# COUNTEREXAMPLE TO A CONJECTURE ON THE STRUCTURE OF BIPARTITE PARTITIONABLE GRAPHS 

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#### Abstract

A graph $G$ is called a prism fixer if $\gamma\left(G \times K_{2}\right)=\gamma(G)$, where $\gamma(G)$ denotes the domination number of $G$. A symmetric $\gamma$-set of $G$ is a minimum dominating set $D$ which admits a partition $D=D_{1} \cup D_{2}$ such that $V(G)-N\left[D_{i}\right]=D_{j}, i, j=1,2, i \neq j$. It is known that $G$ is a prism fixer if and only if $G$ has a symmetric $\gamma$-set.

Hartnell and Rall [On dominating the Cartesian product of a graph and $K_{2}$, Discuss. Math. Graph Theory 24 (2004), 389-402] conjectured that if $G$ is a connected, bipartite graph such that $V(G)$ can be partitioned into symmetric $\gamma$-sets, then $G \cong C_{4}$ or $G$ can be obtained from $K_{2 t, 2 t}$ by removing the edges of $t$ vertex-disjoint 4 -cycles. We construct a counterexample to this conjecture and prove an alternative result on the structure of such bipartite graphs.


Keywords: domination, prism fixer, symmetric dominating set, bipartite graph.
2000 Mathematics Subject Classification: 05C69.

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## 1. Introduction

We follow [6] for domination terminology and [3] for other graph theoretical notation and terminology. Specifically, for any graph $G=(V, E)$ and $v \in V$, the open neighbourhood $N(v)$ of $v$ is defined by $N(v)=\{u \in V: u v \in E\}$, and its closed neighbourhood $N[v]$ by $N(v) \cup\{v\}$. For $S \subseteq V, N(S)=$ $\bigcup_{s \in S} N(s)$ and $N[S]=\bigcup_{s \in S} N[s]$. For $A, B \subseteq V, N_{A}(B)=N(B) \cap A$; when $B=\{u\}$ we write $N_{A}(u)$ instead of $N_{A}(B)$. A set $S \subseteq V$ dominates $G$, written $S \succ G$, if every vertex in $V-S$ is adjacent to a vertex in $S$, i.e., if $V=N[S]$. The domination number $\gamma(G)$ of $G$ is defined by $\gamma(G)=\min \{|S|: S \succ G\}$. A $\gamma$-set of $G$ is a dominating set of $G$ of cardinality $\gamma(G)$. Further, a $\gamma$-set $D$ of $G$ is a symmetric $\gamma$-set if $D$ has a partition $D=D_{1} \cup D_{2}$ such that $V(G)-N\left[D_{i}\right]=D_{j}, i, j=1,2, i \neq j$. (Symmetric $\gamma$-sets are called two-colored $\gamma$-sets in [4, 5].)

A set $S \subseteq V$ is a packing (also called a 2-packing) of $G$ if $N[u] \cap N[v]=\phi$ for all distinct $u, v \in S$. A dominating set $D$ of $G$ is an efficient dominating set (also known as a perfect code, or a perfect single-error-correcting code) if $|D \cap N[v]|=1$ for each $v \in V(G)$. Thus $D$ is an efficient dominating set if and only if $D$ is a dominating set and a packing. As shown in [1] and [10], respectively, deciding whether a general graph and a bipartite graph, respectively, has an efficient dominating set, is NP-complete.

The cartesian product $G \times K_{2}$ is also called the prism of $G$. It is easy to see that $\gamma(G) \leq \gamma\left(G \times K_{2}\right) \leq 2 \gamma(G)$ for all graphs $G$. If the lower bound is satisfied, then $G$ is called a prism fixer. It is evident from the characterization of prism fixers as graphs that possess symmetric $\gamma$-sets (Theorem 2, $[5,7]$ ) that if $G$ is a prism fixer, then $G \times K_{2}$ has an efficient dominating set, i.e., a perfect code. (Note that the converse of this statement is not true. For example, the hypercube $Q_{7}$ is known to have a perfect code [6, Theorem 4.8] and $\gamma\left(Q_{7}\right)=16$. Also, $Q_{7}=Q_{6} \times K_{2}$, but $Q_{6}$ is not a prism fixer because $\gamma\left(Q_{6}\right)=12[8]$.) Thus the desirability of a graph possessing a perfect code serves as partial motivation for studying prism fixers.

Domination in prisms of graphs has been studied in $[2,4,5,7,9]$. In particular, the structure of prism fixers and the relation between prism fixers and Vizing's famous conjecture on the domination number of the cartesian products of graphs were investigated in $[4,5]$.

Conjecture 1 (Vizing's Conjecture) [11]. For any graphs $G$ and $H$, $\gamma(G \times H) \geq \gamma(G) \gamma(H)$.

Hartnell and Rall [4] constructed infinite classes of graphs to show that Vizing's conjecture, if true, is sharp. Many of these graphs have the property that their vertex sets partition into symmetric $\gamma$-sets; such a partition is called a symmetric partition and graphs with symmetric partitions are said to be partitionable. This connection between prism fixers and Vizing's conjecture serves as further motivation for the study of prism fixers. In [5] Hartnell and Rall further investigated the structure of prism fixers and closed with the following conjecture on the structure of bipartite partitionable graphs.

Conjecture 2 [5]. If $G$ is a connected, bipartite, partitionable graph, then $G \cong C_{4}$ or $G$ can be obtained from $K_{2 t, 2 t}$ by removing the edges of $t$ vertex-disjoint 4 -cycles.

We provide a counterexample to Conjecture 2 and prove a suitably amended result instead.

## 2. Prism Fixers and Symmetric $\gamma$-Sets

We begin by stating properties of symmetric $\gamma$-sets and a characterization of prism fixers.

Proposition $1[5,7]$. If $A$ is a symmetric $\gamma$-set of $G$, then
(a) $A$ is independent;
(b) $A_{i}, i=1,2$, is a maximal packing of $G$;
(c) each vertex in $V-A$ is adjacent to exactly one vertex in $A_{i}, i=1,2$;
(d) for each vertex $u \in V-A$ there exists a vertex $v \in V-A$ such that $N_{A}(u)=N_{A}(v)=\{x, y\}$ (say) and $\langle u, v, x, y\rangle=C_{4}$;
(e) $\delta(G) \geq 2$.

Theorem 2 [5, 7]. The graph $G$ is a prism fixer if and only if $G$ has a symmetric $\gamma$-set.

Note that $C_{4}$ is a prism fixer and, indeed, a bipartite partitionable graph. The following result on bipartite partitionable graphs was proved by Hartnell and Rall.

Proposition 3 [5]. Let $G \neq C_{4}$ be a bipartite graph such that $V(G)$ can be partitioned into $t$ symmetric $\gamma$-sets $A^{1}, \ldots, A^{t}$. Then $G$ is $2(t-1)$-regular, $\gamma(G)=4 k$ for some integer $k$ and for each $i=1, \ldots, t,\left|A_{1}^{i}\right|=\left|A_{2}^{i}\right|=2 k$.

We now define notation for prism fixers that will be used in the rest of the paper. See Figure 1. For a prism fixer $G$ and a symmetric $\gamma$-set $A$ of $G$, let $G^{*}$ be the graph with vertex set $V\left(G^{*}\right)=A$ and edge set $E\left(G^{*}\right)=\{u v$ : $\left.N_{G}(u) \cap N_{G}(v) \neq \phi\right\}$. Let $F_{1}^{*}, \ldots, F_{n}^{*}$ be the components of $G^{*}$. We say $F_{1}^{*}, \ldots, F_{n}^{*}$ are the graphs used in the construction of $G$ with respect to $A$. It follows from Proposition 1 that $F_{i}^{*}$ is bipartite for each $i$ (regardless of whether $G$ is bipartite or not). Further, for each $F_{i}^{*}$ let $F_{i}$ be the subgraph of $G$ induced by $N_{G}\left[V\left(F_{i}^{*}\right)\right]$.


Figure 1. The graphs $F_{1}^{*}, F_{2}^{*}$ used in the construction of $G$, and the graphs $F_{1}$ and $F_{2}$.

## 3. Counterexample

A counterexample to Conjecture 2 is given by the graph $G$ in Figure 2 with vertex set $V(G)=\{0,1, \ldots, 15\} \cup\left\{0^{\prime}, 1^{\prime}, \ldots, 15^{\prime}\right\}$ and the following (abbreviated) adjacency list:

| $v$ | $N(v)$ | $v$ | $N(v)$ |
| :---: | :--- | :--- | :--- |
| $0,0^{\prime}$ | $4,4^{\prime}, 5,5^{\prime}, 6,6^{\prime}$ | $5,5^{\prime}$ | $0,0^{\prime}, 12,12^{\prime}, 13,13^{\prime}$ |
| $1,1^{\prime}$ | $7,7^{\prime}, 8,8^{\prime}, 9,9^{\prime}$ | $6,6^{\prime}$ | $0,0^{\prime}, 10,10^{\prime}, 14,14^{\prime}$ |
| $2,2^{\prime}$ | $10,10^{\prime}, 11,11^{\prime}, 12,12^{\prime}$ | $7,7^{\prime}$ | $1,1^{\prime}, 12,12^{\prime}, 14,14^{\prime}$ |
| $3,3^{\prime}$ | $13,13^{\prime}, 14,14^{\prime}, 15,15^{\prime}$ | $8,8^{\prime}$ | $1,1^{\prime}, 10,10^{\prime}, 15,15^{\prime}$ |
| $4,4^{\prime}$ | $0,0^{\prime}, 11,11^{\prime}, 15,15^{\prime}$ | $9,9^{\prime}$ | $1,1^{\prime}, 11,11^{\prime}, 13,13^{\prime}$ |



Figure 2. A counterexample to Conjecture 2.
Note that $G$ is a connected, bipartite graph. We have verified by computer that $\gamma(G)=8$; an analytical proof is not difficult, just tedious. Moreover, $V(G)$ can be partitioned into the $\gamma$-sets $A^{1}=\left\{0,0^{\prime}, 1,1^{\prime}, 2,2^{\prime}, 3,3^{\prime}\right\}$, $A^{2}=\left\{4,4^{\prime}, 7,7^{\prime}, 10,10^{\prime}, 13,13^{\prime}\right\}, A^{3}=\left\{5,5^{\prime}, 8,8^{\prime}, 11,11^{\prime}, 14,14^{\prime}\right\}$ and $A^{4}=$ $\left\{6,6^{\prime}, 9,9^{\prime}, 12,12^{\prime}, 15,15^{\prime}\right\}$, which are easily seen to be symmetric $\gamma$-sets. Also note that if $G$ could be obtained from $K_{16,16}$ by removing 8 vertexdisjoint 4-cycles, then $\operatorname{deg} v=14$ for all $v \in V(G)$. However, $\operatorname{deg} v=6$ for all $v \in V(G)$ and thus Conjecture 2 does not hold for $G$.

## 4. Structural Results

However, a revised statement of Conjecture 2 does hold. Denote the disjoint union of $n$ copies of the graph $H$ by $n H$ and note that $l C_{4}$ is a spanning subgraph of $K_{2 l, 2 l}$. We shall prove:

Theorem 4. Let $G$ be a connected, bipartite, partitionable graph. Then there exist pairwise edge-disjoint subgraphs $H_{1} \cong \ldots \cong H_{\lambda} \cong l C_{4}$ of $K_{2 l, 2 l}$ such that $G$ can be obtained from $K_{2 l, 2 l}$ by removing the edges in $\bigcup_{i=1}^{\lambda} E\left(H_{i}\right)$.

We first prove several other results about the structure of bipartite partitionable graphs. The first result concerns the way in which one $\gamma$-set in a symmetric partition $\mathcal{P}$ dominates another $\gamma$-set in $\mathcal{P}$.

Proposition 5. Let $G$ be a bipartite, partitionable graph, $\mathcal{P}$ a symmetric partition of $V(G), A, B \in \mathcal{P}$ and $x \in A$. If $u_{1}, u_{2} \in B \cap N(x)$, then $N_{A}\left(u_{1}\right)=N_{A}\left(u_{2}\right)$.

Proof. Note that $A \cap B=\phi$ since $\mathcal{P}$ is a partition. Without loss of generality, assume $x \in A_{1}$ and $u_{1} \in B_{1}$.

Suppose to the contrary that $N_{A}\left(u_{1}\right) \neq N_{A}\left(u_{2}\right)$; say $N_{A}\left(u_{1}\right)=\left\{x, y_{1}\right\}$ and $N_{A}\left(u_{2}\right)=\left\{x, y_{2}\right\}$. Note that $y_{1}, y_{2} \in A_{2}$, hence $y_{1}, y_{2} \notin B$. Let $S=\{b \in$ $B_{1}: u b \in E(G)$ for some $\left.u \in N(x) \cap N\left(y_{1}\right)\right\}$ and $T=\left\{a \in A_{2}: a b \in E(G)\right.$ for some $b \in S\}$. Since $S \subseteq B, A \cap S=\phi$. Since $B_{1}$ is a packing (Proposition 1(b)), no two vertices in $S$ share a neighbour. Also, every vertex in $S$ has exactly one neighbour in $A_{2}$ and hence in $T$. Therefore $|S|=|T|$. Finally, note that the only vertices not dominated by $A-T$ are the vertices in $T$ and that $S \succ N(x) \cap N\left(y_{1}\right)-\left\{u_{1}\right\}$.

Suppose there exists a vertex $a \in T$ such that $N(x) \cap N(a) \neq \phi$. Then there exist vertices $b \in S$ and $u \in N(x) \cap N\left(y_{1}\right)$ such that $a b, b u \in E(G)$. If $a=y_{1}$, then $b=u_{1}$ and $x, b, u, x$ is an odd cycle in $G$; a contradiction since $G$ is bipartite. If $a \neq y_{1}$, then $b \neq u_{1}$ and there exists a vertex $w \in N(x) \cap N(a)$; thus $w \neq b, u$. But then $x, u, b, a, w, x$ is an odd cycle in $G$; a contradiction. Therefore $N(x) \cap N(a)=\phi$ for all $a \in T$ and thus $y_{1} \notin T$.

It follows that the only vertices not dominated by $A^{\prime}=A-T-\left\{x, y_{1}\right\}$ are the vertices of $T \cup\left\{x, y_{1}\right\} \cup\left(N(x) \cap N\left(y_{1}\right)\right)$. But then $A^{\prime \prime}=A^{\prime} \cup S \cup\left\{u_{1}\right\} \succ G$ and $\left|A^{\prime \prime}\right|=|A|-|T|-2+|S|+1=|A|-1=\gamma-1$; a contradiction.
Using Proposition 5 we now prove that if $G$ is a bipartite, partitionable graph, then with respect to any $\gamma$-set in a symmetric partition of $G, F_{i}^{*}=K_{2}$ for all $i$.

Theorem 6. Let $G$ be a bipartite, partitionable graph and $\mathcal{P}$ a symmetric partition of $V(G)$. If $A \in \mathcal{P}$ and $F_{1}^{*}, \ldots, F_{n}^{*}$ are the graphs used in the construction of $G$ with respect to $A$, then $F_{i}^{*}=K_{2}$ for all $i \in\{1, \ldots, n\}$.

Proof. Suppose to the contrary that $F_{1}^{*} \neq K_{2}$. Then without loss of generality there exists a vertex $x \in A_{1} \cap V\left(F_{1}\right)$ such that $N_{A_{2}}(N(x)) \supseteq$ $\{y, z\}, y \neq z$.

Let $B \in \mathcal{P}-\{A\}$; thus $x, y, z \notin B$. By Proposition $1(\mathrm{c}), x$ has exactly two neighbours in $B$; say $N_{B}(x)=\{v, w\}$. We may assume without loss of generality that $v \in N(y)$. Then by Proposition $5, N_{A}(v)=N_{A}(w)$ and so $w \in N(y)$. Without loss of generality $v \in B_{1}$ and $w \in B_{2}$. Since $N_{B}(x)=\{v, w\}$,

$$
\begin{equation*}
N(x) \cap N(z) \cap B=\phi . \tag{1}
\end{equation*}
$$

Therefore each vertex $u \in N(x) \cap N(z)$ has exactly one neighbour in $B_{1}$.
Let $S=\left\{b \in B_{1}: b u \in E(G)\right.$ for some $\left.u \in N(x) \cap N(z)\right\}$ and $T=\{a \in$ $A_{2}: a b \in E(G)$ for some $\left.b \in S\right\}$. Since $A \cap B=\phi, S \cap A=\phi$, so every vertex in $S$ has exactly one neighbour in $A_{2}$ and since $B_{1}$ is a packing, no two vertices of $S$ share the same neighbour. It follows that $|S|=|T|$. Note that $S \succ N(x) \cap N(z)$.

Suppose there exists a vertex $a \in T$ such that $N(x) \cap N(a) \neq \phi$; say $w \in N(x) \cap N(a)$. Then there exist vertices $b \in S, u \in N(x) \cap N(z)$ such that $a b, b u \in E(G)$. By (1), $b \notin N(x) \cap N(z)$. If $a \neq z$, then $x, w, a, b, u, x$ is an odd cycle in $G$; a contradiction. If $a=z$, then $z, b, u, z$ is an odd cycle in $G$; a contradiction. Therefore $N(x) \cap N(a)=\phi$ for all $a \in T$ and it also follows that $z \notin T$.

Now the only vertices not dominated by $A^{\prime}=A-T-\{x, z\}$ are the vertices of $T \cup\{x, z\} \cup(N(x) \cap N(z))$. But then letting $u \in N(x) \cap N(z)$, we have $A^{\prime \prime}=A^{\prime} \cup S \cup\{u\} \succ G$ and $\left|A^{\prime \prime}\right|=|A|-|T|-2+|S|+1=|A|-1=\gamma-1$; a contradiction. Therefore $F_{i}^{*}=K_{2}$ for all $i \in\{1, \ldots, n\}$.

In our final lemma before the proof of Theorem 4 we compare the cardinalities of the sets $A_{i} \cap V_{j}, i, j=1,2$, where $G$ has bipartition $\left(V_{1}, V_{2}\right)$ and $A=A_{1} \cup A_{2}$ is a set in a symmetric partition of $V(G)$.

Lemma 7. Let $G$ be a bipartite, partitionable graph with bipartition $\left(V_{1}, V_{2}\right)$ and symmetric partition $\mathcal{P}$. If $A \in \mathcal{P}$, then
(a) $\left|A_{1} \cap V_{i}\right|=\left|A_{2} \cap V_{i}\right|, i=1,2$,
(b) $\left|A_{i} \cap V_{1}\right|=\left|A_{i} \cap V_{2}\right|, i=1,2$.

Proof. (a) Let $F_{1}^{*}, \ldots, F_{n}^{*}$ be the graphs used in the construction of $G$ with respect to $A$. Then by Theorem $6, F_{i}^{*}=K_{2}$ for all $i$. Thus each vertex $x \in A_{1} \cap V_{1}$ has a unique vertex $y \in A_{2} \cap V_{1}$ such that $N(x)=N(y)$ and therefore $\left|A_{1} \cap V_{1}\right|=\left|A_{2} \cap V_{1}\right|$. Similarly for $V_{2}$, we have $\left|A_{1} \cap V_{2}\right|=\left|A_{2} \cap V_{2}\right|$.
(b) Note that $\bigcup_{x \in A_{1} \cap V_{1}} N(x)=V_{2}-A$ and $A_{1}$ is a packing. By Proposition $3, G$ is $2(t-1$ )-regular (where $t=|\mathcal{P}|$ ), hence

$$
\left|A_{1} \cap V_{1}\right|=\frac{\left|V_{2}-A\right|}{2(t-1)}
$$

and similarly

$$
\left|A_{1} \cap V_{2}\right|=\frac{\left|V_{1}-A\right|}{2(t-1)}
$$

Let $H=\langle V-A\rangle$. Then $H$ is bipartite with bipartition $\left(H_{1}, H_{2}\right)=\left(V_{1}-\right.$ $\left.A, V_{2}-A\right)$. Since every vertex in $V-A$ is adjacent in $G$ to exactly two vertices of $A, \operatorname{deg}_{H} v=\operatorname{deg}_{G} v-2$ for all $v \in V(H)$. Since $G$ is regular, $H$ is also regular. Hence $\left|H_{1}\right|=\left|H_{2}\right|$ and so $\left|V_{1}-A\right|=\left|V_{2}-A\right|$. It follows that $\left|A_{1} \cap V_{1}\right|=\left|A_{1} \cap V_{2}\right|$. A similar argument shows that $\left|A_{2} \cap V_{1}\right|=\left|A_{2} \cap V_{2}\right|$.

We are now ready to prove Theorem 4. For vertices $a, b, c, d \in V\left(K_{2 l, 2 l}\right)$ with $a, c \in V_{1}, b, d \in V_{2}$, we write the 4-cycle $a, b, c, d, a$ in $K_{2 l, 2 l}$ simply as $a b c d$.

Proof of Theorem 4. Let $G$ have bipartition $\left(V_{1}, V_{2}\right)$ and symmetric partition $\mathcal{P}=\left\{A^{1}, \ldots, A^{t}\right\}$. By Proposition 3 and Lemma $7, G$ is a spanning subgraph of $K_{2 l, 2 l}$ for some $l$. If $G=C_{4}$, let $\lambda=0$ and we are done. So assume $G \nsupseteq C_{4}$ (thus $t \geq 3$ ). Let $F_{i, 1}^{*}, \ldots, F_{i, n}^{*}$ be the graphs used in the construction of $G$ with respect to $A^{i}$. By Theorem $6, F_{i, j}^{*}=K_{2}$ for all $i, j$. Let $a=\left|A_{1}^{i} \cap V_{1}\right|=\left|A_{1}^{i} \cap V_{2}\right|=\left|A_{2}^{i} \cap V_{1}\right|=\left|A_{2}^{i} \cap V_{2}\right|\left(=\frac{\gamma}{4}\right)$. For $i \in\{1, \ldots, t\}$, $q \in\{1,2\}$, let

$$
A_{1}^{i} \cap V_{q}=\left\{v_{1, q}^{i}, v_{2, q}^{i}, \ldots, v_{a, q}^{i}\right\} \text { and } A_{2}^{i} \cap V_{q}=\left\{w_{1, q}^{i}, w_{2, q}^{i}, \ldots, w_{a, q}^{i}\right\}
$$

so that $N\left(v_{j, q}^{i}\right)=N\left(w_{j, q}^{i}\right)$ for all $j$.
For each $i=1, \ldots, t$, we first define $a$ mutually disjoint sets, each containing $a$ mutually disjoint 4 -cycles with vertex sets in $A^{i}$ and edge sets in $E(\bar{G})$. For each $k \in\{1, \ldots, a\}$, define

$$
\mathcal{C}_{k}^{i}=\left\{v_{p, 1}^{i} v_{p+k(\bmod a), 2}^{i} w_{p, 1}^{i} w_{p+k(\bmod a), 2}^{i}: 1 \leq p \leq a\right\}
$$

For the graph in Figure 2 the sets $\mathcal{C}_{1}^{1}$ (solid black lines) and $\mathcal{C}_{2}^{1}$ (broken black lines) are shown in Figure 3. Since $A^{i}$ is independent, all of the edges in each of the 4 -cycles in $\mathcal{C}_{k}^{i}$ are in $E(\bar{G})$. Also,
(2) for each $k$, every vertex of $A^{i}$ is in exactly one 4-cycle of $\mathcal{C}_{k}^{i}$
and

$$
\begin{equation*}
\mathcal{C}_{k}^{i} \cap \mathcal{C}_{k^{\prime}}^{i}=\phi \text { when } k \neq k^{\prime} . \tag{3}
\end{equation*}
$$

For $j \in\{1, \ldots, t\}-\{i\}$, each vertex of $A^{i}$ has exactly two neighbours in $A^{j}$. For $i$ fixed and each $p \in\{1, \ldots, a\}$, let $A^{j} \cap N\left(v_{p, q}^{i}\right)=\left\{r_{p, q}^{j}, s_{p, q}^{j}\right\}=A^{j} \cap$ $N\left(w_{p, q}^{i}\right)$. For each $i \in\{1, \ldots, t\}$ and each $j \in\{1, \ldots, t\}-\{i\}$, we now define $a-1$ mutually disjoint sets, each containing $2 a$ mutually disjoint 4-cycles with vertex sets in $A^{i} \cup A^{j}$ and edge sets in $E(\bar{G})$. For each $k \in\{1, \ldots, a-1\}$, define

$$
\mathcal{C}_{k}^{(i, j)}=\left\{v_{p, q}^{i} r_{p+k(\bmod a), q}^{j} w_{p, q}^{i} s_{p+k(\bmod a), q}^{j}: 1 \leq p \leq a, 1 \leq q \leq 2\right\}
$$

For the graph in Figure 2 the set $\mathcal{C}_{1}^{(1,2)}$ (with solid black lines for $q=1$ and broken black lines for $q=2$ ) is shown in Figure 4. Since $r_{p+k(\bmod a), q}^{j}, s_{p+k(\bmod a), q}^{j} \notin N\left(\left\{v_{p, q}^{i}, w_{p, q}^{i}\right\}\right)$ for all $k \in\{1, \ldots, a-1\}$, it follows that all of the edges in each of the 4 -cycles of $\mathcal{C}_{k}^{(i, j)}$ are in $E(\bar{G})$. Also note that
(4) every vertex of $A^{i} \cup A^{j}$ is in exactly one 4-cycle of $\mathcal{C}_{k}^{(i, j)}$,

$$
\begin{equation*}
\mathcal{C}_{k}^{(i, j)} \cap \mathcal{C}_{k^{\prime}}^{(i, j)}=\phi \text { when } k \neq k^{\prime} \tag{5}
\end{equation*}
$$

and for each $i \in\{1, \ldots, t\}, j \in\{1, \ldots, a\}, q \in\{1,2\}$,

$$
\begin{align*}
& N_{K_{2 l, 2 l}}\left(v_{j, q}^{i}\right)-N_{G}\left(v_{j, q}^{i}\right) \\
& =\left(\bigcup_{p=1}^{a}\left\{v_{p, q+1(\bmod 2)}^{i}, w_{p, q+1(\bmod 2)}^{i}\right\}\right) \cup\left(\bigcup_{\substack{h=1 \\
h \neq i}}^{t} \bigcup_{\substack{p=1 \\
p \neq j}}^{a}\left\{r_{p, q}^{h}, s_{p, q}^{h}\right\}\right) \tag{6}
\end{align*}
$$

Thus the vertices "missing" from the neighbourhood of $v_{j, q}^{i}$ are precisely the vertices adjacent to $v_{j, q}^{i}$ in the 4 -cycles contained in all of the $\mathcal{C}_{k}^{i}$ and the $\mathcal{C}_{k}^{(i, j)}$. We now consider two cases depending on the parity of $t$.


Figure 3. Sets $\mathcal{C}_{1}^{1}$ (solid black lines) and $\mathcal{C}_{2}^{1}$ (broken lines) for the graph in Figure 2.

Case 1. $t$ is even. Then $K_{t}$ is 1-factorable (see [3, Theorem 9.19]). Let $V\left(K_{t}\right)=\{1, \ldots, t\}$ and let $M_{1}, \ldots, M_{t-1}$ be the edge sets of a 1-factorization of $K_{t}$. For each $h \in\{1, \ldots, t-1\}$, we obtain the sets $\mathcal{S}_{1}^{h}, \ldots, \mathcal{S}_{a-1}^{h}$ as follows. For each $k \in\{1, \ldots, a-1\}$, define

$$
\mathcal{S}_{k}^{h}=\bigcup_{i j \in M_{h}, i<j} \mathcal{C}_{k}^{(i, j)}
$$

Since $M_{h}$ is a perfect matching in $K_{t}$, it follows from (4) that each vertex of $V(G)=\bigcup_{i=1}^{t} A^{i}$ is in exactly one 4-cycle of $\mathcal{S}_{k}^{h}$ and thus $\left\langle\mathcal{S}_{k}^{h}\right\rangle \cong l C_{4}$. Also, by (5), $\mathcal{S}_{k}^{h} \cap \mathcal{S}_{k^{\prime}}^{h}=\phi$ when $k \neq k^{\prime}$. Moreover, each $i j \in E\left(K_{t}\right)$ is in exactly one $M_{h}$ and so $\mathcal{S}_{k}^{h} \cap \mathcal{S}_{k^{\prime}}^{h^{\prime}}=\phi$ when $h \neq h^{\prime}$.

Further, for each $k \in\{1, \ldots, a\}$, define

$$
\mathcal{S}_{k}=\bigcup_{i=1}^{t} \mathcal{C}_{k}^{i} .
$$



Figure 4. Set $\mathcal{C}_{1}^{(1,2)}$ for the graph in Figure 2.
By (2), every vertex of $V(G)$ is in exactly one 4-cycle in $\mathcal{S}_{k}$ and thus $\left\langle\mathcal{S}_{k}\right\rangle \cong$ $l C_{4}$. Also, by (3), $\mathcal{S}_{k} \cap \mathcal{S}_{k^{\prime}}=\phi$ when $k \neq k^{\prime}$. Let

$$
\mathfrak{C}=\left(\bigcup_{k=1}^{a}\left\langle\mathcal{S}_{k}\right\rangle\right) \cup\left(\bigcup_{h=1}^{t-1} \bigcup_{k=1}^{a-1}\left\langle\mathcal{S}_{k}^{h}\right\rangle\right)
$$

Then $\mathfrak{C}$ consists of $a+(a-1)(t-1)=t(a-1)+1$ disjoint copies of $l C_{4}$. Also, $\cup \mathfrak{C}$ is precisely all of the 4 -cycles in all of the $\mathcal{C}_{k}^{i}$ and $\mathcal{C}_{k}^{(i, j)}$. Thus by (6), $G$ can be obtained from $K_{2 l, 2 l}$ by removing the edges of the copies of $l C_{4}$ in $\mathfrak{C}$.

Case 2. $t$ is odd. Let $M_{1}, \ldots, M_{t}$ be the edge sets of a 1-factorization of $K_{t+1}$, where $V\left(K_{t+1}\right)=\{1, \ldots, t+1\}$. For each $h \in\{1, \ldots, t\}$, we obtain the sets $\mathcal{S}_{1}^{h}, \ldots, \mathcal{S}_{a-1}^{h}$ as follows. For each $k \in\{1, \ldots, a-1\}$, define

$$
\mathcal{S}_{k}^{h}=\bigcup_{i j \in M_{h}, i<j<t+1} \mathcal{C}_{k}^{(i, j)} \cup \mathcal{C}_{a}^{m} \text { where } m(t+1) \in M_{h} .
$$

Since $M_{h}$ is a perfect matching in $K_{t+1}$, (2) and (4) imply that each vertex of $V(G)$ is in exactly one 4 -cycle of $\mathcal{S}_{k}^{h}$ and thus $\left\langle\mathcal{S}_{k}^{h}\right\rangle \cong l C_{4}$. Since each
vertex in $\{1, \ldots, t\}$ is adjacent to vertex $t+1$ in exactly one $M_{h},(3)$ and (5) imply that $\mathcal{S}_{k}^{h} \cap \mathcal{S}_{k^{\prime}}^{h}=\phi$ when $k \neq k^{\prime}$. Also, $\mathcal{S}_{k}^{h} \cap \mathcal{S}_{k^{\prime}}^{h^{\prime}}=\phi$ when $h \neq h^{\prime}$. Further, for each $k \in\{1, \ldots, a-1\}$, define

$$
\mathcal{S}_{k}=\bigcup_{i=1}^{t} \mathcal{C}_{k}^{i}
$$

Then by (2), every vertex of $V(G)$ is in exactly one 4-cycle in $\mathcal{S}_{k}$ and thus $\left\langle\mathcal{S}_{k}\right\rangle \cong l C_{4}$. Note that we do not have an $\mathcal{S}_{a}$ because the sets $\mathcal{C}_{a}^{i}$ were included in the $\mathcal{S}_{k}^{h}$ above. By (3), $\mathcal{S}_{k} \cap \mathcal{S}_{k^{\prime}}=\phi$ when $k \neq k^{\prime}$. Let

$$
\mathfrak{C}=\left(\bigcup_{k=1}^{a-1} \mathcal{S}_{k}\right) \cup\left(\bigcup_{h=1}^{t} \bigcup_{k=1}^{a-1} \mathcal{S}_{k}^{h}\right)
$$

Then $\mathfrak{C}$ consists of $a-1+t(a-1)=(t+1)(a-1)$ disjoint copies of $l C_{4}$. Also, $\bigcup \mathfrak{C}$ is precisely all of the 4 -cycles in all of the $\mathcal{C}_{k}^{i}$ and $\mathcal{C}_{k}^{(i, j)}$. Thus by (6), $G$ can be obtained from $K_{2 l, 2 l}$ by removing the edges of the copies of $l C_{4}$ in $\mathfrak{C}$.

In the proof of Theorem 4, a given bipartite graph whose vertex set partitions into $t$ symmetric $\gamma$-sets was obtained by deleting the edges of $t(a-1)+1$ or $(t+1)(a-1)$, depending on whether $t$ is even or odd, pairwise disjoint copies of $l C_{4}$ from $K_{2 l, 2 l}$, where $a=\gamma(G) / 4$ and $t=\frac{l}{a}$. We close with the following problem.

Problem 1. Consider $K_{2 l, 2 l}$ and let $a \geq 1$ be a divisor of $l$ such that $t=$ $\frac{l}{a} \geq 3$. For which values of $l$ and $a$ is it possible to remove the edges of $t(a-1)+1$ if $t$ is even, or $(t+1)(a-1)$ if $t$ is odd, pairwise disjoint copies of $l C_{4}$ from $K_{2 l, 2 l}$ and obtain a connected, bipartite, partitionable graph?

Note that it is possible to remove edges as described and obtain a bipartite graph whose vertex set partitions into dominating sets with the same properties as symmetric $\gamma$-sets (Proposition 1), except that they are not necessarily $\gamma$-sets.

For example, if $l=6$ and $a=2$, there are two ways of removing edges of four disjoint copies of $6 C_{4}$ from $K_{12,12}$ to obtain a bipartite graph $G$ whose vertex set partitions into three dominating sets, each of which satisfies Proposition 1 and $F_{i}^{*}=K_{2}$ for each $i$. In one case $\gamma\left(G_{1}\right)=4 a=8$ and $G_{1}$ is partitionable but not connected. In the other case $\gamma\left(G_{2}\right)=6$, and the dominating sets in the partition are thus not $\gamma$-sets. See Figure 5.


Figure 5. $G_{1}$ is partitionable but disconnected; $G_{2}$ is not partitionable.
As a final remark we note that the graph $G$ in Figure 2 with $\gamma(G)=8$ can be obtained as a "duplication" of its induced subgraph $H=\langle\{0,1, \ldots, 15\}\rangle$; that is, for each vertex $v \in V(H)$ we add a duplicate vertex $v^{\prime}$, joining $v^{\prime}$ to all vertices $u, u^{\prime}$, where $u \in N(v)$ and $u^{\prime}$ is the duplication of $u$. The set $\{0,1,2,3\}$ is an efficient dominating set of $H$, hence $\gamma(H)=4[6$, Theorem 4.2]. However, it is not true in general that if $G$ is a duplication of a graph $G^{\prime}$ with efficient dominating set of size $k$, then $\gamma(G)=2 k$. It is an obvious upper bound, but the graph $G_{2}$ in Figure 5 presents a counterexample to equality in this bound. It is a duplication of $C_{12}$, which has efficient dominating sets of size 4 , but $\gamma\left(G_{2}\right)=6$ as shown.

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Received 21 August 2006
Revised 21 February 2007
Accepted 7 March 2007


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    ${ }^{\dagger}$ Supported by the Natural Sciences and Engineering Research Council of Canada.

