COUNTEREXAMPLE TO A CONJECTURE ON THE STRUCTURE OF BIPARTITE PARTITIONABLE GRAPHS

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Abstract

A graph G is called a prism fixer if $\gamma(G \times K_2) = \gamma(G)$, where $\gamma(G)$ denotes the domination number of G. A symmetric γ -set of G is a minimum dominating set D which admits a partition $D = D_1 \cup D_2$ such that $V(G) - N[D_i] = D_j$, $i, j = 1, 2, i \neq j$. It is known that G is a prism fixer if and only if G has a symmetric γ -set.

Hartnell and Rall [On dominating the Cartesian product of a graph and K_2 , *Discuss. Math. Graph Theory* **24** (2004), 389–402] conjectured that if G is a connected, bipartite graph such that V(G) can be partitioned into symmetric γ -sets, then $G \cong C_4$ or G can be obtained from $K_{2t,2t}$ by removing the edges of t vertex-disjoint 4-cycles. We construct a counterexample to this conjecture and prove an alternative result on the structure of such bipartite graphs.

Keywords: domination, prism fixer, symmetric dominating set, bipartite graph.

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1. INTRODUCTION

We follow [6] for domination terminology and [3] for other graph theoretical notation and terminology. Specifically, for any graph G = (V, E) and $v \in V$, the open neighbourhood N(v) of v is defined by $N(v) = \{u \in V : uv \in E\}$, and its closed neighbourhood N[v] by $N(v) \cup \{v\}$. For $S \subseteq V$, $N(S) = \bigcup_{s \in S} N(s)$ and $N[S] = \bigcup_{s \in S} N[s]$. For $A, B \subseteq V$, $N_A(B) = N(B) \cap A$; when $B = \{u\}$ we write $N_A(u)$ instead of $N_A(B)$. A set $S \subseteq V$ dominates G, written $S \succ G$, if every vertex in V - S is adjacent to a vertex in S, i.e., if V = N[S]. The domination number $\gamma(G)$ of G is defined by $\gamma(G) = \min\{|S| : S \succ G\}$. A γ -set of G is a dominating set of G of cardinality $\gamma(G)$. Further, a γ -set D of G is a symmetric γ -set if D has a partition $D = D_1 \cup D_2$ such that $V(G) - N[D_i] = D_j$, $i, j = 1, 2, i \neq j$. (Symmetric γ -sets are called two-colored γ -sets in [4, 5].)

A set $S \subseteq V$ is a packing (also called a 2-packing) of G if $N[u] \cap N[v] = \phi$ for all distinct $u, v \in S$. A dominating set D of G is an *efficient dominating* set (also known as a perfect code, or a perfect single-error-correcting code) if $|D \cap N[v]| = 1$ for each $v \in V(G)$. Thus D is an efficient dominating set if and only if D is a dominating set and a packing. As shown in [1] and [10], respectively, deciding whether a general graph and a bipartite graph, respectively, has an efficient dominating set, is NP-complete.

The cartesian product $G \times K_2$ is also called the *prism* of G. It is easy to see that $\gamma(G) \leq \gamma(G \times K_2) \leq 2\gamma(G)$ for all graphs G. If the lower bound is satisfied, then G is called a *prism fixer*. It is evident from the characterization of prism fixers as graphs that possess symmetric γ -sets (Theorem 2, [5, 7]) that if G is a prism fixer, then $G \times K_2$ has an efficient dominating set, i.e., a perfect code. (Note that the converse of this statement is not true. For example, the hypercube Q_7 is known to have a perfect code [6, Theorem 4.8] and $\gamma(Q_7) = 16$. Also, $Q_7 = Q_6 \times K_2$, but Q_6 is not a prism fixer because $\gamma(Q_6) = 12$ [8].) Thus the desirability of a graph possessing a perfect code serves as partial motivation for studying prism fixers.

Domination in prisms of graphs has been studied in [2, 4, 5, 7, 9]. In particular, the structure of prism fixers and the relation between prism fixers and Vizing's famous conjecture on the domination number of the cartesian products of graphs were investigated in [4, 5].

Conjecture 1 (Vizing's Conjecture) [11]. For any graphs G and H, $\gamma(G \times H) \geq \gamma(G)\gamma(H)$.

Hartnell and Rall [4] constructed infinite classes of graphs to show that Vizing's conjecture, if true, is sharp. Many of these graphs have the property that their vertex sets partition into symmetric γ -sets; such a partition is called a *symmetric partition* and graphs with symmetric partitions are said to be *partitionable*. This connection between prism fixers and Vizing's conjecture serves as further motivation for the study of prism fixers. In [5] Hartnell and Rall further investigated the structure of prism fixers and closed with the following conjecture on the structure of bipartite partitionable graphs.

Conjecture 2 [5]. If G is a connected, bipartite, partitionable graph, then $G \cong C_4$ or G can be obtained from $K_{2t,2t}$ by removing the edges of t vertex-disjoint 4-cycles.

We provide a counterexample to Conjecture 2 and prove a suitably amended result instead.

2. PRISM FIXERS AND SYMMETRIC γ -Sets

We begin by stating properties of symmetric γ -sets and a characterization of prism fixers.

Proposition 1 [5, 7]. If A is a symmetric γ -set of G, then

- (a) A is independent;
- (b) A_i , i = 1, 2, is a maximal packing of G;
- (c) each vertex in V A is adjacent to exactly one vertex in A_i , i = 1, 2;
- (d) for each vertex $u \in V A$ there exists a vertex $v \in V A$ such that $N_A(u) = N_A(v) = \{x, y\}$ (say) and $\langle u, v, x, y \rangle = C_4$;
- (e) $\delta(G) \geq 2$.

Theorem 2 [5, 7]. The graph G is a prism fixer if and only if G has a symmetric γ -set.

Note that C_4 is a prism fixer and, indeed, a bipartite partitionable graph. The following result on bipartite partitionable graphs was proved by Hartnell and Rall. **Proposition 3** [5]. Let $G \neq C_4$ be a bipartite graph such that V(G) can be partitioned into t symmetric γ -sets A^1, \ldots, A^t . Then G is 2(t-1)-regular, $\gamma(G) = 4k$ for some integer k and for each $i = 1, \ldots, t$, $|A_1^i| = |A_2^i| = 2k$.

We now define notation for prism fixers that will be used in the rest of the paper. See Figure 1. For a prism fixer G and a symmetric γ -set A of G, let G^* be the graph with vertex set $V(G^*) = A$ and edge set $E(G^*) = \{uv : N_G(u) \cap N_G(v) \neq \phi\}$. Let F_1^*, \ldots, F_n^* be the components of G^* . We say F_1^*, \ldots, F_n^* are the graphs used in the construction of G with respect to A. It follows from Proposition 1 that F_i^* is bipartite for each i (regardless of whether G is bipartite or not). Further, for each F_i^* let F_i be the subgraph of G induced by $N_G[V(F_i^*)]$.



Figure 1. The graphs F_1^* , F_2^* used in the construction of G, and the graphs F_1 and F_2 .

3. Counterexample

A counterexample to Conjecture 2 is given by the graph G in Figure 2 with vertex set $V(G) = \{0, 1, ..., 15\} \cup \{0', 1', ..., 15'\}$ and the following (abbreviated) adjacency list:

| v | N(v) | v | N(v) |
|-------|--|-------|-------------------------|
| 0, 0' | 4, 4', 5, 5', 6, 6' | 5, 5' | 0, 0', 12, 12', 13, 13' |
| 1, 1' | $7,7^{\prime},8,8^{\prime},9,9^{\prime}$ | 6, 6' | 0, 0', 10, 10', 14, 14' |
| 2, 2' | 10, 10', 11, 11', 12, 12' | 7,7' | 1, 1', 12, 12', 14, 14' |
| 3,3' | 13, 13', 14, 14', 15, 15' | 8, 8' | 1, 1', 10, 10', 15, 15' |
| 4, 4' | 0, 0', 11, 11', 15, 15' | 9,9' | 1, 1', 11, 11', 13, 13' |



Figure 2. A counterexample to Conjecture 2.

Note that G is a connected, bipartite graph. We have verified by computer that $\gamma(G) = 8$; an analytical proof is not difficult, just tedious. Moreover, V(G) can be partitioned into the γ -sets $A^1 = \{0, 0', 1, 1', 2, 2', 3, 3'\}$, $A^2 = \{4, 4', 7, 7', 10, 10', 13, 13'\}$, $A^3 = \{5, 5', 8, 8', 11, 11', 14, 14'\}$ and $A^4 = \{6, 6', 9, 9', 12, 12', 15, 15'\}$, which are easily seen to be symmetric γ -sets. Also note that if G could be obtained from $K_{16,16}$ by removing 8 vertexdisjoint 4-cycles, then deg v = 14 for all $v \in V(G)$. However, deg v = 6 for all $v \in V(G)$ and thus Conjecture 2 does not hold for G.

4. Structural Results

However, a revised statement of Conjecture 2 does hold. Denote the disjoint union of n copies of the graph H by nH and note that lC_4 is a spanning subgraph of $K_{2l,2l}$. We shall prove:

Theorem 4. Let G be a connected, bipartite, partitionable graph. Then there exist pairwise edge-disjoint subgraphs $H_1 \cong \cdots \cong H_\lambda \cong lC_4$ of $K_{2l,2l}$ such that G can be obtained from $K_{2l,2l}$ by removing the edges in $\bigcup_{i=1}^{\lambda} E(H_i)$. We first prove several other results about the structure of bipartite partitionable graphs. The first result concerns the way in which one γ -set in a symmetric partition \mathcal{P} dominates another γ -set in \mathcal{P} .

Proposition 5. Let G be a bipartite, partitionable graph, \mathcal{P} a symmetric partition of V(G), $A, B \in \mathcal{P}$ and $x \in A$. If $u_1, u_2 \in B \cap N(x)$, then $N_A(u_1) = N_A(u_2)$.

Proof. Note that $A \cap B = \phi$ since \mathcal{P} is a partition. Without loss of generality, assume $x \in A_1$ and $u_1 \in B_1$.

Suppose to the contrary that $N_A(u_1) \neq N_A(u_2)$; say $N_A(u_1) = \{x, y_1\}$ and $N_A(u_2) = \{x, y_2\}$. Note that $y_1, y_2 \in A_2$, hence $y_1, y_2 \notin B$. Let $S = \{b \in B_1 : ub \in E(G) \text{ for some } u \in N(x) \cap N(y_1)\}$ and $T = \{a \in A_2 : ab \in E(G) \text{ for some } b \in S\}$. Since $S \subseteq B, A \cap S = \phi$. Since B_1 is a packing (Proposition 1(b)), no two vertices in S share a neighbour. Also, every vertex in S has exactly one neighbour in A_2 and hence in T. Therefore |S| = |T|. Finally, note that the only vertices not dominated by A - T are the vertices in T and that $S \succ N(x) \cap N(y_1) - \{u_1\}$.

Suppose there exists a vertex $a \in T$ such that $N(x) \cap N(a) \neq \phi$. Then there exist vertices $b \in S$ and $u \in N(x) \cap N(y_1)$ such that $ab, bu \in E(G)$. If $a = y_1$, then $b = u_1$ and x, b, u, x is an odd cycle in G; a contradiction since G is bipartite. If $a \neq y_1$, then $b \neq u_1$ and there exists a vertex $w \in N(x) \cap N(a)$; thus $w \neq b, u$. But then x, u, b, a, w, x is an odd cycle in G; a contradiction. Therefore $N(x) \cap N(a) = \phi$ for all $a \in T$ and thus $y_1 \notin T$.

It follows that the only vertices not dominated by $A' = A - T - \{x, y_1\}$ are the vertices of $T \cup \{x, y_1\} \cup (N(x) \cap N(y_1))$. But then $A'' = A' \cup S \cup \{u_1\} \succ G$ and $|A''| = |A| - |T| - 2 + |S| + 1 = |A| - 1 = \gamma - 1$; a contradiction.

Using Proposition 5 we now prove that if G is a bipartite, partitionable graph, then with respect to any γ -set in a symmetric partition of G, $F_i^* = K_2$ for all *i*.

Theorem 6. Let G be a bipartite, partitionable graph and \mathcal{P} a symmetric partition of V(G). If $A \in \mathcal{P}$ and F_1^*, \ldots, F_n^* are the graphs used in the construction of G with respect to A, then $F_i^* = K_2$ for all $i \in \{1, \ldots, n\}$.

Proof. Suppose to the contrary that $F_1^* \neq K_2$. Then without loss of generality there exists a vertex $x \in A_1 \cap V(F_1)$ such that $N_{A_2}(N(x)) \supseteq \{y, z\}, y \neq z$.

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Let $B \in \mathcal{P} - \{A\}$; thus $x, y, z \notin B$. By Proposition 1(c), x has exactly two neighbours in B; say $N_B(x) = \{v, w\}$. We may assume without loss of generality that $v \in N(y)$. Then by Proposition 5, $N_A(v) = N_A(w)$ and so $w \in N(y)$. Without loss of generality $v \in B_1$ and $w \in B_2$. Since $N_B(x) = \{v, w\}$,

(1)
$$N(x) \cap N(z) \cap B = \phi.$$

Therefore each vertex $u \in N(x) \cap N(z)$ has exactly one neighbour in B_1 .

Let $S = \{b \in B_1 : bu \in E(G) \text{ for some } u \in N(x) \cap N(z)\}$ and $T = \{a \in A_2 : ab \in E(G) \text{ for some } b \in S\}$. Since $A \cap B = \phi$, $S \cap A = \phi$, so every vertex in S has exactly one neighbour in A_2 and since B_1 is a packing, no two vertices of S share the same neighbour. It follows that |S| = |T|. Note that $S \succ N(x) \cap N(z)$.

Suppose there exists a vertex $a \in T$ such that $N(x) \cap N(a) \neq \phi$; say $w \in N(x) \cap N(a)$. Then there exist vertices $b \in S$, $u \in N(x) \cap N(z)$ such that $ab, bu \in E(G)$. By (1), $b \notin N(x) \cap N(z)$. If $a \neq z$, then x, w, a, b, u, x is an odd cycle in G; a contradiction. If a = z, then z, b, u, z is an odd cycle in G; a contradiction. Therefore $N(x) \cap N(a) = \phi$ for all $a \in T$ and it also follows that $z \notin T$.

Now the only vertices not dominated by $A' = A - T - \{x, z\}$ are the vertices of $T \cup \{x, z\} \cup (N(x) \cap N(z))$. But then letting $u \in N(x) \cap N(z)$, we have $A'' = A' \cup S \cup \{u\} \succ G$ and $|A''| = |A| - |T| - 2 + |S| + 1 = |A| - 1 = \gamma - 1$; a contradiction. Therefore $F_i^* = K_2$ for all $i \in \{1, \ldots, n\}$.

In our final lemma before the proof of Theorem 4 we compare the cardinalities of the sets $A_i \cap V_j$, i, j = 1, 2, where G has bipartition (V_1, V_2) and $A = A_1 \cup A_2$ is a set in a symmetric partition of V(G).

Lemma 7. Let G be a bipartite, partitionable graph with bipartition (V_1, V_2) and symmetric partition \mathcal{P} . If $A \in \mathcal{P}$, then

- (a) $|A_1 \cap V_i| = |A_2 \cap V_i|, i = 1, 2,$
- (b) $|A_i \cap V_1| = |A_i \cap V_2|, i = 1, 2.$

Proof. (a) Let F_1^*, \ldots, F_n^* be the graphs used in the construction of G with respect to A. Then by Theorem 6, $F_i^* = K_2$ for all i. Thus each vertex $x \in A_1 \cap V_1$ has a unique vertex $y \in A_2 \cap V_1$ such that N(x) = N(y) and therefore $|A_1 \cap V_1| = |A_2 \cap V_1|$. Similarly for V_2 , we have $|A_1 \cap V_2| = |A_2 \cap V_2|$.

(b) Note that $\bigcup_{x \in A_1 \cap V_1} N(x) = V_2 - A$ and A_1 is a packing. By Proposition 3, G is 2(t-1)-regular (where $t = |\mathcal{P}|$), hence

$$|A_1 \cap V_1| = \frac{|V_2 - A|}{2(t - 1)}$$

and similarly

$$|A_1 \cap V_2| = \frac{|V_1 - A|}{2(t - 1)}.$$

Let $H = \langle V - A \rangle$. Then H is bipartite with bipartition $(H_1, H_2) = (V_1 - V_1)$ $A, V_2 - A$). Since every vertex in V - A is adjacent in G to exactly two vertices of A, $\deg_H v = \deg_G v - 2$ for all $v \in V(H)$. Since G is regular, H is also regular. Hence $|H_1| = |H_2|$ and so $|V_1 - A| = |V_2 - A|$. It follows that $|A_1 \cap V_1| = |A_1 \cap V_2|$. A similar argument shows that $|A_2 \cap V_1| = |A_2 \cap V_2|$.

We are now ready to prove Theorem 4. For vertices $a, b, c, d \in V(K_{2l,2l})$ with $a, c \in V_1$, $b, d \in V_2$, we write the 4-cycle a, b, c, d, a in $K_{2l,2l}$ simply as abcd.

Proof of Theorem 4. Let G have bipartition (V_1, V_2) and symmetric partition $\mathcal{P} = \{A^1, \ldots, A^t\}$. By Proposition 3 and Lemma 7, G is a spanning subgraph of $K_{2l,2l}$ for some l. If $G = C_4$, let $\lambda = 0$ and we are done. So assume $G \ncong C_4$ (thus $t \ge 3$). Let $F_{i,1}^*, \ldots, F_{i,n}^*$ be the graphs used in the construction of G with respect to A^i . By Theorem 6, $F_{i,j}^* = K_2$ for all i, j. Let $a = |A_1^i \cap V_1| = |A_1^i \cap V_2| = |A_2^i \cap V_1| = |A_2^i \cap V_2| (= \frac{\gamma}{4})$. For $i \in \{1, \dots, t\}$, $q \in \{1, 2\},$ let

$$A_1^i \cap V_q = \{v_{1,q}^i, v_{2,q}^i, \dots, v_{a,q}^i\}$$
 and $A_2^i \cap V_q = \{w_{1,q}^i, w_{2,q}^i, \dots, w_{a,q}^i\}$

so that $N(v_{j,q}^i) = N(w_{j,q}^i)$ for all j. For each $i = 1, \ldots, t$, we first define a mutually disjoint sets, each containing a mutually disjoint 4-cycles with vertex sets in A^i and edge sets in $E(\overline{G})$. For each $k \in \{1, \ldots, a\}$, define

$$\mathcal{C}_{k}^{i} = \left\{ v_{p,1}^{i} v_{p+k(\text{mod }a),2}^{i} w_{p,1}^{i} w_{p+k(\text{mod }a),2}^{i} : 1 \le p \le a \right\}.$$

For the graph in Figure 2 the sets \mathcal{C}_1^1 (solid black lines) and \mathcal{C}_2^1 (broken black lines) are shown in Figure 3. Since A^i is independent, all of the edges in each of the 4-cycles in \mathcal{C}_k^i are in $E(\overline{G})$. Also,

(2) for each k, every vertex of A^i is in exactly one 4-cycle of \mathcal{C}_k^i

and

(3)
$$\mathcal{C}_k^i \cap \mathcal{C}_{k'}^i = \phi \text{ when } k \neq k'.$$

For $j \in \{1, \ldots, t\} - \{i\}$, each vertex of A^i has exactly two neighbours in A^j . For i fixed and each $p \in \{1, \ldots, a\}$, let $A^j \cap N(v_{p,q}^i) = \{r_{p,q}^j, s_{p,q}^j\} = A^j \cap N(w_{p,q}^i)$. For each $i \in \{1, \ldots, t\}$ and each $j \in \{1, \ldots, t\} - \{i\}$, we now define a - 1 mutually disjoint sets, each containing 2a mutually disjoint 4-cycles with vertex sets in $A^i \cup A^j$ and edge sets in $E(\overline{G})$. For each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{C}_{k}^{(i,j)} = \left\{ v_{p,q}^{i} r_{p+k(\text{mod }a),q}^{j} w_{p,q}^{i} s_{p+k(\text{mod }a),q}^{j} : 1 \le p \le a, 1 \le q \le 2 \right\}.$$

For the graph in Figure 2 the set $C_1^{(1,2)}$ (with solid black lines for q = 1and broken black lines for q = 2) is shown in Figure 4. Since $r_{p+k(\text{mod }a),q}^j, s_{p+k(\text{mod }a),q}^j \notin N(\{v_{p,q}^i, w_{p,q}^i\})$ for all $k \in \{1, \ldots, a-1\}$, it follows that all of the edges in each of the 4-cycles of $C_k^{(i,j)}$ are in $E(\overline{G})$. Also note that

(4) every vertex of $A^i \cup A^j$ is in exactly one 4-cycle of $\mathcal{C}_k^{(i,j)}$,

(5)
$$\mathcal{C}_{k}^{(i,j)} \cap \mathcal{C}_{k'}^{(i,j)} = \phi \text{ when } k \neq k',$$

and for each $i \in \{1, \ldots, t\}, j \in \{1, \ldots, a\}, q \in \{1, 2\},\$

$$N_{K_{2l,2l}}(v_{j,q}^{i}) - N_{G}(v_{j,q}^{i})$$

$$(6) \qquad = \left(\bigcup_{p=1}^{a} \left\{ v_{p,q+1(\text{mod }2)}^{i}, w_{p,q+1(\text{mod }2)}^{i} \right\} \right) \cup \left(\bigcup_{\substack{h=1\\h\neq i}}^{t} \bigcup_{\substack{p=1\\p\neq j}}^{a} \left\{ r_{p,q}^{h}, s_{p,q}^{h} \right\} \right).$$

Thus the vertices "missing" from the neighbourhood of $v_{j,q}^i$ are precisely the vertices adjacent to $v_{j,q}^i$ in the 4-cycles contained in all of the C_k^i and the $C_k^{(i,j)}$. We now consider two cases depending on the parity of t.



Figure 3. Sets C_1^1 (solid black lines) and C_2^1 (broken lines) for the graph in Figure 2.

Case 1. t is even. Then K_t is 1-factorable (see [3, Theorem 9.19]). Let $V(K_t) = \{1, \ldots, t\}$ and let M_1, \ldots, M_{t-1} be the edge sets of a 1-factorization of K_t . For each $h \in \{1, \ldots, t-1\}$, we obtain the sets $\mathcal{S}_1^h, \ldots, \mathcal{S}_{a-1}^h$ as follows. For each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{S}_k^h = \bigcup_{ij \in M_h, i < j} \mathcal{C}_k^{(i,j)}.$$

Since M_h is a perfect matching in K_t , it follows from (4) that each vertex of $V(G) = \bigcup_{i=1}^t A^i$ is in exactly one 4-cycle of \mathcal{S}_k^h and thus $\langle \mathcal{S}_k^h \rangle \cong lC_4$. Also, by (5), $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^h = \phi$ when $k \neq k'$. Moreover, each $ij \in E(K_t)$ is in exactly one M_h and so $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^{h'} = \phi$ when $h \neq h'$.

Further, for each $k \in \{1, \ldots, a\}$, define

$$\mathcal{S}_k = \bigcup_{i=1}^t \mathcal{C}_k^i.$$



Figure 4. Set $C_1^{(1,2)}$ for the graph in Figure 2.

By (2), every vertex of V(G) is in exactly one 4-cycle in \mathcal{S}_k and thus $\langle \mathcal{S}_k \rangle \cong lC_4$. Also, by (3), $\mathcal{S}_k \cap \mathcal{S}_{k'} = \phi$ when $k \neq k'$. Let

$$\mathfrak{C} = \left(\bigcup_{k=1}^{a} \langle \mathcal{S}_k \rangle \right) \cup \left(\bigcup_{h=1}^{t-1} \bigcup_{k=1}^{a-1} \left\langle \mathcal{S}_k^h \right\rangle \right).$$

Then \mathfrak{C} consists of a + (a - 1)(t - 1) = t(a - 1) + 1 disjoint copies of lC_4 . Also, $\bigcup \mathfrak{C}$ is precisely all of the 4-cycles in all of the \mathcal{C}_k^i and $\mathcal{C}_k^{(i,j)}$. Thus by (6), G can be obtained from $K_{2l,2l}$ by removing the edges of the copies of lC_4 in \mathfrak{C} .

Case 2. t is odd. Let M_1, \ldots, M_t be the edge sets of a 1-factorization of K_{t+1} , where $V(K_{t+1}) = \{1, \ldots, t+1\}$. For each $h \in \{1, \ldots, t\}$, we obtain the sets S_1^h, \ldots, S_{a-1}^h as follows. For each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{S}_k^h = \bigcup_{ij \in M_h, i < j < t+1} \mathcal{C}_k^{(i,j)} \cup \mathcal{C}_a^m \text{ where } m(t+1) \in M_h.$$

Since M_h is a perfect matching in K_{t+1} , (2) and (4) imply that each vertex of V(G) is in exactly one 4-cycle of \mathcal{S}_k^h and thus $\langle \mathcal{S}_k^h \rangle \cong lC_4$. Since each

vertex in $\{1, \ldots, t\}$ is adjacent to vertex t + 1 in exactly one M_h , (3) and (5) imply that $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^h = \phi$ when $k \neq k'$. Also, $\mathcal{S}_k^h \cap \mathcal{S}_{k'}^{h'} = \phi$ when $h \neq h'$.

Further, for each $k \in \{1, \ldots, a-1\}$, define

$$\mathcal{S}_k = \bigcup_{i=1}^t \mathcal{C}_k^i.$$

Then by (2), every vertex of V(G) is in exactly one 4-cycle in \mathcal{S}_k and thus $\langle \mathcal{S}_k \rangle \cong lC_4$. Note that we do not have an \mathcal{S}_a because the sets \mathcal{C}_a^i were included in the \mathcal{S}_k^h above. By (3), $\mathcal{S}_k \cap \mathcal{S}_{k'} = \phi$ when $k \neq k'$. Let

$$\mathfrak{C} = \left(\bigcup_{k=1}^{a-1} \mathcal{S}_k\right) \cup \left(\bigcup_{h=1}^t \bigcup_{k=1}^{a-1} \mathcal{S}_k^h\right).$$

Then \mathfrak{C} consists of a - 1 + t(a - 1) = (t + 1)(a - 1) disjoint copies of lC_4 . Also, $\bigcup \mathfrak{C}$ is precisely all of the 4-cycles in all of the \mathcal{C}_k^i and $\mathcal{C}_k^{(i,j)}$. Thus by (6), G can be obtained from $K_{2l,2l}$ by removing the edges of the copies of lC_4 in \mathfrak{C} .

In the proof of Theorem 4, a given bipartite graph whose vertex set partitions into t symmetric γ -sets was obtained by deleting the edges of t(a-1) + 1or (t+1)(a-1), depending on whether t is even or odd, pairwise disjoint copies of lC_4 from $K_{2l,2l}$, where $a = \gamma(G)/4$ and $t = \frac{l}{a}$. We close with the following problem.

Problem 1. Consider $K_{2l,2l}$ and let $a \ge 1$ be a divisor of l such that $t = \frac{l}{a} \ge 3$. For which values of l and a is it possible to remove the edges of t(a-1)+1 if t is even, or (t+1)(a-1) if t is odd, pairwise disjoint copies of lC_4 from $K_{2l,2l}$ and obtain a connected, bipartite, partitionable graph?

Note that it is possible to remove edges as described and obtain a bipartite graph whose vertex set partitions into dominating sets with the same properties as symmetric γ -sets (Proposition 1), except that they are not necessarily γ -sets.

For example, if l = 6 and a = 2, there are two ways of removing edges of four disjoint copies of $6C_4$ from $K_{12,12}$ to obtain a bipartite graph Gwhose vertex set partitions into three dominating sets, each of which satisfies Proposition 1 and $F_i^* = K_2$ for each *i*. In one case $\gamma(G_1) = 4a = 8$ and G_1 is partitionable but not connected. In the other case $\gamma(G_2) = 6$, and the dominating sets in the partition are thus not γ -sets. See Figure 5.



Figure 5. G_1 is partitionable but disconnected; G_2 is not partitionable.

As a final remark we note that the graph G in Figure 2 with $\gamma(G) = 8$ can be obtained as a "duplication" of its induced subgraph $H = \langle \{0, 1, \ldots, 15\} \rangle$; that is, for each vertex $v \in V(H)$ we add a duplicate vertex v', joining v'to all vertices u, u', where $u \in N(v)$ and u' is the duplication of u. The set $\{0, 1, 2, 3\}$ is an efficient dominating set of H, hence $\gamma(H) = 4$ [6, Theorem 4.2]. However, it is not true in general that if G is a duplication of a graph G'with efficient dominating set of size k, then $\gamma(G) = 2k$. It is an obvious upper bound, but the graph G_2 in Figure 5 presents a counterexample to equality in this bound. It is a duplication of C_{12} , which has efficient dominating sets of size 4, but $\gamma(G_2) = 6$ as shown.

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