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DISTANCE DEFINED BY SPANNING TREES IN GRAPHS

Gary Chartrand

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

Ladislav Nebeský

Faculty of Philosophy & Arts
Charles University, Prague
J. Palacha 2, CZ – 116 38 Praha 1, Czech Republic

and

Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

Abstract

For a spanning tree T in a nontrivial connected graph G and for vertices u and v in G, there exists a unique u–v path $u = u_0, u_1, u_2, \ldots, u_k = v$ in T. A $u - v$ T-path in G is a $u - v$ path $u = v_0, v_1, \ldots, v_\ell = v$ in G that is a subsequence of the sequence $u = u_0, u_1, u_2, \ldots, u_k = v$. A $u - v$ T-path of minimum length is a $u - v$ T-geodesic in G. The T-distance $d_{G|T}(u, v)$ from u to v in G is the length of a $u - v$ T-geodesic. Let $\mathrm{geo}(G)$ and $\mathrm{geo}(G|T)$ be the set of geodesics and the set of T-geodesics respectively in G. Necessary and sufficient conditions are established for (1) $\mathrm{geo}(G) = \mathrm{geo}(G|T)$ and (2) $\mathrm{geo}(G|T) = \mathrm{geo}(G|T^*)$, where T and T^* are two spanning trees of G. It is shown for a connected graph G that $\mathrm{geo}(G|T) = \mathrm{geo}(G)$ for every spanning tree T of G if and only if G is a block graph. For a spanning tree T of a connected graph G, it is also shown that $\mathrm{geo}(G|T)$ satisfies seven of the eight axioms of the characterization of $\mathrm{geo}(G)$. Furthermore, we study the relationship between the distance d and T-distance $d_{G|T}$ in graphs and present several realization results on parameters and subgraphs defined by these two distances.

Keywords: distance, geodesic, T-path, T-geodesic, T-distance.

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1. T-Distance in Graphs

Let G be a nontrivial connected graph. The standard *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest $u-v$ path in G and a $u-v$ path of minimum length is a $u-v$ *geodesic* in G. Let T be a spanning tree of G. For vertices u and v of G, there exists a unique $u-v$ path

$$u = u_0, u_1, u_2, \ldots, u_k = v \tag{1}$$

in T. A $u-v$ *T-path* in G is a $u-v$ path

$$u = v_0, v_1, \ldots, v_\ell = v \tag{2}$$

in G such that the sequence (2) is a subsequence of the sequence (1). First, we present two lemmas, the first of which is a consequence of the definitions.

Lemma 1.1. *Let T be a spanning tree of a connected graph G and let P : u, v, w be a path in G. Then P is a T-path in G if and only if v lies on the $u-w$ path in T.*

Lemma 1.2. *Let T be a spanning tree of a connected graph G and let*

$$P : u_0, u_1, \ldots, u_k$$

be a path in G, where $k \geq 2$. Then P is a T-path in G if and only if

(3) $\quad u_i, u_{i+1}, u_{i+2}$ *is a T-path in G for each integer i with $0 \leq i \leq n-2$.*

Proof. We proceed by induction on k. The case when $k = 2$ is obvious. Let $k \geq 3$. Clearly, if P is a T-path, then (3) holds. Conversely, let (3) hold. By the induction hypothesis,

$$P^* : u_0, u_1, \ldots, u_{k-1}$$

is a T-path in G. Hence P^* is a subsequence of the $u_0 - u_{k-1}$ path in T. Since u_{k-2}, u_{k-1}, u_k is a T-path, we see that u_{k-1} belongs to the $u_{k-2} - u_k$ path in T. Since $u_{k-1} \neq u_{k-2}$, we see that that P is a subsequence of the $u_0 - u_k$ path in T. Hence P is a T-path in G. ■

A $u-v$ T-path of minimum length is a $u-v$ *T-geodesic* in G. The *T-distance* $d_{G|T}(u,v)$ from u to v in G is the length of any $u-v$ T-geodesic, that is, $d_{G|T}(u,v)$ is the minimum length of a $u-v$ T-path in G. In particular, if $d_G(u,v) \leq 1$, then $d_{G|T}(u,v) = d_G(u,v)$. Hence if P is the $u-v$ path in (1) and Q is the $u-v$ T-path in (2), then Q is obtained from P by possibly deleting some interior vertices of P and adding some edges of $G - E(T)$. Thus for each connected graph G and each spanning tree T of G,

$$d_G(u,v) \leq d_{G|T}(u,v) \leq d_T(u,v) \tag{4}$$

for every two vertices u and v of G.

For example, consider the graph G of Figure 1 and the spanning tree T of G, where the edges of T are indicated in bold. For the vertices u and v of G,

$$d_G(u,v) = 3,\ d_{G|T}(u,v) = 6,\ \text{and } d_T(u,v) = 9.$$

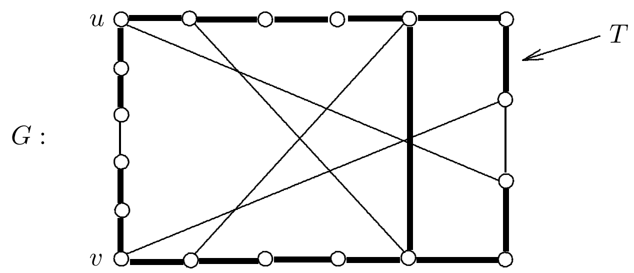

Figure 1. T-Distance in a graph

For a connected graph G and a spanning tree T of G, it follows for every two vertices u and v of G that

(i) $d_{G|T}(u,v) \geq 0$,

(ii) $d_{G|T}(u,v) = 0$ if and only if $u = v$, and

(iii) $d_{G|T}(u,v) = d_{G|T}(v,u)$.

Despite the fact that T-distance satisfies properties (i)–(iii), $d_{G|T}$ is not a metric on $V(G)$ as it does not satisfy the triangle inequality. For example, for the graph $G = C_5$ and the spanning tree T of G shown in Figure 2,

$$d_{G|T}(u, v) = 3 > 1 + 1 = d_{G|T}(u, w) + d_{G|T}(w, v).$$

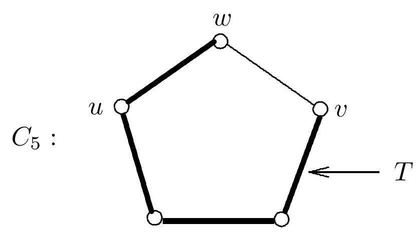

Figure 2. Failure of the triangle inequality for T-distance.

2. T-Geodesics in Graphs

We refer to the book [3] for graph theory notation and terminology not described in this paper. Let G be a connected graph and T a spanning tree of G. We denote by geo(G) and geo$(G|T)$ the set of all geodesics and the set of T-geodesics respectively in G. First, we make two observations.

Observation 2.1. *Let T and T^* be two spanning trees of a connected graph G.*

$$\textit{If } \text{geo}(G|T) = \text{geo}(G|T^*), \textit{ then } d_{G|T} = d_{G|T^*}.$$

The converse of Observation 2.1 is not true, however. For example, consider the graph $G = K_4 - e$ of Figure 3 and the two spanning trees T and T^* of G. Observe that $d_{G|T} = d_{G|T^*}$, while geo$(G|T) \neq$ geo$(G|T^*)$, as u, x, v is a $u - v$ T^*-geodesic in G that is not a $u - v$ T-geodesic in G.

The following result provides a necessary and sufficient condition for the set of T-geodesics and the set of T^*-geodesics of a connected graph G to be the same for spanning trees T and T^* of G.

Theorem 2.2. *Let T and T^* be spanning trees of a connected graph G. Then*

$$\text{geo}(G|T) = \text{geo}(G|T^*)$$

if and only if every T-geodesic of length 2 in G is a T^*-path in G and every T^*-geodesic of length 2 in G is a T-path in G.

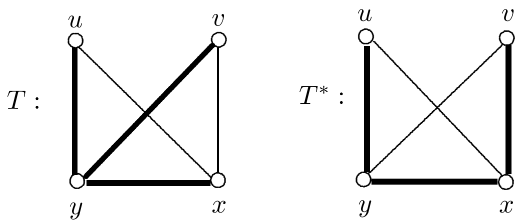

Figure 3. The converse of Observation 2.1 is false.

Proof. If $\text{geo}(G|T) = \text{geo}(G|T^*)$, then obviously every T-geodesic of length 2 in G is a T^*-path in G and every T^*-geodesic of length 2 in G is a T-path in G. It remains to verify the converse. Assume that every T-geodesic of length 2 in G is a T^*-path in G and every T^*-geodesic of length 2 in G is a T-path in G. Let

$$P : u = u_0, u_1, \ldots, u_k = v \quad (k \geq 0)$$

be a T-geodesic in G. We show that P is also a T^*-geodesic in G. If $k \in \{0, 1\}$, then certainly P is a T^*-path (indeed, a T^*-geodesic) in G. Thus we may assume that $k \geq 2$. By Lemma 1.2, u_i, u_{i+1}, u_{i+2} is a $u_i - u_{i+2}$ T-path in G. Furthermore, it is a $u_i - u_{i+2}$ T-geodesic in G, for otherwise, P is not a T-geodesic in G. By assumption, it follows that u_i, u_{i+1}, u_{i+2} is a $u_i - u_{i+2}$ T^*-path in G. Again, by Lemma 1.2, P is a T^*-path in G. This implies that $d_{G|T^*}(u, v) \leq d_{G|T}(u, v)$. Similarly, every $u - v$ T^*-geodesic in G is a $u - v$ T-path in G and so $d_{G|T}(u, v) \leq d_{G|T^*}(u, v)$. Thus $d_{G|T}(u, v) = d_{G|T^*}(u, v)$, implying that P is a T^*-geodesic in G and so $\text{geo}(G|T) \subseteq \text{geo}(G|T^*)$. Similarly, $\text{geo}(G|T^*) \subseteq \text{geo}(G|T)$. Thus $\text{geo}(G|T) = \text{geo}(G|T^*)$. ■

For a spanning tree T of a connected graph G, let $\text{geo}_2(G|T)$ denote the set of all T-geodesics of length 2 in G. The following is a consequence of Theorem 2.2.

Corollary 2.3. *Let T and T^* be spanning trees of a connected graph G. Then*

$$\text{geo}(G|T) = \text{geo}(G|T^*) \textit{ if and only if } \text{geo}_2(G|T) = \text{geo}_2(G|T^*).$$

We now study necessary and sufficient conditions for the sets of geodesics and T-geodesics of a connected graph G to be the same for a spanning tree T of G.

Theorem 2.4. *Let T be a spanning tree of a connected graph G. Then*

$$\mathrm{geo}(G|T) = \mathrm{geo}(G)$$

if and only if every geodesic of length 2 in G is a T-path in G.

Proof. If $\mathrm{geo}(G|T) = \mathrm{geo}(G)$, then every geodesic of length 2 in G is a T-geodesic of length 2 in G and therefore, a T-path in G.

For the converse, assume that every geodesic of length 2 is a T-path in G. We show that $\mathrm{geo}(G|T) = \mathrm{geo}(G)$. We first show that $\mathrm{geo}(G) \subseteq \mathrm{geo}(G|T)$. Let

$$P: \ u = u_0, u_1, u_2, \ldots, u_k = v$$

be a $u - v$ geodesic in G. Thus $P_i : u_i, u_{i+1}, u_{i+2}$ is a $u_i - u_{i+2}$ geodesic of length 2 for every integer i $(0 \le i \le k-2)$. By hypothesis, P_i is a T-path in G for $0 \le i \le k-2$. By Lemma 1.2 , P is a $u - v$ T-path in G. Thus $d_{G|T}(u,v) \le d_G(u,v)$. However, since $d_G(u,v) \le d_{G|T}(u,v)$, it follows that $d_G(u,v) = d_{G|T}(u,v)$ and P is a $u - v$ T-geodesic in G. Hence $\mathrm{geo}(G) \subseteq \mathrm{geo}(G|T)$.

Next, we show that $\mathrm{geo}(G|T) \subseteq \mathrm{geo}(G)$. Let

$$Q: u = v_0, v_1, v_2, \ldots, v_\ell = v$$

be a $u - v$ T-geodesic in G. Since $\mathrm{geo}(G) \subseteq \mathrm{geo}(G|T)$, it follows that $d_G(u,v) = d_{G|T}(u,v) = \ell$ and so Q is a geodesic in G. Therefore, $\mathrm{geo}(G|T) \subseteq \mathrm{geo}(G)$, which completes the proof. ∎

For a connected graph G, let $\mathrm{geo}_2(G)$ denote the set of all geodesics of length 2 in G. The following two corollaries are consequences of Theorem 2.4.

Corollary 2.5. *Let T be a spanning tree of a connected graph G. Then $\mathrm{geo}(G) = \mathrm{geo}(G|T)$ if and only if $\mathrm{geo}_2(G) = \mathrm{geo}_2(G|T)$.*

For a graph F, the *square* F^2 of F is the graph whose vertex set is $V(F^2) = V(F)$ such that $uv \in E(F^2)$ if and only if $1 \le d_F(u,v) \le 2$.

Corollary 2.6. *Let T be a nontrivial tree. If G is the square of T, then $\mathrm{geo}(G) = \mathrm{geo}(G|T)$.*

Proof. Consider an arbitrary path P of length 2 in G. Observe that if P is not a T-path in G, then P is not a geodesic in G. Hence the result follows. ■

A connected graph G is a *block graph* if every block of G is complete. In fact, block graphs are the only connected graphs G for which $\mathrm{geo}(G|T) = \mathrm{geo}(G)$ for every spanning tree T of G. In order to show that, we first establish some preliminary results. Let T be a spanning tree of a connected graph G. If $u, v, w \in V(G)$ and w belongs to a $u - v$ T-geodesic, then $d_{G|T}(u,v) = d_{G|T}(u,w) + d_{G|T}(w,v)$. The converse of this statement is not true. For example, for the 4-cycle $C_4 : u, u^*, v, w, u$ and the spanning tree $T = C_4 - uw$ of C_4 in Figure 4,

$$d_{G|T}(u,v) = 2 = 1 + 1 = d_{G|T}(u,w) + d_{G|T}(w,v),$$

but w belongs to no $u - v$ T-geodesic in G.

C_4 :

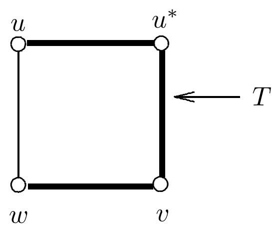

Figure 4. The 4-cycle C_4 and a spanning tree T in C_4.

In general, we have the following.

Lemma 2.7. *Let v be a cut-vertex of a connected graph G and let T be a spanning tree of G. Let*

$$P' :\ u_0, u_1, u_2, \ldots, u_{i-1}, u_i = v \quad \textit{and} \quad P'' :\ v = u_i, u_{i+1}, \ldots, u_k$$

be paths in G such that u_{i-1} and u_{i+1} belong to distinct components of $G - v$. Then

$$P :\ u_0, u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n$$

is a T-path in G if and only if P' and P'' are T-paths in G. Furthermore, P is a T-geodesic in G if and only if P' and P'' are T-geodesics in G.

Theorem 2.8. *Let G be a connected graph. Then* $\mathrm{geo}(G|T) = \mathrm{geo}(G)$ *for every spanning tree T of G if and only if G is a block graph.*

Proof. First, assume that G is not a block graph. Then there exists a block B of G such that B is not complete. Necessarily, B is 2-connected. Then there exists a 2-connected induced subgraph H of minimum order p in B that is not complete. Either $p \geq 4$ and $H = C_p$ or $p = 4$ and $H = K_4 - e$. We consider these two cases.

Case 1. $H = C_p$. Let $H : u_1, u_2, \ldots, u_p, u_1$, where $p \geq 4$. Then there exists a spanning tree T_1 of G such that

$$u_1u_2, u_2u_3, \ldots, u_{p-1}u_p \in E(T_1) \text{ and } u_pu_1 \notin E(T_1).$$

Since $p \geq 4$, the path

$$u_{p-1}, u_p, u_1$$

is a geodesic in G but not a T_1-geodesic in G.

Case 2. $H = K_4 - e$. Let $V(H) = \{u, v, x, y\}$ such that $E(H) = \{ux, uy, vx, vy, xy\}$. Then there exists a spanning tree T_2 of G such that

$$ux, vx, xy \in E(T_2) \text{ and } uy, vy \notin E(T_2).$$

Since H is an induced subgraph of G, it follows that

$$P : \; u, y, v$$

is a geodesic in G. On the other hand, P is not a T_2-path in G and so it is not a T_2-geodesic in G.

For the converse, let G be a block graph. Let X be the set of all paths u, v, w in G, where v is a cut-vertex of G and u and w belong to distinct components of $G - v$. Consider an arbitrary spanning tree T of G. Since $\mathrm{geo}_2(G) = X = \mathrm{geo}_2(G|T)$, it follows by Corollary 2.5 that $\mathrm{geo}(G|T) = \mathrm{geo}(G)$. ■

Next, we present a necessary condition for a subset of the set of all paths in a nontrivial connected graph G to be $\mathrm{geo}(G|T)$ for some spanning tree T of G.

Theorem 2.9. *Let G be a nontrivial connected graph and let T be a spanning tree of G. Put $\mathbf{A} = \mathrm{geo}(G|T)$. Then $\mathbf{A}$ is a subset of the set of all paths in G and satisfies the following seven axioms:*

Axiom 1: *If $uv \in E(G)$, then the path u, v belongs to $\mathbf{A}$.*

Axiom 2: *If the path $u_0, u_1, \ldots, u_p$ $(p \geq 1)$ belongs to $\mathbf{A}$, then the path $u_p, u_{p-1}, \ldots, u_0$ also belongs to $\mathbf{A}$.*

Axiom 3: *If the path $u_0, u_1, \ldots, u_p$ $(p \geq 1)$ belongs to $\mathbf{A}$, then the path $u_0, u_1, \ldots, u_{p-1}$ also belongs to $\mathbf{A}$.*

Axiom 4: *If $P^{(1)} : u_0, u_1, \ldots, u_p$ and $P^{(2)} : v_0, v_1, \ldots, v_q$ $(p, q \geq 1)$ are paths belonging to $\mathbf{A}$ such that $v_0 = u_i$ and $v_q = u_j$ for some pair i, j with $0 \leq i < j \leq p$, then the path*

$$P^{(3)} : u_0, u_1, \ldots, u_i = v_0, v_1, \ldots, v_q = u_j, u_{j+1}, \ldots, u_p$$

also belongs to $\mathbf{A}$.

Axiom 5: *If u and v are two distinct vertices of G, then there exist vertices w_0, w_1, …, w_k $(k \geq 1)$ in G such that $u = w_0$, $v = w_k$, and the sequence*

$$u = w_0, w_1, \ldots, v = w_k$$

also belongs to $\mathbf{A}$.

Axiom 6: *If $p \geq 2$ and the path $u_0, u_1, \ldots, u_p$ belongs to $\mathbf{A}$, then the sequence u_0, u_p does not belong to $\mathbf{A}$.*

Axiom 7: *If the paths*

$$\begin{aligned} &Q^{(1)} : v_0, u_0, u_1, \ldots, u_p, \\ &Q^{(2)} : u_0, v_0, v_1, \ldots, v_q, \quad \textit{and} \\ &Q^{(3)} : u_p, v_q, v_{q-1}, \ldots, v_0, \end{aligned}$$

all belong to $\mathbf{A}$, then the sequence

$$v_q, u_p, u_{p-1}, \ldots, u_0$$

also belongs to $\mathbf{A}$.

Proof. It is trivial to verify Axioms 1, 2, 3, 5, and 6. To verify Axiom 4, let $P^{(1)}$ and $P^{(2)}$ be two T-geodesics in G. Hence $p = j - i$. By Axioms 2 and 3,

$$P^* : u_i, u_{i+1}, \ldots, u_j$$

is also a T-geodesic in G. Since $P^{(1)}$ and $P^{(2)}$ are T-paths in G, the path $P^{(3)}$ is also a T-path in G. Hence $P^{(3)}$ is a T-geodesic in G as well.

To verify Axiom 7, let $Q^{(1)}$ and $Q^{(2)}$ be T-geodesics in G. Then $Q^{(3)}$ is not a T-path in G and so $Q^{(3)}$ is not a T-geodesic in G. ■

Remark. With the aid of Theorem 1 in [5], we have the following characterization of the set of all geodesics in a nontrivial connected graph: Let G be a nontrivial connected graph and let $\mathbf{A}$ be a subset of the set of all paths in G. Then $\mathbf{A} = \mathrm{geo}(G)$ if and only if $\mathbf{A}$ satisfies Axioms 1-7 in Theorem 2.9 as well as

Axiom 8: If $u_0, u_1, \ldots, u_k, v \in V(G)$, $k \geq 2$, and the paths

$$u_0, u_1, \ldots, u_k \quad \text{and} \quad u_k, v$$

belong to $\mathbf{A}$, then at least one of the statements (a), (b), and (c) holds:

(a) there exist $v_0, \ldots, v_p \in V(G)$, $p \geq 1$, such that $v_0 = u_0$, $v_p = v$, and the sequence $v_0, \ldots, v_p$, u_k belongs to $\mathbf{A}$,

(b) there exist $w_0, \ldots, w_q \in V(G)$, $q \geq 1$, such that $w_0 = u_1$, $w_q = v$, and the sequence $u_0, w_0, \ldots, w_q$ belongs to $\mathbf{A}$,

(c) the sequence $u_1, u_2, \ldots, u_k, v$ belongs to $\mathbf{A}$.

(Also, see [6] and [7].)

Let us compare properties of $\mathrm{geo}(G)$ and $\mathrm{geo}(G|T)$ for a nontrivial connected graph G and a spanning tree T of G. Although the following result is a consequence of Theorem 2.9 and Remark, we are able to present an alternative proof, which is independent of Theorem 2.9 and Remark.

Proposition 2.10. *Let G be a nontrivial connected graph and let T be a spanning tree of G. If* $\mathrm{geo}(G) \neq \mathrm{geo}(G|T)$*, then* $\mathrm{geo}(G|T)$ *does not satisfy Axiom* 8.

Proof. Assume that $\mathrm{geo}(G) \neq \mathrm{geo}(G|T)$. By Corollary 2.5, $\mathrm{geo}_2(G) \neq \mathrm{geo}_2(G|T)$. If $\mathrm{geo}_2(G) \subseteq \mathrm{geo}_2(G|T)$, then it is easy to see that $\mathrm{geo}_2(G) = \mathrm{geo}_2(G|T)$; a contradiction. Thus there exists a path $P : x, y, z$ of order 3 such that P belongs to $\mathrm{geo}(G) - \mathrm{geo}(G|T)$. It is clear that z does not belong to the $x-y$ path in T or x does not belong to the $y-z$ path in T. Without loss of generality, we assume that z does not belong to the $x-y$ path in T.

Put $k = d_{G|T}(x, z)$. By virtue of (4), $k \geq 2$. There exist $u_0, u_1, \ldots, u_k \in V(G)$ such that $u_0 = x$, $u_k = z$, and $Q : x = u_0, u_1, \ldots, u_k = z$ is a T-geodesic in G. Since $zy \in E(G)$, the path z, y of order two is also a T-geodesic in G. Since $xy \in E(G)$, no $x - y$ T-geodesic in G contains u_1. Obviously, P is not a T-path in G. It is easy to see that no $x - z$ T-geodesic in G contains y.

Assume that $R : u_1, \ldots, u_k = z, y$ is a T-geodesic in G. Then z belongs to $u_1 - y$ path in T. Since Q is a T-geodesic in G, it follows that u_1 belongs to the $x - z$ path in T. This implies that z belongs to the $x - y$ path in T, which is a contradiction. Thus R is not a T-geodesic in G. We see that $\text{geo}(G|T)$ does not satisfy Axiom 8. ■

Next we show that if G is a connected triangle-free graph and T and T^* are spanning trees of G, then $\text{geo}(G|T) = \text{geo}(G|T^*)$ only when $T = T^*$. To order to do this, we first present a lemma.

Lemma 2.11. *Let T be a spanning tree of a connected graph G and let u and v be adjacent vertices of G belonging to no triangle in G. Then $uv \notin E(T)$ if and only if there exist $k \geq 2$ vertices $u_1, u_2, \ldots, u_k$ of G such that both*

$$u, u_1, u_2, \ldots, u_k \quad \textit{and} \quad u_1, u_2, \ldots, u_k, v$$

are T-geodesics in G.

Proof. First suppose that $uv \notin E(T)$. Then T is a spanning tree of $G - uv$. Since uv does not belong to any triangle in G, every $u - v$ path in $G - uv$ has length at least 3. Let

$$P : \; u = u_0, u_1, u_2, \ldots, u_k, u_{k+1} = v,$$

be a $u - v$ T-geodesic P in $G - uv$, where $k \geq 2$. Then by Axioms 2 and 3 in Theorem 2.9, both

$$u = u_0, u_1, u_2, \ldots, u_k \text{ and } u_1, u_2, \ldots, u_k, u_{k+1} = v$$

are T-geodesics in G.

We now verify the converse. Assume, to the contrary, that $uv \in E(T)$. Then either u belongs to the $u_1 - v$ path in T or v belongs to the $u_1 - u$ path in T. In either case, u_{k-1} belongs to the $u_k - v$ path in T. Hence $u_{k-1}, u_k, u_{k+1} = v$ is not a T-path in G, which is a contradiction. Therefore, $uv \notin E(T)$. ■

Proposition 2.12. *Let G be a connected triangle-free graph and T and T^* be spanning trees of G. Then* $\mathrm{geo}(G|T) = \mathrm{geo}(G|T^*)$ *if and only if* $T = T^*$.

Proof. If $T = T^*$, then obviously $\mathrm{geo}(G|T) = \mathrm{geo}(G|T^*)$. It remains to verify the converse. Since G is triangle-free, it follows by Lemma 2.11 that $E(T)$ is determined by $\mathrm{geo}(G|T)$. Similarly, $E(T^*)$ is determined by $\mathrm{geo}(G|T^*)$. Thus if $\mathrm{geo}(G|T) = \mathrm{geo}(G|T^*)$, then $T = T^*$. ■

3. Realization Results

For every connected graph G, for every spanning tree T of G, and for every pair u, v of vertices of G,

(1) we have referred to three distances defined between u and v,

(2) we will see several distance parameters defined for G, and

(3) we will describe two induced subgraphs of G defined in terms of T.

In this section, we show that every three positive integers satisfying some expected conditions can be realized as the distances referred to in (1), combinations of numbers satisfying some necessary conditions can be realized as the values of the parameters referred to in (2), and that graphs, one of which satisfies some prescribed conditions, can be realized as the subgraphs referred to in (3). We begin with (1).

Proposition 3.1. *Let a, b, and c be positive integers with $a \le b \le c$. There exist a connected graph G, a spanning tree T of G, and two vertices u and v of G such that $d_G(u,v) = a$, $d_{G|T}(u,v) = b$, and $d_T(u,v) = c$ if and only if $a \ne 1$ or $b = 1$.*

Proof. Suppose first that $a = 1$ and $b \ne 1$. We have seen that for every connected graph G and a spanning tree T of G such that $d_G(u,v) = 1$ for $u, v \in V(G)$, we have $d_{G|T}(u,v) = 1$. Thus, there is no connected graph G with a spanning tree T and two vertices u and v for which $d_G(u,v) = 1$ and $d_{G|T}(u,v) \ne 1$.

For the converse, assume that $a \ne 1$ or $b = 1$. Suppose first that $b = 1$. Then $a = 1$. If $c = 1$, then $G = T = P_2$ with $V(G) = \{u, v\}$ has the desired properties. If $c \ne 1$, then let $G = C_{c+1}$, where $e = uv \in E(G)$, and let $T = G - e$, which have the desired properties. We now assume that $b \ne 1$. Thus $a \ge 2$. We consider four cases.

Case 1. $a = b = c$. Let $G = P_{a+1}$ and let u and v be the two end-vertices of G. Then $d_G(u, v) = d_{G|T}(u, v) = d_T(u, v)$.

Case 2. $2 \leq a < b < c$. Let G be the graph obtained from the path

$$P : u = u_0, u_1, \ldots, u_c = v \tag{5}$$

of length c by (i) adding a new vertex w and joining w to u_{a-2} and v, and (ii) adding the edge $u_{b-1}u_c$. Let T be the spanning tree of G obtained from P by adding the edge wu_{a-2}. Then $d_G(u, v) = a$, $d_{G|T}(u, v) = b$, and $d_T(u, v) = c$.

Case 3. $1 \leq a = b < c$. Let G be the graph obtained from the path P in (5) by adding the edge $u_{a-1}u_c$ and let $T = P$ be a spanning tree of G. Then $d_G(u, v) = d_{G|T}(u, v) = a$ and $d_T(u, v) = c$.

Case 4. $2 \leq a < b = c$. Let G be the graph obtained from the path P in (5) by adding a new vertex w and joining w to u_{a-2} and v. Let T be the spanning tree of G obtained from P by adding the edge wv. Thus $d_G(u, v) = a$ and $d_{G|T}(u, v) = d_T(u, v) = c$. ■

Let T be a spanning tree of a connected graph G. For $v \in V(G)$, the *T-eccentricity* $e_{G|T}(v)$ of v in G is defined as the T-distance from v to a vertex farthest from v. Define the *T-radius* $\mathrm{rad}_T(G)$ of G as

$$\mathrm{rad}_T(G) = \min\{e_{G|T}(v) : v \in V(G)\}$$

and the *T-diameter* $\mathrm{diam}_T(G)$ of G as

$$\mathrm{diam}_T(G) = \max\{e_{G|T}(v) : v \in V(G)\}.$$

For example, consider the graph G of Figure 5 and the spanning tree $T = G - u_3v - u_5v$ of G. Since $P : u_0, u_1, u_2, \ldots, u_6$ is a T-path of greatest length in G, it follows that $\mathrm{diam}_T(G) = e_{G|T}(u_0) = 6$. On the other hand, $e_{G|T}(v) = 2$ and $e_{G|T}(u_i) \geq 3$ for $0 \leq i \leq 6$. Thus $\mathrm{rad}_T(G) = 2$.

For the standard radius $\mathrm{rad}(G)$ and diameter $\mathrm{diam}(G)$ of a connected graph G, it is well-known that

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G).$$

As the example in Figure 5 illustrates, these same bounds do not hold for T-radius and T-diameter. Indeed, there are no restrictions on $\mathrm{rad}_T(G)$ and $\mathrm{diam}_T(G)$ other than $\mathrm{rad}_T(G) \le \mathrm{diam}_T(G)$.

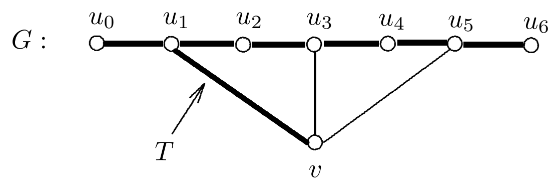

Figure 5. A graph G and a spanning tree T of G for which $\mathrm{rad}_T(G) = 2$ and $\mathrm{diam}_T(G) = 6$.

Proposition 3.2. *For each pair (r, d) of positive integers with $r \le d$, there exist a connected graph G and a spanning tree T of G such that*

$$\mathrm{rad}_T(G) = r \text{ and } \mathrm{diam}_T(G) = d.$$

Proof. We consider two cases.

Case 1. $1 \le r \le (d+1)/2$. Let G be the graph obtained from the $u - v$ path $P : u = u_0, u_1, \ldots, u_d = v$ of length d by adding a new vertex w and joining w to u_i for every i with $r - 1 \le i \le d$. Let T be the spanning tree of G obtained from P by adding the edge wu_{r-1}. Since $r \le \frac{d+1}{2}$, it follows that $d_{G|T}(u_{r-1}, u_d) = d - r + 1 \ge r$. Thus $e_{G|T}(w) = r$, $e_{G|T}(u) = e_{G|T}(v) = d$, and $r \le e_{G|T}(x) \le d$ for all $x \in V(G)$. Therefore, $\mathrm{rad}_T(G) = r$ and $\mathrm{diam}_T(G) = d$.

Case 2. $r \ge d/2$. We consider two subcases, according to whether $r = d$ or $r < d$.

Subcase 2.1. $r = d$. Let $S_{r+1}(P_{2r+1})$ be the $(r+1)$-step graph of P_{2r+1} obtained from the path $P_{2r+1} : x_0, x_1, \ldots, x_{2r}$ of order $2r + 1$ by adding an edge between every two vertices of P_{2r+1} whose distance is $r + 1$ in P_{2r+1}, and let $T = P_{2r+1}$. Observe that $e_{G|T}(v) = r$ for every vertex v of G. Thus $\mathrm{rad}_T(G) = \mathrm{diam}_T(G) = r$. In the case where $r = d = 4$, the graph $S_5(P_9)$ is shown in Figure 6.

Subcase 2.2. $d/2 \le r < d$. Let $d = r + k$, where $0 < k \le r$. Let G be the graph obtained from the graph $S_{r+1}(P_{2r+1})$ of Subcase 2.1 and the path $P'_k : y_1, y_2, \ldots, y_k$ of order k by adding the edge $x_{2r}y_1$. Let T be the spanning path x_0, x_1, …, x_{2r}, y_1, y_2, …, y_k in G.

Since $r \geq d/2$, it follows that $e_{G|T}(x_{2r}) = r$, $e_{G|T}(y_k) = r + k = d$, and $r \leq e_{G|T}(z) \leq d$ for all $z \in V(G)$. Therefore, $\mathrm{rad}_T(G) = r$ and $\mathrm{diam}_T(G) = d$, as desired. ■

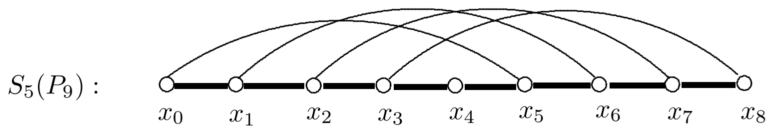

Figure 6. The graph $S_5(P_9)$ and a spanning tree T of $S_5(P_9)$ with $\mathrm{rad}_T(S_5(P_9)) = \mathrm{diam}_T(S_5(P_9)) = 4$ in Subcase 2.1.

Let G be a connected graph and T a spanning tree of G. For a vertex v of G, it follows by (4) that

$$d_G(v, x) \leq d_{G|T}(v, x) \leq d_T(v, x)$$

for all $x \in V(G)$. Thus

$$e_G(v) \leq e_{G|T}(v) \leq e_T(v)$$

for every vertex v of G. Therefore,

$$\mathrm{rad}(G) \leq \mathrm{rad}_T(G) \leq \mathrm{rad}(T)$$

and

$$\mathrm{diam}(G) \leq \mathrm{diam}_T(G) \leq \mathrm{diam}(T).$$

Next, we determine all positive integers a, b, c with $a \leq b \leq c$ that are realizable as $\mathrm{rad}(G)$, $\mathrm{rad}_T(G)$, and $\mathrm{rad}(T)$ (and $\mathrm{diam}(G)$, $\mathrm{diam}_T(G)$, and $\mathrm{diam}(T)$), respectively, for some connected graph G and a spanning tree T of G.

Theorem 3.3. *For every three positive integers a, b, c with $a \leq b \leq c$, there exists a connected graph G and a spanning tree T of G such that*

$$\mathrm{rad}(G) = a,\ \mathrm{rad}_T(G) = b,\ \textit{and}\ \mathrm{rad}(T) = c.$$

Proof. We consider four cases.

Case 1. $a = b = c$. Let $G = T = P_{2a+1}$. Then $\text{rad}(G) = \text{rad}_T(G) = \text{rad}(T) = a$.

Case 2. $a = b < c$. Let T be the tree obtained by adding the edge ww_1 to the two paths $P_a : w_1, w_2, \ldots, w_a$ and

$$P_{2c+1}\colon\ u_c,\ u_{c-1},\ \ldots,\ u_1,\ w,\ v_1,\ v_2,\ \ldots,\ v_c.$$

Let G be the graph obtained by adding the edges u_iw and v_iw to T for $i \geq a+1$. Then $e_G(w) = a$ and $e_G(x) \geq a$ for all $x \in V(G)$, implying that $\text{rad}(G) = a$. Then $\text{rad}(T) = c$. Since $e_{G|T}(w) = a$ and $e_{G|T}(x) \geq a$ for all $x \in V(G)$, it follows that $\text{rad}_T(G) = a$.

Case 3. $a < b = c$. Let $k = b - a \geq 1$. Consider the graph $H = P_{2k+1} + K_1$, where $P_{2k+1} : u_0, u_1, \ldots, u_{2k}$ and $V(K_1) = \{v\}$. Let G be the graph obtained from H by subdividing each edge u_iv a total of $a-1$ times. Suppose that we insert $a-1$ vertices $u_{i,1}$, $u_{i,2}$, $\ldots$, $u_{i,a-1}$ of degree 2 into each edge u_iv for $0 \leq i \leq 2k$ such that u_i, $u_{i,1}$, $u_{i,2}$, $\ldots$, $u_{i,a-1}$, v is a path for each i $(0 \leq i \leq 2k)$. Then $T = G - \{u_{i,a-1}v :\ 1 \leq i \leq 2k\}$ is a spanning tree of G. The graph G and the spanning path T of G are shown in Figure 7 for $a = 3$ and $b = 5$ (where $k = 2$).

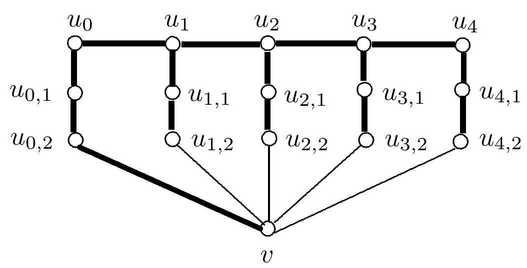

Figure 7. The graph G and a spanning tree T of G with $\text{rad}(G) = 3$ and $\text{rad}_T(G) = \text{rad}(T) = 5$ in Case 3.

Since $e_G(v) = a$ and $e_G(x) \geq a$ for all $x \in V(G)$, it follows that $\text{rad}(G) = a$. Since $e_{G|T}(u_k) = k+(a-1)+1 = k+a = b$ and $e_{G|T}(x) \geq b$ for all $x \in V(G)$, it follows that $\text{rad}_T(G) = b$. Furthermore, $e_T(u_k) = b$ and $e(x) \geq b$ for all $x \in V(T)$, it follows that $\text{rad}(T) = b$.

Case 4. $a < b < c$. Let $b - a = k$. We start with the graph G in Case 3. Let G' be the graph obtained from G by replacing the vertex u_k by the complete graph K_{2c+1} and joining each vertex of K_{2c+1} to every vertex in the neighborhood $N_G(u_k)$ of u_k in G. Let

$$V(K_{2c+1}) = \{y_c, y_{c-1}, \ldots, y_1, x, z_1, z_2, \ldots, z_c\}$$

and let T' be the spanning tree obtained from the spanning tree T of G and the path

$$P : y_c, y_{c-1}, \ldots, y_1, x, z_1, z_2, \ldots, z_c$$

by identifying u_k in T and x in P. Observe that $e_{G'}(v) = a$ and $e_{G'}(u) \geq a$ for all $u \in V(G)$, implying that $\mathrm{rad}(G') = a$. Since $e_{G'|T'}(z_i) = k + a = b$ for $1 \leq i \leq \ell+1$ and $e_{G'|T'}(u) \geq a$ for all $u \in V(G)$, it follows that $\mathrm{rad}_{T'}(G') = b$. Furthermore, $e_{T'}(x) = c$ and $e_{T'}(u) \geq c$ for all $u \in V(T')$, implying that $\mathrm{rad}(T') = c$. ■

Theorem 3.4. *For every three positive integers* a, b, c *with* $a \leq b \leq c$, *there exists a connected graph* G *and a spanning tree* T *of* G *such that*

$$\mathrm{diam}(G) = a,\ \mathrm{diam}_T(G) = b,\ \textit{and}\ \mathrm{diam}(T) = c$$

if and only if $a \neq 1$ *or* $b = 1$.

Proof. Let G be a connected graph and let T be a spanning tree of G such that $\mathrm{diam}(G) = a$, $\mathrm{diam}_T(G) = b$, and $\mathrm{diam}(T) = c$. Then $1 \leq a \leq b \leq c$. If $a = 1$, then G is complete and so $\mathrm{diam}_T(G) = 1$ for every spanning tree T of G. Thus $b = 1$.

For the converse, let a, b, c be positive integers with $a \leq b \leq c$. Suppose first that $b = 1$. Let $G = K_n$ for some integer $n \geq c + 1$. Then $\mathrm{diam}(G) = \mathrm{diam}_T(G) = 1$ for every spanning tree T of G. Since $n \geq c + 1$, there exists a tree of order n with diameter c and so G contains a spanning tree with diameter c. Therefore, $G = K_n$ has the desired property. Hence we may assume that $a \neq 1$. Thus $2 \leq a \leq b \leq c$. We consider four cases.

Case 1. $a = b = c$. Let $G = T = P_{a+1}$. Then $\mathrm{diam}(G) = \mathrm{diam}_T(G) = \mathrm{diam}(T) = a$.

Case 2. $2 \leq a = b < c$. Let G be the graph obtained from the path P_{c+1}: v_0, v_1, …, v_c of order $c + 1$ by adding all edges v_iv_j with $a - 1 \leq i < j \leq c$

and let $T = P_{c+1}$. Then $\mathrm{diam}(G) = a$ and $\mathrm{diam}(T) = c$. Since $e_{G|T}(v_0) = a$ and $e_{G|T}(x) \le a$ for all $x \in V(G)$, it follows that $\mathrm{diam}_T(G) = a$.

Case 3. $2 \le a < b = c$. Let $k = b - a \ge 1$. We start with the wheel $C_{k+4} + K_1$, where $C_{k+4} : u_0, u_1, \ldots, u_{k+3}, u_0$ and $V(K_1) = \{w\}$. The graph $G = C_{k+4} + K_1$ if $a = 2$, while if $a \ge 3$, then G is obtained from $C_{k+4} + K_1$ and the path $P_{a-1} : v_0, v_1, \ldots, v_{a-2}$ of order $a-1$ by identifying the vertices u_0 and v_0. Let

$$T = G - (\{wu_i : \ 0 \le i \le k+2\} \cup \{u_{k+2}u_{k+3}\})$$

be a spanning tree of G. The graph G and the spanning tree T of G are shown in Figure 8 for $a = 5$ and $b = 9$ (for $k = 4$).

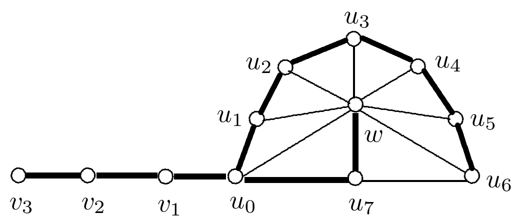

Figure 8. The graph G and a spanning tree T of G with $\mathrm{diam}(G) = 5$ and $\mathrm{diam}_T(G) = \mathrm{diam}(T) = 9$.

Since $e_G(v_{a-2}) = a$ and $e_G(x) \le a$ for all $x \in V(G)$, it follows that $\mathrm{diam}(G) = a$. Because $e_{G|T}(v_{a-2}) = (a-2) + (k+2) = a + k = b$ and $e_{G|T}(x) \le b$ for all $x \in V(G)$, it follows that $\mathrm{diam}_T(G) = b$. Furthermore, $e_T(v_{a-2}) = b$ and $e_T(x) \le b$ for all $x \in V(T)$. Thus $\mathrm{diam}(T) = b$.

Case 4. $2 \le a < b < c$. Let $b - a = k$ and $c - b = \ell$. We consider two subcases.

Subcase 4.1. $a = 2$. We construct the graph G from the wheel $C_{k+4}+K_1$, where $C_{k+4} : u_0, u_1, \ldots, u_{k+3}, u_0$ and $V(K_1) = \{w\}$, by replacing u_0 by the complete graph $K_{\ell+1}$ and joining each vertex of $K_{\ell+1}$ to every vertex in the neighborhood of u_0 in $C_{k+4} + K_1$. Let $V(K_{\ell+1}) = \{z_1, z_2, \ldots, z_{\ell+1}\}$ and let T be the spanning tree of G obtained from the path

$$z_1, z_2, \ldots, z_{\ell+1}, u_1, u_2, \ldots, u_{k+2}, w$$

by adding the edge $z_{\ell+1}u_{k+3}$. Then $\mathrm{diam}(G) = 2$. Since $e_{G|T}(z_i) = k+2 = b$ $(1 \le i \le \ell+1)$ and $e_{G|T}(x) \le b$ for all $x \in V(G)$, it follows that $\mathrm{diam}_T(G) = b$. Furthermore, $e_T(z_1) = b + \ell = c$ and $e_T(x) \le c$ for all $x \in V(T)$. Thus $\mathrm{diam}(T) = c$.

Subcase 4.2. $a \ge 3$. In this case, we start with the graph G in Case 3 and construct the graph G' from G by replacing the vertex v_{a-2} by the complete graph $K_{\ell+1}$ and joining each vertex of $K_{\ell+1}$ to the vertex in $N_G(v_{a-2})$ (see Figure 9 for $\ell = 3$, $a = 5$, and $b = 9$).

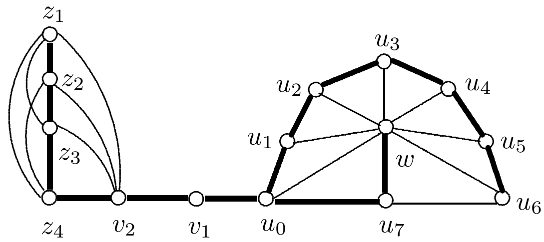

Figure 9. The graph G' and a spanning tree T' of G' with $\mathrm{diam}(G') = 5$, $\mathrm{diam}_{T'}(G') = 9$, and $\mathrm{diam}(T') = 12$.

Let $V(K_{\ell+1}) = \{z_1, z_2, \ldots, z_{\ell+1}\}$ and let T' be the spanning tree of G' obtained from the spanning tree T (with v_{a-2} being relabeled as $z_{\ell+1}$) of G in Case 3 and the path $P : z_1, z_2, \ldots, z_\ell$ by adding the edge $z_\ell z_{\ell+1}$. Then $e_{G'}(z_i) = a$ $(1 \le i \le \ell + 1)$ and $e_{G'}(x) \le a$ for all $x \in V(G')$, implying that $\mathrm{diam}(G') = a$. Since $e_{G'|T'}(z_{\ell+1}) = b$ and $e_{G'|T'}(x) \le b$, it follows that $\mathrm{diam}_{T'}(G') = b$. Furthermore, $e_{T'}(z_1) = b + \ell = c$ and $e_{T'}(x) \le c$ for all $x \in V(T')$, implying that $\mathrm{diam}(T') = c$. ■

A vertex v of a connected graph G with a spanning tree T is a *T-central vertex* of G if $e_{G|T}(v) = \mathrm{rad}_T(G)$. By the *$T$-center* $\mathrm{Cen}_T(G)$ of G, we mean the subgraph of G induced by its T-central vertices. Harary and Norman [4] proved, for standard distance in graphs, that the center of every connected graph G lies in a single block of G. This is true for T-distance as well.

Theorem 3.5. *Let T be a spanning tree of a nontrivial connected graph G. Then the T-center of G lies in a single block of G.*

Proof. Assume, to the contrary, that there exists a nontrivial connected graph G and a spanning tree T of G such that two T-central vertices of G

lies in distinct blocks of G. Then G contains a cut-vertex v such that two T-central vertices of G lies in distinct components of $G - v$. Let u be a vertex of G such that $d_{G|T}(u, v) = e_{G|T}(v)$ and let P'' be a $u - v$ T-geodesic in G. Then some component G' of $G - v$ contains a T-central vertex w but contains no vertices of P''. Let P' be a $w - v$ T-geodesic in G. By Lemma 2.7, the path P obtained from P' followed by P'' is a T-geodesic in G. Thus $e_{G|T}(v) > e_{G|T}(w)$, producing a contradiction. ∎

Hedetniemi (see [2]) showed that every graph is the center of some connected graph. This result is now extended to T-centers.

Theorem 3.6. *For every graph G, there exists a connected graph H and a spanning tree T of H such that* $\mathrm{Cen}(H) = \mathrm{Cen}_T(H) = G$.

Proof. First, add two new vertices u and v to G and join them to every vertex of G but not to each other. Next, we add two additional vertices u_1 and v_1, where we join u_1 to u and join v_1 to v. The resulting graph is denoted by H, as shown in Figure 10. By the proof of the Hedetniemi theorem, $\mathrm{Cen}(H) = G$. Let T be the spanning tree of H shown in Figure 10 whose edges are indicated in bold. Since $e_{H|T}(u_1) = e_{H|T}(v_1) = 4$, $e_{H|T}(u) = e_{H|T}(v) = 3$, and $e_{H|T}(x) = 2$ for every vertex x in G, it follows that $V(G)$ is the set of T-central vertices of H and so $\mathrm{Cen}_T(H) = G$ as well. ∎

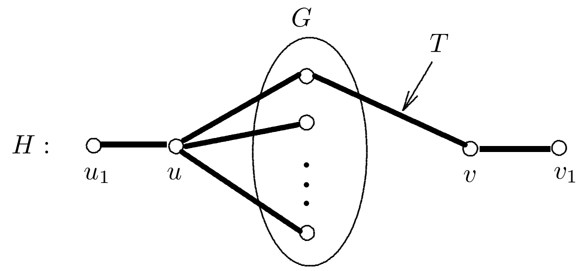

Figure 10. The graph H in the proof of Proposition 3.6.

A vertex v of a connected graph G with a spanning tree T is a *T-peripheral vertex* of G if $e_{G|T}(v) = \mathrm{diam}_T(G)$. By the *$T$-periphery* $\mathrm{Per}_T(G)$ of G, we mean the subgraph of G induced by its T-peripheral vertices. Bielak and Sysło [1] showed that a nontrivial graph G is the periphery of some connected graph if and only if every vertex of G has eccentricity 1 or no vertex of G

has eccentricity 1. We next show that this theorem can be extended to T-peripheries.

Theorem 3.7. *Let G be a nontrivial graph. Then the following statements are equivalent:*

(1) *Every vertex of G has eccentricity* 1 *or no vertex of G has eccentricity* 1*.*

(2) *There exists a connected graph F such that* $\mathrm{Per}(F) = G$*.*

(3) *There exists a connected graph H and a spanning tree T of H such that* $\mathrm{Per}(H) = \mathrm{Per}_T(H) = G$*.*

Proof. That (1) and (2) are equivalent is a restatement of the theorem of Bielak and Sysło. We show that (1) and (3) are equivalent.

Assume first that every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1. If every vertex of G has eccentricity 1, then G is complete and $\mathrm{Per}(G) = G$. Furthermore, $\mathrm{Per}_T(G) = G$ for every spanning tree T of G. Now assume that *no* vertex of G has eccentricity 1. This implies that for every vertex u of G, there is a vertex v in G that is not adjacent to u. Let H be the graph obtained by adding a new vertex w and joining w to every vertex of G. It is a consequence of the proof of the Bielak-Sysło theorem that $\mathrm{Per}(H) = G$. Let T be the spanning tree of H that is the star with central vertex w. Then $e_{H|T}(w) = 1$. Since $e_{H|T}(x) = 2$ for every vertex x of G, it follows that every vertex of G is a T-peripheral vertex of H and so $\mathrm{Per}_T(H) = G$ as well. Hence (1) implies (3).

We now show that (3) implies (1). Let G be a graph that contains some vertices of eccentricity 1 and some vertices whose eccentricity is not 1. It is a consequence of Bielak-Sysło theorem that $\mathrm{Per}(H^*) \neq G$ for every connected graph H^*. It remains to show that for every connected graph H' and every spanning tree T' of H', $\mathrm{Per}_{T'}(H') \neq G$. Assume, to the contrary, that there exists a connected graph H and a spanning tree T of H such that $\mathrm{Per}_T(H) = G$. Necessarily, G is an induced subgraph of H. Furthermore, G is a proper subgraph of H since for every vertex y of G, $e_{H|T}(y) = 1$ if and only if $e_G(y) = 1$. Thus there exists an integer $k \geq 2$ such that $e_{H|T}(v) = k$ for every vertex v of G; while $e_{H|T}(v) < k$ for every vertex v of H that is not in G. Let x be a vertex of G such that $e_G(x) = 1$ and let w be a vertex of H such that $d_{H|T}(x, w) = e_{H|T}(x) = k \geq 2$. Since w is not adjacent to x, it follows that w is not in G. However, $d_{H|T}(w, x) = k$ and so $e_{H|T}(w) \geq k$, which implies that $e_{H|T}(w) = k$. This, however, contradicts the fact that w is not in the T-periphery of H. ■

Acknowledgments

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