

ON (k, l) -KERNELS IN D -JOIN OF DIGRAPHS

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Abstract

In [5] the necessary and sufficient conditions for the existence of (k, l) -kernels in a D -join of digraphs were given if the digraph D is without circuits of length less than k . In this paper we generalize these results for an arbitrary digraph D . Moreover, we give the total number of (k, l) -kernels, k -independent sets and l -dominating sets in a D -join of digraphs.

Keywords: (k, l) -kernel, k -independent set, l -dominating set, D -join, counting.

2000 Mathematics Subject Classification: 05C20.

1. INTRODUCTION

For concepts not defined here see [2]. Let D be a finite, directed graph (for short: a digraph) without loops and multiple arcs, where $V(D)$ is the set of vertices and $A(D)$ is the set of arcs of D . By a *path* from a vertex x_1 to a vertex x_n , $n \geq 2$, we mean a sequence of vertices $x_1, \dots, x_n$ and arcs $(x_i, x_{i+1}) \in A(D)$ for $i = 1, 2, \dots, n-1$ and for simplicity we denote it by $x_1 \dots x_n$. A *circuit* is a path with $x_1 = x_n$. By $d_D(x_i, x_j)$ we denote the length of the shortest path from x_i to x_j in D . If there does not exist a path from x_i to x_j in D , then we put $d_D(x_i, x_j) = \infty$. For any $X \subseteq V(D)$ and $x \in (V(D) \setminus X)$ we put $d_D(x, X) = \min_{y \in X} d_D(x, y)$. By $\mathcal{C}_D^{\eta \leq d \leq \mu}(x_i)$

we denote the family of all circuits in D containing the vertex x_i of length d , where $\eta \leq d \leq \mu$.

We say that a subset $J \subset V(D)$ is a (k, l) -kernel of D if

- (1) for each $x_i, x_j \in J$ and $i \neq j$, $d_D(x_i, x_j) \geq k$ and
- (2) for each $x_i \notin J$ there exists $x_j \in J$ such that $d_D(x_i, x_j) \leq l$.

If the set J satisfies the condition in (1) or in (2), then we shall call it a k -independent set of D (also called a k -stable set of D) or an l -dominating set of D , respectively. We notice that a 2-independent set is an independent set and a 1-dominating set is a dominating set of D . In addition, we assume that a subset containing only one vertex and an empty set is also meant as a k -independent set. The set $V(D)$ is an l -dominating set of D . If an l -dominating set of D has exactly one vertex, then this vertex we shall call an l -dominating vertex of D . Moreover, the l -dominating vertex of D is also a (k, l) -kernel of D for every $k \geq 2$. A digraph D whose every induced subdigraph has a (k, l) -kernel is called a (k, l) -kernel perfect digraph. Sufficient conditions for the existence of kernels and (k, l) -kernels in digraphs have been investigated, for instance in [1, 3, 4, 5]. By $NkI(D)$, $NlD(D)$ and $NklK(D)$ we mean the number of all k -independent sets, l -dominating sets and (k, l) -kernels of the digraph D , respectively. Moreover, by $Nld(D)$ we will denote the number of all l -dominating vertices of D . The total number of k -independent sets and (k, l) -kernels in graphs and in some their products were studied in [6] and [8].

Let D be a digraph with $V(D) = \{x_1, \dots, x_n\}$, $n \geq 2$ and $\alpha = (D_i)_{i \in \{1, \dots, n\}}$ be a sequence of vertex disjoint digraphs on $V(D_i) = \{y_1^i, \dots, y_{p_i}^i\}$, $p_i \geq 1$, $i = 1, \dots, n$. The D -join of the digraph D and the sequence α is a digraph $\sigma(\alpha, D)$ such that $V(\sigma(\alpha, D)) = \bigcup_{i=1}^n (\{x_i\} \times V(D_i))$ and $A(\sigma(\alpha, D)) = \{((x_s, y_j^s), (x_q, y_t^q)) : x_s = x_q \text{ and } (y_j^s, y_t^s) \in A(D_s) \text{ or } (x_s, x_q) \in A(D)\}$. By D_i^c we mean a copy of the digraph D_i in $\sigma(\alpha, D)$.

It may be noted that if all digraphs from the sequence α have the same vertex set, then from the D -join we obtain the *generalized lexicographic product* of the digraph D and the sequence of the digraphs D_i , i.e., $\sigma(\alpha, D) = D[D_1, \dots, D_n]$. If all digraphs from the sequence α are isomorphic to the same digraph H , then from the D -join we obtain the *composition* $D[H]$ of the digraphs D and H .

The existence of (k, l) -kernels in the lexicographic product $D[D_1, \dots, D_n]$ was studied in [7]. Moreover, in [8] the total number of k -independent sets of a lexicographic product of graphs were determined using the concept of

the Fibonacci polynomial of graphs. In [5] the necessary and sufficient conditions for the existence of (k, l) -kernels in D -join were given, where D is a digraph without circuits of length less than k . It was proved:

Theorem 1 [5]. *Let D be a digraph without circuits of length less than k . A subset $S^* \subset V(\sigma(\alpha, D))$ is a k -independent set of $\sigma(\alpha, D)$ if and only if there exists a k -independent set $S \subset V(D)$ such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i; x_i \in S\}$, $S_i \subseteq V(D_i^c)$ and S_i is a k -independent set of D_i^c for every $i \in \mathcal{I}$.*

Theorem 2 [5]. *Let $Q \subseteq V(D)$, $\mathcal{I} = \{i : x_i \in Q\}$ and $Q_i \subseteq V(D_i)$. If Q is an l -dominating set of D and Q_i is an l -dominating set of D_i^c for every $i \in \mathcal{I}$, then $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$ is an l -dominating set of $\sigma(\alpha, D)$.*

Theorem 3 [5]. *Let $k \geq 2$, $l \leq k - 1$ be integers. Let D be a digraph without circuits of length less than k . The subset J^* is a (k, l) -kernel of the $\sigma(\alpha, D)$ if and only if there exists a (k, l) -kernel $J \subseteq V(D)$ of the digraph D such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and J_i is a (k, l) -kernel of D_i^c for every $i \in \mathcal{I}$.*

In this paper, we generalize these results for an arbitrary digraph D . Moreover, we determine the total number of k -independent sets, l -dominating sets and (k, l) -kernels in $\sigma(\alpha, D)$.

2. THE EXISTENCE OF (k, l) -KERNELS IN D -JOIN

In this section, we give the necessary and sufficient conditions for the existence of (k, l) -kernels in D -join if D is an arbitrary digraph on n , $n \geq 2$ vertices and $\alpha = (D_i)_{i \in \{1, \dots, n\}}$ is an arbitrary sequence of vertex disjoint digraphs on p_i , $p_i \geq 1$ vertices.

Theorem 4. *Let $(x_i, y_p^i), (x_j, y_q^j) \in V(\sigma(\alpha, D))$. Then*

$$d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) = \begin{cases} d_D(x_i, x_j) & \text{for } i \neq j, \\ \min\{d_{D_i}(y_p^i, y_q^i), d_D(x_i)\} & \text{for } i = j, \end{cases}$$

where $d_D(x_i)$ denotes the length of the shortest circuit containing the vertex x_i in D .

Proof. Assume that $(x_i, y_p^i), (x_j, y_q^j)$ are two different vertices of $V(\sigma(\alpha, D))$ and distinguish two possible cases:

1. $i \neq j$. Then the theorem follows immediately from the definition of $\sigma(\alpha, D)$.

2. $i = j$. Using the definition of $\sigma(\alpha, D)$ we have that there exists a path from (x_i, y_p^i) to (x_i, y_q^i) in $\sigma(\alpha, D)$ of the same length as the path from y_p to y_q in D_i . Moreover, if there exists a circuit in D which includes a vertex x_i , then by the definition of $\sigma(\alpha, D)$ it follows that there also exists a path from (x_i, y_p^i) to (x_i, y_q^i) of length $d_D(x_i)$ equal to the length of the shortest circuit in D , which includes a vertex x_i . Otherwise, if there does not exist a circuit in D which includes a vertex x_i , then we put $d_D(x_i) = \infty$. Evidently $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) = \min\{d_{D_i}(y_p, y_q), d_D(x_i)\}$.

Thus the theorem is proved. ■

Theorem 5. *A subset $S^* \subset V(\sigma(\alpha, D))$ is a k -independent set of $\sigma(\alpha, D)$ if and only if $S \subset V(D)$ is a k -independent set of D such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i : x_i \in S\}$, $S_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

- (a) S_i is a k -independent set of D_i^c if $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$ or
- (b) S_i is 1-element set containing an arbitrary vertex from $V(D_i^c)$, otherwise.

Proof. I. Let S^* be a k -independent set of the D -join $\sigma(\alpha, D)$. Denote $S = \{x_i \in V(D) : S^* \cap V(D_i^c) \neq \emptyset\}$. First, we shall prove that S is a k -independent set of D . Let $x_i, x_j \in S$ be two different vertices. Then by the definition of the set S there exist $1 \leq r \leq p_i$ and $1 \leq s \leq p_j$ such that $(x_i, y_r^i), (x_j, y_s^j) \in S^*$. By Theorem 4 and from the assumption of the set S^* we obtain that $d_D(x_i, x_j) = d_{\sigma(\alpha, D)}((x_i, y_r^i), (x_j, y_s^j)) \geq k$. The definition of the set S implies that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i : x_i \in S\}$. We consider the following cases.

I.1. Let $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$.

Because S^* is k -independent so by the definition of $\sigma(\alpha, D)$ and by assumption it follows immediately that S_i is a k -independent set of D_i^c .

I.2. Let $\mathcal{C}_D^{d \leq k-1}(x_i) \neq \emptyset$.

We shall prove that S_i contains exactly one arbitrary vertex from $V(D_i^c)$. By Theorem 4 we obtain that for arbitrary two vertices from $V(D_i^c)$ the distance between them in $\sigma(\alpha, D)$ is less than k . Consequently, S_i must contain exactly one arbitrary vertex from $V(D_i^c)$.

Hence from the above cases we obtain that S_i is a k -independent set of D_i^c if there does not exist in D a circuit containing x_i of length less than k or S_i contains exactly one arbitrary vertex from $V(D_i^c)$, otherwise.

II. Let $S \subset V(D)$ be a k -independent set of the digraph D . Let $\mathcal{I} = \{i : x_i \in S\}$ and let S_i be as in the assumption. We shall prove that $S^* = \bigcup_{i \in \mathcal{I}} S_i$ is a k -independent set of the D -join $\sigma(\alpha, D)$. Let $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ be two distinct vertices. Consider the possible cases:

II.1. $(x_i, y_p^i) \in S_i$ and $(x_j, y_q^j) \in S_j$, where $i \neq j$.

Since S is k -independent in D , so by Theorem 4 it follows that $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) = d_D(x_i, x_j) \geq k$.

II.2. $(x_i, y_p^i), (x_i, y_q^i) \in S_i$, where $p \neq q$ for some $i \in \mathcal{I}$.

Since S_i contains at least two vertices, so by the assumption, S_i is k -independent of D_i^c and $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$. To prove that S^* is a k -independent set of $\sigma(\alpha, D)$ assume on the contrary that $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_i, y_q^i)) < k$. If $k = 2$, then a contradiction with the independence of S_i in D_i^c . Let $k \geq 3$. This means that there exists a path $(x_i, y_p^i) \dots (x_i, y_q^i)$ in $\sigma(\alpha, D)$ of length less than k such that at least one inner vertex of this path does not belong to $V(D_i^c)$. Hence there exists in D a circuit containing the vertex x_i of length less than k , a contradiction to the assumption.

Taking the two above cases into considerations we obtain that for distinct $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ there holds $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) \geq k$, hence S^* is a k -independent set of $\sigma(\alpha, D)$.

Thus the theorem is proved. ■

If D is a digraph without circuits of length less than k , then we obtain Theorem 1.

Theorem 6. *A subset $Q^* \subseteq V(\sigma(\alpha, D))$ is an l -dominating set of $\sigma(\alpha, D)$ if and only if $Q \subseteq V(D)$ is an l -dominating set of D such that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$, $Q_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

- (a) Q_i is an l -dominating set of D_i^c if $\mathcal{C}_D^{d \leq l}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$, there holds $d_D(x_i, x_j) > l$ or
- (b) Q_i is an arbitrary nonempty subset of $V(D_i^c)$, otherwise.

Proof. I. Let Q^* be an l -dominating set of the D -join $\sigma(\alpha, D)$. Denote $Q = \{x_i \in V(D); Q^* \cap V(D_i^c) \neq \emptyset\}$. First, we shall prove that Q is an

l -dominating set of D . Let $x_j \notin Q$. By the definition of the set Q we have that for each $1 \leq r \leq p_j$ there holds $(x_j, y_r^j) \notin Q^*$. Since Q^* is l -dominating so there exists $(x_i, y_s^i) \in Q^*$, where $i \neq j$ such that $d_{\sigma(\alpha, D)}((x_j, y_r^j), (x_i, y_s^i)) \leq l$. Evidently, $x_i \in Q$, so using Theorem 4 we obtain that $d_D(x_j, x_i) = d_{\sigma(\alpha, D)}((x_j, y_r^j), (x_i, y_s^i)) \leq l$. The definition of the set Q implies that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i; x_i \in Q\}$. Consider the following cases:

I.1. Assume that $\mathcal{C}_D^{d \leq l}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$.

Since Q^* is l -dominating so from the definition of $\sigma(\alpha, D)$ and by our assumptions immediately follows that Q_i is an l -dominating set of D_i^c .

I.2. Assume that case I.1 does not hold.

We shall prove that Q_i is an arbitrary nonempty subset of D_i^c . If $\mathcal{C}_D^{d \leq l}(x_i) \neq \emptyset$, then there exists in D a circuit which includes the vertex x_i of length less than or equal to l . So for arbitrary two vertices $(x_i, y_q^i), (x_i, y_p^i) \in V(D_i^c)$ there holds $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_i, y_q^i)) \leq l$. If there exists $j \in \mathcal{I}$ and $j \neq i$ such that there exists in D a path $x_i \dots x_j$ of length less than or equal to l , then for an arbitrary vertex $(x_i, y_p^i) \in V(D_i^c)$ holds $d_{\sigma(\alpha, D)}((x_i, y_p^i), Q_j) \leq l$. Hence $d_{\sigma(\alpha, D)}((x_i, y_p^i), Q^*) \leq l$. All this implies that Q_i is an arbitrary nonempty subset of $V(D_i^c)$.

II. Let $Q \subseteq V(D)$ be an l -dominating set of the digraph D , where $\mathcal{I} = \{i : x_i \in Q\}$ and let Q_i be as in the theorem. We shall prove that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$ is an l -dominating set of the D -join. We distinguish the following cases:

II.1. Let $(x_j, y_p^j) \notin Q^*$ and $j \notin \mathcal{I}$.

Then by the definition of the set Q we have that $x_j \notin Q$. Since Q is an l -dominating set of D , so there exists $i \in \mathcal{I}$ such that $x_i \in Q$ and $d_D(x_j, x_i) \leq l$. Hence there is $1 \leq q \leq p_i$ such that $(x_i, y_q^i) \in Q^*$. By Theorem 4 we obtain that $d_{\sigma(\alpha, D)}((x_j, y_p^j), Q^*) \leq l$.

II.2. Let $(x_j, y_p^j) \notin Q^*$ and $j \in \mathcal{I}$.

If Q_j is an l -dominating set of D_j^c , then $d_{D_j^c}((x_j, y_p^j), Q_j) \leq l$. So $d_{\sigma(\alpha, D)}((x_j, y_p^j), Q^*) \leq l$. If Q_j is a nonempty subset of $V(D_j^c)$, then from the assumption of the theorem we have that there exists $t \in \mathcal{I}$ and $t \neq j$ such that there exists a path $x_j \dots x_t$ in D of length less than or equal to l or $\mathcal{C}_D^{d \leq l}(x_j) \neq \emptyset$. Consequently, $d_{\sigma(\alpha, D)}((x_j, y_p^j), Q_t) \leq l$ or

$d_{\sigma(\alpha, D)}((x_j, y_p^j), Q_j) \leq l$, respectively. Hence $d_{\sigma(\alpha, D)}((x_j, y_p^j), Q^*) \leq l$, so Q^* is an l -dominating set of $\sigma(\alpha, D)$.

Thus the theorem is proved. $\blacksquare$

Theorem 7. *Let $k \geq 2$, $1 \leq l \leq k - 1$ be integers. The subset $J^* \subset V(\sigma(\alpha, D))$ is a (k, l) -kernel of the D -join $\sigma(\alpha, D)$ if and only if there exists a (k, l) -kernel $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

- (a) J_i is a (k, l) -kernel of D_i^c if $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$ or
- (b) J_i is 1-element set containing an arbitrary vertex of $V(D_i^c)$ if $\mathcal{C}_D^{d \leq l}(x_i) \neq \emptyset$ or
- (c) J_i is 1-element set containing an l -dominating vertex of D_i^c , otherwise.

Proof. I. Let $k \geq 2$, $1 \leq l \leq k - 1$ be integers. Let J^* be a (k, l) -kernel of the D -join $\sigma(\alpha, D)$. Denote $J = \{x_i \in V(D); J^* \cap V(D_i^c) \neq \emptyset\}$. First, we shall prove that J is a (k, l) -kernel of D . Let $x_i, x_j \in J$ and $i \neq j$. Then from the definition of the set J we have that there exists $1 \leq p \leq p_i$ and $1 \leq q \leq p_j$ such that $(x_i, y_p^i), (x_j, y_q^j) \in J^*$. By Theorem 4 we have that $d_D(x_i, x_j) = d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) \geq k$. So, J is a k -independent set of D . Now, we will show that J is an l -dominating set of D . Let $x_j \notin J$. Using the definition of the set J for each $1 \leq r \leq p_j$ holds $(x_j, y_r^j) \notin J^*$. Since J^* is l -dominating, hence there exists $(x_i, y_s^i) \in J^*$, where $j \neq i$ such that $d_{\sigma(\alpha, D)}((x_j, y_r^j), (x_i, y_s^i)) \leq l$.

From the definition of the set J we have that $x_i \in J$, so by Theorem 4 there holds $d_D(x_j, x_i) = d_{\sigma(\alpha, D)}((x_j, y_r^j), (x_i, y_s^i)) \leq l$. Consequently, J is an l -dominating set of D , hence J is a (k, l) -kernel of D . The definition of the set J implies that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$. Consider the possible cases:

I.1. Let $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$.

We shall prove that J_i is a (k, l) -kernel of D_i^c in this case. From Theorem 5(a) we obtain that J_i is a k -independent set of D_i^c . Next we shall show that J_i is l -dominating. Since J is a k -independent set of D and $l \leq k - 1$, then for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) \geq k \geq l + 1$. So, there does not exist in D a path $x_i \dots x_j$ of length less than or equal to l . Moreover, $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$ and $l \leq k - 1$, hence $\mathcal{C}_D^{d \leq l}(x_i) = \emptyset$.

From the above and by Theorem 6(a) we obtain that J_i is an l -dominating set of D_i^c . Consequently, J_i is a (k, l) -kernel of D_i^c in this case.

I.2. Let $\mathcal{C}_D^{d \leq k-1}(x_i) \neq \emptyset$.

Then by Theorem 5(b) the set J_i contains exactly one arbitrary vertex from $V(D_i^c)$. So J_i is a k -independent set of D_i^c . Because $l \leq k-1$, then for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) \geq k \geq l+1$. Hence there does not exist in D a path $x_i \dots x_j$ of length $d_D(x_i, x_j) \leq l$. From the assumption there exists in D a circuit containing the vertex x_i of length less than k . We distinguish the following possibilities:

I.2.1. $\mathcal{C}_D^{d \leq l}(x_i) \neq \emptyset$.

Then by Theorem 6(b) it follows immediately that J_i is a 1-element set containing an arbitrary vertex of $V(D_i^c)$.

I.2.2. $\mathcal{C}_D^{d \leq l}(x_i) = \emptyset$ and $\mathcal{C}_D^{l+1 \leq d \leq k-1}(x_i) \neq \emptyset$.

We will show that J_i is a 1-element set containing an l -dominating vertex of D_i^c . Using Theorem 6(a) we obtain that J_i is an l -dominating set of D_i^c . Because J_i contains exactly one vertex, so $J_i = \{(x_i, y_i^j)\}$, where (x_i, y_i^j) is an l -dominating vertex of D_i^c .

II. Let $J \subset V(D)$ be a (k, l) -kernel of the digraph D . Let $\mathcal{I} = \{i : x_i \in J\}$ and J_i be as in the statements of the theorem. We shall prove that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a (k, l) -kernel of $\sigma(\alpha, D)$. Firstly we will prove that J^* is a k -independent set of the D -join $\sigma(\alpha, D)$. Let $(x_i, y_p^i), (x_j, y_q^j) \in J^*$ be two different vertices. Consider the following cases:

II.1. $(x_i, y_p^i) \in J_i$ and $(x_j, y_q^j) \in J_j$, where $i \neq j$.

Evidently, $x_i, x_j \in J$ and because J is k -independent so by Theorem 4 we have that $d_D(x_i, x_j) = d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) \geq k$.

II.2. $(x_i, y_p^i), (x_i, y_q^i) \in J_i$ for some $i \in \mathcal{I}$.

Since J_i contains at least two vertices, so by assumption J_i is a (k, l) -kernel of D_i^c . Hence $d_{D_i^c}((x_i, y_p^i), (x_i, y_q^i)) \geq k$. Assume on the contrary that $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_i, y_q^i)) < k$. If $k = 2$, then there is a contradiction with the independence of J_i in D_i^c . Let $k \geq 3$. This means that there exists a path $(x_i, y_p^i) \dots (x_i, y_q^i)$ in $\sigma(\alpha, D)$ of length less than k such that at least one inner vertex of this path does not belong to $V(D_i^c)$. Hence there exists in D a circuit containing the vertex x_i of length less than k and by Theorem 5(b) the set J_i contains exactly one vertex from $V(D_i^c)$, a contradiction to $(x_i, y_p^i), (x_i, y_q^i) \in J_i$.

Taking the two above cases into consideration we obtain that for distinct $(x_i, y_p^i), (x_j, y_q^j) \in J^*$ there holds $d_{\sigma(\alpha, D)}((x_i, y_p^i), (x_j, y_q^j)) \geq k$. Hence J^* is a k -independent set of $\sigma(\alpha, D)$.

Now we shall prove that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, is an l -dominating set of the D -join $\sigma(\alpha, D)$. Consider the possible cases:

II.3. Let $(x_j, y_p^j) \notin J^*$ and $j \notin \mathcal{I}$.

Then by the definition of the set J we have that $x_j \notin J$. Since J is l -dominating in D , so there exists $i \in \mathcal{I}$ such that $x_i \in J$ and $d_D(x_j, x_i) \leq l$. Consequently, there exists $1 \leq q \leq p_i$ such that $(x_i, y_q^i) \in J^*$ and by Theorem 4 we obtain that $d_{\sigma(\alpha, D)}((x_j, y_p^j), (x_i, y_q^i)) \leq l$.

II.4. Let $(x_j, y_p^j) \notin J^*$ and $j \in \mathcal{I}$.

If J_j is a (k, l) -kernel of D_j^c , then J_j is an l -dominating set of D_j^c , so $d_{D_j^c}((x_j, y_q^j), J_j) \leq l$. Hence $d_{\sigma(\alpha, D)}((x_j, y_q^j), J^*) \leq l$. If J_j contains exactly one arbitrary vertex of $V(D_j^c)$, then by assumption of the theorem there exists in D a circuit containing the vertex x_j of length less than or equal to l . So there exists in $\sigma(\alpha, D)$ a path from (x_j, y_p^j) to J_j and $d_{\sigma(\alpha, D)}((x_j, y_p^j), J_j) \leq l$. Hence $d_{\sigma(\alpha, D)}((x_j, y_p^j), J^*) \leq l$. If J_j is a 1-element set containing an l -dominating vertex of D_j^c , then by the definition of the l -dominating vertex $d_{D_j^c}((x_j, y_p^j), J_j) \leq l$. Hence $d_{\sigma(\alpha, D)}((x_j, y_p^j), J^*) \leq l$. Thus it follows that J^* is an l -dominating set of $\sigma(\alpha, D)$.

Taking the above cases into consideration we obtain that J^* is a (k, l) -kernel of $\sigma(\alpha, D)$.

Thus the theorem is proved. ■

If the digraph D is without circuits of length less than k , then we obtain Theorem 3.

Theorem 8. *Let $k \geq 2$, $l \geq k$ be integers. The subset $J^* \subset V(\sigma(\alpha, D))$ is a (k, l) -kernel of the D -join $\sigma(\alpha, D)$ if and only if there exists a (k, l) -kernel $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and for every $i \in \mathcal{I}$*

- (a) J_i is a (k, l) -kernel of D_i^c if $\mathcal{C}_D^{d \leq l}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$ or
- (b) J_i is a 1-element set containing an arbitrary vertex from $V(D_i^c)$ if $\mathcal{C}_D^{d \leq k-1}(x_i) \neq \emptyset$ or
- (c) J_i is an arbitrary nonempty k -independent set of D_i^c , otherwise.

Proof. I. Let $k \geq 2$, $l \geq k$ be integers. Let J^* be a (k, l) -kernel of the D -join $\sigma(\alpha, D)$. Denote $J = \{x_i \in V(D) : J^* \cap V(D_i^c) \neq \emptyset\}$. Proving analogously as in Theorem 7 we obtain that J is a (k, l) -kernel of the

digraph D . Of course, the definition of the set J implies that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$. Consider the following cases:

I.1. Let $\mathcal{C}_D^{d \leq l}(x_i) = \emptyset$.

Since $l \geq k$, so there does not exist in D a circuit containing the vertex x_i of length less than k . Then from Theorem 5(a) we obtain that J_i is a k -independent set of D_i^c . By our assumption $l \geq k$, so to establish sets J_i we consider the following possibilities:

I.1.1. There exists $j \in \mathcal{I}$ and $j \neq i$ such that $d_D(x_i, x_j) \leq l$.

By Theorem 6 (b) an arbitrary nonempty subset of $V(D_i^c)$ is l -dominating in D_i^c , so J_i is an arbitrary k -independent set of D_i^c .

I.1.2. For each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$.

Then by Theorem 6(a) we obtain that J_i is an l -dominating set of D_i^c . Consequently, J_i is a (k, l) -kernel in this case.

I.2. Let $\mathcal{C}_D^{d \leq l}(x_i) \neq \emptyset$.

Because $l \geq k$, we consider the following possibilities:

I.2.1. $\mathcal{C}_D^{k \leq d \leq l}(x_i) \neq \emptyset$ and $\mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset$.

Then by Theorem 5(a) and Theorem 6(b) we obtain that the set J_i is an arbitrary k -independent set of $V(D_i^c)$.

I.2.2. $\mathcal{C}_D^{d \leq k-1}(x_i) \neq \emptyset$.

We shall prove that J_i is a 1-element set containing an arbitrary vertex from $V(D_i^c)$. By Theorem 5(b) the set J_i contains exactly one vertex from $V(D_i^c)$. Because $l \geq k$, so there exists in D a circuit containing the vertex x_i of length less than or equal to l . Hence by Theorem 6(b) we obtain that J_i contains exactly one arbitrary vertex from $V(D_i^c)$.

II. Let $J \subset V(D)$ be a (k, l) -kernel of the digraph D and let $\mathcal{I} = \{i : x_i \in J\}$. Proving analogously as in Theorem 7 we can show that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a (k, l) -kernel of the D -join $\sigma(\alpha, D)$ where $J_i, i \in \mathcal{I}$ satisfy the assumption of the theorem.

Thus the theorem is proved. ■

3. ON (k, l) -KERNEL PERFECTNESS OF THE D -JOIN

From the definition of the $\sigma(\alpha, D)$ it follows immediately:

Proposition 1. *Every induced subdigraph of $\sigma(\alpha, D)$ is*

- (a) *a digraph of the form $\sigma(\tilde{\alpha}, \tilde{D})$, where $\tilde{D}$ is an induced subdigraph of D with $V(\tilde{D}) = \{x_t : t \in \tilde{\mathcal{I}}\}$, $|\tilde{\mathcal{I}}| > 1$, $\tilde{\mathcal{I}} \subseteq \{1, \dots, n\}$ and $\tilde{\alpha}$ is a family of induced subdigraphs of D_t , where $t \in \tilde{\mathcal{I}}$ or*
- (b) *an induced subdigraph of D_i for some $1 \leq i \leq n$ or*
- (c) *the union of the digraphs from (a) and (b).*

From the definition of the (k, l) -kernel perfect digraph and by Proposition 1 it follows immediately:

Proposition 2. *If $\sigma(\alpha, D)$ is (k, l) -kernel perfect, then D and D_i , $i = 1, \dots, n$ are (k, l) -kernel perfect.*

In [5] it has been proved:

Theorem 9 [5]. *Let D be a digraph without circuits of length less than k and let $\alpha = (D_i)_{i \in \{1, \dots, n\}}$ be a sequence of vertex disjoint digraphs. The D -join $\sigma(\alpha, D)$ is a (k, l) -kernel perfect digraph if and only if the digraph D and the digraphs D_i , $i = 1, \dots, n$ are (k, l) -kernel perfect digraphs.*

In this section, we generalize this result for an arbitrary digraph D .

Theorem 10. *Let D be a (k, l) -kernel perfect digraph. Let D_i , $i = 1, \dots, n$ be a (k, l) -kernel perfect digraph if $C_D^{d \leq k-1}(x_i) = \emptyset$ or every subdigraph of D_i has an l -dominating vertex, otherwise. Then $\sigma(\alpha, D)$ is a (k, l) -kernel perfect digraph.*

Proof. Assume that D and D_i , $i = 1, \dots, n$ are as in the statements of the theorem. We shall show that $\sigma(\alpha, D)$ is a (k, l) -kernel perfect digraph. From Proposition 1 it follows that we need only to prove that $\sigma(\alpha, D)$ has a (k, l) -kernel. By Theorem 7, Theorem 8 and from our assumptions there exists a (k, l) -kernel $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a (k, l) -kernel of the D -join, where $\mathcal{I} = \{i; x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and J_i is a (k, l) -kernel of D_i^c if $C_D^{d \leq k-1}(x_i) = \emptyset$ or J_i is a 1-element set containing an l -dominating vertex of D_i^c .

Thus the theorem is proved. ■

4. THE TOTAL NUMBER OF (k, l) -KERNELS OF THE D -JOIN

In this section, we calculate the number of all k -independent sets, l -dominating sets and (k, l) -kernels of the D -join $\sigma(\alpha, D)$.

Theorem 11. *Let $k \geq 2$, $n \geq 2$ be integers. Let $\sigma(\alpha, D)$ be a D -join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs $(D_i)_{i \in \{1, \dots, n\}}$ on p_i vertices, $p_i \geq 1$. Let $\mathcal{S} = \{S_1, \dots, S_j\}$, $j \geq 1$ be a family of all nonempty k -independent sets of the digraph D and let $\mathcal{S} \ni S_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subset \{1, \dots, n\}$. Then $NkI(\sigma(\alpha, D)) = 1 + \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \varphi(D_i)$, where*

$$\varphi(D_i) = \begin{cases} NkI(D_i) - 1 & \text{if } \mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset, \\ p_i & \text{otherwise.} \end{cases}$$

Proof. Let D be a given digraph on n -vertices, $n \geq 2$. Theorem 4 implies that to obtain a k -independent set of $\sigma(\alpha, D)$ first we have to choose a k -independent set of D . Let $\mathcal{S} = \{S_1, \dots, S_j\}$, $j \geq 1$ be a family of all nonempty k -independent sets of the digraph D . Assume that $\mathcal{S} \ni S_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subset \{1, \dots, n\}$. Next by Theorem 5 in each of the D_i^c , $i \in \mathcal{I}_r$ we have to choose an nonempty k -independent set if there does not exist in D a circuit containing the vertex x_i of length less than k or we choose an arbitrary vertex from $V(D_i^c)$, otherwise. Evidently we can do it on $NkI(D_i) - 1$ or p_i ways, respectively. Hence from the fundamental combinatorial statement we have $\sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \varphi(D_i)$ sets being k -independent, where

$$\varphi(D_i) = \begin{cases} NkI(D_i) - 1 & \text{if } \mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset, \\ p_i & \text{otherwise.} \end{cases}$$

Moreover, the empty set also is a k -independent set of $\sigma(\alpha, D)$. Consequently, $NkI(\sigma(\alpha, D)) = 1 + \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \varphi(D_i)$.

Thus the theorem is proved. ■

Theorem 12. *Let $l \geq 1$, $n \geq 2$ be integers. Let $\sigma(\alpha, D)$ be a D -join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs $(D_i)_{i \in \{1, \dots, n\}}$ on p_i vertices, $p_i \geq 1$. Let $\mathcal{Q} = \{Q_1, \dots, Q_j\}$, $j \geq 1$ be a family of all l -dominating sets of the digraph D and let $\mathcal{Q} \ni Q_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subseteq \{1, \dots, n\}$. Then $NlD(\sigma(\alpha, D)) = \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \psi(D_i)$, where*

$$\psi(D_i) = \begin{cases} NlD(D_i) & \text{if } \mathcal{C}_D^{d \leq l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \\ & \text{holds } d_D(x_i, x_j) > l, \\ 2^{p_i} - 1 & \text{otherwise.} \end{cases}$$

Proof. Let D be a given digraph on n vertices, $n \geq 2$. By Theorem 4 we have that to obtain an l -dominating set of $\sigma(\alpha, D)$ first we have to choose an l -dominating set of D . Let $\mathcal{Q} = \{Q_1, \dots, Q_j\}$, $j \geq 1$ be a family of all l -dominating sets of the digraph D . Assume that $\mathcal{Q} \ni Q_r = \{x_i : i \in \mathcal{I}_r\}$ and $\mathcal{I}_r \subseteq \{1, \dots, n\}$. Next by Theorem 6 in each of the D_i^c , $i \in \mathcal{I}_r$ we have to choose an l -dominating set if for each $j \in \mathcal{I}_r$ and $j \neq i$ there does not exist a path $x_i \dots x_j$ of length less than or equal to l and there does not exist in D a circuit containing the vertex x_i of length less than or equal to l or we have to choose in D_i^c an arbitrary nonempty subset of $V(D_i^c)$. Evidently, we can do it on $NlD(D_i)$ or $2^{p_i} - 1$ ways, respectively. Hence from the fundamental combinatorial statement we have $\sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \psi(D_i)$ sets being l -dominating sets of $\sigma(\alpha, D)$, where

$$\psi(D_i) = \begin{cases} NlD(D_i) & \text{if } \mathcal{C}_D^{d \leq l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \\ & \text{holds } d_D(x_i, x_j) > l, \\ 2^{p_i} - 1 & \text{otherwise.} \end{cases}$$

Thus the theorem is proved. ■

Using the same method as in Theorems 11 and 12 we can prove:

Theorem 13. Let $k \geq 2$, $1 \leq l \leq k-1$, $n \geq 2$ be integers. Let $\sigma(\alpha, D)$ be a D -join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs $(D_i)_{i \in \{1, \dots, n\}}$ on p_i vertices, $p_i \geq 1$. Let $\mathcal{J} = \{J_1, \dots, J_j\}$, $j \geq 1$ be a family of all (k, l) -kernels of the digraph D and let $\mathcal{J} \ni J_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subseteq \{1, \dots, n\}$. Then $NklK(\sigma(\alpha, D)) = \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \mu(D_i)$, where

$$\mu(D_i) = \begin{cases} NklK(D_i) & \text{if } \mathcal{C}_D^{d \leq k-1}(x_i) = \emptyset, \\ p_i & \text{if } \mathcal{C}_D^{d \leq l}(x_i) \neq \emptyset, \\ Nld(D_i) & \text{otherwise.} \end{cases}$$

Theorem 14. Let $k \geq 2$, $l \geq k$, $n \geq 2$ be integers. Let $\sigma(\alpha, D)$ be a D -join of the digraph D on n vertices and α be a sequence of vertex disjoint digraphs

$(D_i)_{i \in \{1, \dots, n\}}$ on p_i vertices, $p_i \geq 1$. Let $\mathcal{J} = \{J_1, \dots, J_j\}$, $j \geq 1$ be a family of all (k, l) -kernels of the digraph D and let $\mathcal{J} \ni J_r = \{x_i : i \in \mathcal{I}_r\}$, where $\mathcal{I}_r \subseteq \{1, \dots, n\}$. Then $NklK(\sigma(\alpha, D)) = \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \eta(D_i)$, where $\eta(D_i) =$

$$\begin{cases} NklK(D_i) & \text{if } \mathcal{C}_D^{d \leq l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \text{ holds } d_D(x_i, x_j) > l, \\ p_i & \text{if } \mathcal{C}_D^{d \leq k-1}(x_i) \neq \emptyset, \\ NkI(D_i) - 1 & \text{otherwise.} \end{cases}$$

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Received 29 April 2006

Revised 18 May 2007

Accepted 18 May 2007