ-doubler*. In [4] it is conjectured that the only universal γ-fixers are graphs without edges.

This paper is concerned with the fractional analogue of the conjecture mentioned above. To develop this, some elementary tools are needed.

As discussed in [2], for a function $f : V \to [0, 1]$ define the sets $B_f = \{v \in V : f[v] = 1\}$ and $P_f = \{v \in V : f(v) > 0\}$.

Lemma 3 [2]**.** *A dominating function f is a minimal dominating function if and only if B_f dominates P_f.*

If f is a fractional dominating function of the prism πG, then define the *condensation* $f_\pi : V(G) \to [0, 1]$ of f by

$$f_\pi(v) = \min\{1, f((v, 0)) + f((\pi(v), 1))\}$$

for all $v \in V(G)$.

Lemma 4. *If f is a fractional dominating function on πG, then its condensation f_π is a fractional dominating function on G with $|f_\pi| \leq |f|$.*

Proof. Let $v \in V(G)$ and consider $\sum_{u \in N_G[v]} f_\pi(u)$. If $f_\pi(v) = 1$ then clearly this sum exceeds 1; otherwise, for each $u \in N_G(v)$ we have that $f_\pi(u) \geq f((u,0))$, and $f_\pi(v) = f((v,0)) + f((\pi(v),1))$. Hence

$$\begin{aligned}
\sum_{u \in N_G[v]} f_\pi(u) &= f_\pi(v) + \sum_{u \in N_G(v)} f_\pi(u) \\
&\geq f((v,0)) + f((\pi(v),1)) + \sum_{u \in N_G(v)} f((u,0)) \\
&= \sum_{x \in N_{\pi G}[(v,0)]} f(x) \\
&\geq 1.
\end{aligned}$$

A similar calculation shows that $|f_\pi| \leq |f|$. ■

Corollary 5. *For any graph G and any permutation π of its vertex set, $\gamma_f(G) \leq \gamma_f(\pi G) \leq 2\gamma_f(G)$, and these bounds are sharp.*

Proof. The lower bound follows from Lemma 4. To show the upper bound, let f be a minimum fractional dominating function of G. Then the function $f' : V(\pi G) \to [0,1]$ defined by $f'((u,i)) = f(u)$ is fractional dominating with $|f'| = 2|f|$.

An example of the lower bound occurs when G contains no edges and π is an arbitrary permutation: $\gamma_f(G) = \gamma_f(\pi G) = |V(G)|$. For the upper bound, let $G = K_{1,n}$ for $n \geq 2$ and let π be any automorphism of G; then $\gamma_f(G) = 1$ and $\gamma_f(\pi G) = 2$. ■

The fractional version of Mynhardt's question is then: *For which graphs G is it true that, for any permutation π of $V(G)$, $\gamma_f(\pi G) = \gamma_f(G)$?* Such a graph would naturally be termed a universal γ_f-fixer. As it turns out, this question can be answered without considering any permutations other than the identity.

Lemma 6. *Let f be fractional dominating on $1G$ with condensation f_1 such that $|f_1| = |f|$. Then for any vertex $v \in V(G)$, $f_1[v] = f[(v,0)] + f[(v,1)] - f_1(v)$.*

Proof. Since $|f_1| = |f|$ is follows that $f_1(v) = f((v,0)) + f((v,1))$ for all vertices v. The result then follows from a simple computation using the fact that $f[(v,i)] = f(\{(u,i) : u \in N_G[v]\} + f((v,1-i))$ for $i = 0,1$. ■

Lemma 7. *Let $1G$ be the prism of a simple graph G with vertex set $V = \{v_1, \dots, v_n\}$. Then the collection of sets $\{(v_i, 0), (v_i, 1)\}_{i=1}^{n}$ forms an equitable partition of the vertices of $1G$.*

Proof. Let P_i denote the set containing the images of v_i in the prism. Each $1G[P_i]$ consists of a single edge (and is thus 1-regular); the bipartite graph between P_i and P_j will either be edgeless (if v_i and v_j are not adjacent) or 1-regular. ■

Theorem 8. *Let G be a graph such that $\gamma_f(1G) = \gamma_f(G)$. Then $G = \overline{K_n}$ for some positive integer n.*

Proof. Let G be a graph such that $\gamma_f(1G) = \gamma_f(G)$, and suppose that f is a minimum fractional dominating function of $1G$ with condensation f_1. Let us assume (by Theorem 1 and Lemma 7) that for any $v \in V(G)$, $f((v, 0)) = f((v, 1))$. By Lemma 4 f_1 is a fractional dominating function of G with $|f_1| \leq \gamma_f(1G) = \gamma_f(G)$, and hence f_1 is in fact a minimum fractional dominating function of G. Further, by this equality we know that $f((v, 0)) + f((v, 1)) \leq 1$, and hence that $f(x) \leq \frac{1}{2}$ for any vertex $x \in V(1G)$.

Suppose that v is a vertex in G such that $f_1(v) = 0$. Then by Lemma 6, $f_1(N[v]) = f(N[(v, 0)]) + f(N[(v, 1)])$, and since f is fractional dominating in $1G$ the two right-hand terms are each at least 1; hence, $f_1(N[v]) \geq 2$ for any vertex v receiving a weight of 0.

Let $v^* \in V(G)$ be such that $f_1[v^*] = 1$; such a vertex exists from Lemma 3. It follows that $f[(v^*, 0)] = \frac{1}{2}f_1[v^*] + \frac{1}{2}f_1(v^*) = \frac{1}{2} + \frac{1}{2}f_1(v^*) \geq 1$ since f is dominating; hence $f_1(v^*) \geq 1$ so $f_1(v^*) = 1$. Moreover, $f_1(u) = 0$ for all $u \sim v^*$.

By Lemma 3, if $f_1(w) > 0$ then there exists $v^* \in N[w]$ such that $f_1[v^*] = 1$; if $v^* \in N(w)$ then $f_1(w) = 0$, contradicting our premise, and hence $w = v^*$ and so $f_1(w) = 1$. Therefore, f_1 is the characteristic function of an independent 2-dominating set of G. (A 2-dominating set S is one where, for every vertex $u \notin S$, $|N(u) \cap S| \geq 2$. The 2-domination comes from the fact that f only takes the values 0 and $\frac{1}{2}$; any vertex in $1G$ which receives a weight of 0 must therefore be adjacent to two vertices in the support of f, and this carries over into the condensation.)

So let $d = d_G(v^*)$ for some vertex v^* such that $f_1(v^*) = 1$, and suppose that $d > 0$. Pick some $w \in V(G)$ that is distance 2 from v such that $f_1(w) > 0$; this exists by fact that the support of f_1 is 2-dominating.

Define the function $f^* : V(G) \to [0,1]$ as follows:

$$f^*(v) = \begin{cases} 0 & \text{if } v = v^*, \\ \frac{1}{d} & \text{if } v \sim v^*, \\ 1 - \frac{1}{d} & \text{if } v = w, \\ f_1(v) & \text{otherwise.} \end{cases}$$

f^* is a fractional dominating function of G: If v is a vertex such that $f_1(v) = 1$, then clearly $f^*(v) = 1$. Otherwise $f_1(v) = 0$ and hence $f_1[v] \geq 2$ as v has at least two neighbours with weight 1. If its only two such neighbours are v^* and w, then $f^*[v] = f^*(v) + f^*(w) = 1$; otherwise, it is clear that $f^*[v] \geq 1$.

But $|f^*| < |f_1|$, so the latter is not minimum, and hence $\gamma_f(G) < \gamma_f(1G)$. This fails only when there is no v^* with neighbouring vertices, and hence only when G contains no edges. ∎

Corollary 9. *The only universal γ_f-fixers are the edgeless graphs.*

One consequence of this result to the original conjecture is that if G is a γ-fixer with respect to the identity permutation and not empty then it must be the case that $\gamma_f(G) < \gamma(G)$, and hence this must be true of any universal γ-fixer.

Much of the power in the proof of Theorem 8 comes from the fact that the equitable partition in $1G$ guaranteed by Lemma 7 allows us to restrict our choice of fractional dominating functions significantly. This can be exploited for more general permutations π.

Theorem 10. *Let G be a graph that admits the equitable partition $P_1, \ldots, P_k$, and let π be a permutation of $V(G)$ that fixes each P_i setwise. Then $\gamma_f(G) = \gamma_f(\pi G)$ if and only if G is edgeless.*

Proof. The images $\{(v, j) : v \in P_i, j \in \{0, 1\}\}$ of the partition cells P_i form an equitable partition in πG, so we find a minimum fractional dominating function f of πG that is constant on each of these sets. Using this, we can show (analogously to Lemma 6) that if f_π is the condensation of f to G, then $f_\pi(N[v]) = f(N[(v,0)]) + f(N[(\pi v, 1)]) - f_\pi(v)$. The proof then echoes that of Theorem 8. ∎

Finally, here is a construction for γ_f-fixers with respect to restricted classes of permutations. Construct the corona $\mathrm{cor}(G)$ of a graph G by adjoining a pendant vertex to every node of G.

Theorem 11. *For any graph G, let $V = V(G)$ and $V^* = V(\mathrm{cor}(G)) - V$. Let π be any permutation of $V(\mathrm{cor}(G))$ such that $\pi(V) = V^*$. Then $\gamma_f(\mathrm{cor}(G)) = \gamma_f(\pi\,\mathrm{cor}(G))$.*

Proof. Since the closed neighbourhoods of pendant vertices in $\mathrm{cor}(G)$ are disjoint, $\gamma_f(\mathrm{cor}(G)) = |V|$. Define f on $V(\pi\,\mathrm{cor}(G))$ by

$$f((v,i)) = \begin{cases} \frac{1}{2} & \text{if } v \in V, \\ 0 & \text{if } v \in V^*. \end{cases}$$

Then f is fractional dominating, and $|f| = |V|$. ■

An example of this construction is shown in Figure 1, with $P_4 = \mathrm{cor}(P_2)$.

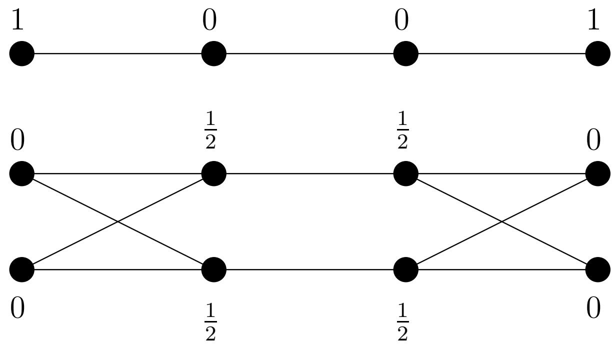

Figure 1. P_4 and its prism πP_4, where $\pi = (12)(34)$, with minimum fractional dominating functions.

The author would like to thank the anonymous referees for their helpful suggestions, and also R. Rubalcaba who commented on an early version of this paper.

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Received 28 September 2006
Revised 24 April 2007
Accepted 25 April 2007