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SOME TOTALLY 4-CHOOSABLE MULTIGRAPHS

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Abstract

It is proved that if G is multigraph with maximum degree 3, and every submultigraph of G has average degree at most $2\frac{1}{2}$ and is different from one forbidden configuration C_4^+ with average degree exactly $2\frac{1}{2}$, then G is totally 4-choosable; that is, if every element (vertex or edge) of G is assigned a list of 4 colours, then every element can be coloured with a colour from its own list in such a way that no two adjacent or incident elements are coloured with the same colour. This shows that the List-Total-Colouring Conjecture, that $\mathrm{ch}''(G) = \chi''(G)$ for every multigraph G, is true for all multigraphs of this type. As a consequence, if G is a graph with maximum degree 3 and girth at least 10 that can be embedded in the plane, projective plane, torus or Klein bottle, then $\mathrm{ch}''(G) = \chi''(G) = 4$. Some further total choosability results are discussed for planar graphs with sufficiently large maximum degree and girth.

Keywords: maximum average degree, planar graph, total choosability, list total colouring.

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1. Introduction

We use standard terminology, as defined in the references: for example, [2] or [10]. A *multigraph* may have multiple edges but not loops. If G is a multigraph, then $\chi'(G)$, $\chi''(G)$, $\mathrm{ch}'(G)$, $\mathrm{ch}''(G)$ and $\Delta(G)$ denote the edge chromatic number (or chromatic index), total (vertex-edge) chromatic number, edge choosability (or list edge chromatic number), total choosability, and maximum degree of G, respectively. So $\mathrm{ch}''(G)$ is the smallest k for which G is totally k-choosable. Note that $\mathrm{ch}'(G)$ and $\mathrm{ch}''(G)$ are the same as what were called $\chi'_{\mathrm{list}}(G)$ and $\chi''_{\mathrm{list}}(G)$ in [2]. The *maximum average degree* $\mathrm{mad}(G)$ of G is the maximum value of $2|E(H)|/|V(H)|$ taken over all submultigraphs H of G.

Clearly $\mathrm{ch}'(G) \geqslant \chi'(G) \geqslant \Delta(G)$ and $\mathrm{ch}''(G) \geqslant \chi''(G) \geqslant \Delta(G)+1$, for every multigraph G. Part of ([2], Theorem 7) can be summarized in tabular form as follows.

Theorem 1 [2]**.** *Let G be a multigraph with maximum degree Δ such that*

if	$\Delta =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
then	$\mathrm{mad}(G) \leqslant$	$2\frac{1}{2}$	3	$3\frac{1}{3}$	$3\frac{2}{3}$	4	$4\frac{1}{4}$	$4\frac{1}{2}$	$4\frac{3}{4}$	5	$5\frac{1}{5}$	$5\frac{2}{5}$	$5\frac{3}{5}$	$5\frac{4}{5}$	6	...

respectively. Then $\mathrm{ch}'(G) = \chi'(G) = \Delta$, *and if* $\Delta \geqslant 6$ *then* $\mathrm{ch}''(G) = \chi''(G) = \Delta + 1$.

It is conjectured in [2] that the final statement holds even if $\Delta < 6$. This conjecture now remains open only for $\Delta = 5$. For $\Delta = 4$, the truth of this conjecture follows from a slight rewording of the proof in [2], or alternatively by arguments given in [3]; the case when $\mathrm{mad}(G) = 3$ also needs a result from [6] (see the next section). For $\Delta = 3$, this conjecture is false as stated. For, let C_4^+ consist of a 4-cycle in which one edge has been replaced by two parallel edges. Then $\mathrm{mad}(C_4^+) = 2\frac{1}{2}$, and it is easy to see that C_4^+ is not totally 4-colourable and hence not totally 4-choosable. In the present paper we will prove the result for $\Delta = 3$ subject to the extra condition that G does not contain a copy of C_4^+. Specifically, we will prove the following two results, the second of which follows immediately from the first.

Theorem 2. *Let G be a multigraph with maximum degree* 3 *such that* $\mathrm{mad}(G) \leqslant 2\frac{1}{2}$. *Suppose that Λ is an assignment of a list of four colours to every element* (*vertex and edge*) *of G, and suppose that if H is any submultigraph of G that is isomorphic to C_4^+ then Λ does not assign the same list to every element of H. Then G is totally Λ-colourable.*

Corollary 2.1. *Let G be a multigraph with maximum degree* 3 *such that* $\mathrm{mad}(G) \leqslant 2\frac{1}{2}$ *and G does not have C_4^+ as a submultigraph. Then* $\mathrm{ch}''(G) = \chi''(G) = 4$.

There is no suggestion that Theorem 1 is sharp, even for multigraphs, and it is even less likely to be sharp for (simple) graphs. In particular, Chetwynd [4] lists all the minimal non-totally-4-colourable graphs with maximum degree 3 on up to ten vertices. They include two (one with six vertices, one with nine) with $\mathrm{mad}(G) = 2\frac{2}{3}$, but none with smaller $\mathrm{mad}(G)$, and the same may well be true for non-totally-4-choosable graphs of all orders. But to prove this would probably need new ideas.

In the next section we will discuss some related results involving planar graphs. We will prove Theorem 2 for $\mathrm{mad}(G) < 2\frac{1}{2}$ in Section 3, and we will extend the result to $\mathrm{mad}(G) = 2\frac{1}{2}$ in Section 4. For brevity, when considering total colourings of a multigraph G, we will sometimes say that a vertex and an edge incident to it are *adjacent* or *neighbours*, since they correspond to adjacent or neighbouring vertices of the total graph $T(G)$ of G. An *element* is a vertex or an edge. A *k-vertex* is a vertex with degree k.

2. Planar Graphs with Large Degree and Girth

If G is a (simple) graph with girth (i.e., length of shortest circuit) g that can be embedded in a surface S with characteristic ϵ, and if G has n vertices, m edges and r regions (faces), then $2m \geqslant gr$, and Euler's formula $n-m+r \geqslant \epsilon$ implies that

$$gn - (g-2)m \geqslant gn - (g-2)m - 2m + gr = g(n-m+r) \geqslant g\epsilon.$$

Thus if $\epsilon > 0$ (i.e., S is the plane or projective plane) then $gn > (g-2)m$ and $\frac{m}{n} < \frac{g}{g-2}$, which (applied to all subgraphs of G) shows that $\mathrm{mad}(G) < \frac{2g}{g-2}$. And if $\epsilon \geqslant 0$ (i.e., S is the plane, projective plane, torus or Klein bottle) then $\mathrm{mad}(G) \leqslant \frac{2g}{g-2}$. Thus the following result follows immediately from Corollary 2.1, with no need for further proof.

Corollary 2.2. *Let G be a graph with maximum degree 3 and girth at least 10 that can be embedded in a surface of nonnegative characteristic. Then* $\mathrm{ch}''(G) = \chi''(G) = 4$.

The current state of knowledge about the total colourability and total choosability of planar (simple) graphs with large maximum degree and girth can be summarized as follows.

Theorem 3. *Let G be a planar graph with maximum degree Δ and girth g. Then*

(i) $\chi''(G) = \Delta + 1$ *if*

- (a) $\Delta \geqslant 10$;
- (b) $\Delta \geqslant 7$ *and* $g \geqslant 4$;
- (c) $\Delta \geqslant 5$ *and* $g \geqslant 5$;
- (d) $\Delta \geqslant 4$ *and* $g \geqslant 6$;
- (e) $\Delta \geqslant 3$ *and* $g \geqslant 10$;

(ii) $\mathrm{ch}''(G) = \Delta + 1$ *if*

- (a) $\Delta \geqslant 12$;
- (b) $\Delta \geqslant 7$ *and* $g \geqslant 4$;
- (c) $\Delta \geqslant 5$ *and* $g \geqslant 5$;
- (d) $\Delta \geqslant 4$ *and* $g \geqslant 6$;
- (e) $\Delta \geqslant 3$ *and* $g \geqslant 10$.

Moreover, with the possible exception of (i)(a) *and* (ii)(c), *all these results hold if G is not planar but can be embedded in a surface of nonnegative characteristic.*

Part (i)(a) of Theorem 3 is proved in [9], and part (ii)(a) in [2]. These results are better than the condition $\Delta \geqslant 16$ that can be deduced from Theorem 1 with $\mathrm{mad}(G) \leqslant 6$ (which holds since $g \geqslant 3$).

Part (ii)(b), which implies part (i)(b), is proved in [2] as a corollary of Theorem 1 with $\Delta \geqslant 7$.

Parts (i)(c) and (i)(d) are proved in [3]. However, the proofs work equally well for (ii)(c) and (ii)(d), with only minor changes such as replacing 'we can totally colour ... with $\Delta + 1$ colours' by 'we can totally colour ... from its lists'. One also needs to find a different way around the following difficulty: if G is a minimal counterexample to the theorem, and H is a proper subgraph of G, one wants to be sure that H is totally $(\Delta + 1)$-colourable; however, the fact that G is a minimal counterexample does not ensure this when $\Delta(H) < \Delta$. We avoided this difficulty in [3] by quoting the known result that $\chi''(H) \leqslant \Delta(H) + 2$ if $\Delta(H) < 5$. For list colourings, this works when $\Delta = 4$, since Juvan, Mohar and Škrekovski [6] proved that $\mathrm{ch}''(H) \leqslant \Delta(H) + 2$ if $\Delta(H) < 4$; but it is not known whether this holds whenever $\Delta(H) < 5$. Another way of avoiding this difficulty, which *does* work for list colourings, is to prove the theorem for all graphs with maximum degree *at most* Δ; then there is no need to refer to any other results in order to deduce that H is totally $(\Delta + 1)$-colourable, or totally $(\Delta + 1)$-choosable in the list-colouring version of the theorem. But then one cannot assume that G contains a vertex with degree Δ, which we used in order to contradict the possibility that $n - m + r = 0$. Thus, with the minor modifications just mentioned, the proofs of (i)(c) and (i)(d) in [3] work equally well for list colourings, and prove (ii)(c) and (ii)(d) in surfaces of positive characteristic, and (ii)(d) (only) in surfaces of characteristic zero.

The results in (c) and (d) are what one could deduce from the conjectured results for $\Delta = 4$ and 5 in Theorem 1. The proof of (c) uses the embedding and girth of G in an intrinsic way and so does not prove this conjecture. In contrast, the proof of (d) uses the embedding and girth of G only to prove that $\mathrm{mad}(G) \leqslant 3$, and so (with the minor modifications mentioned above) it proves the conjecture, that $\mathrm{ch}''(G) = 5$ for every multigraph G such that $\Delta(G) = 4$ and $\mathrm{mad}(G) \leqslant 3$.

Part (i)(e) of Theorem 3 is also proved in [3], but the proof involves recolouring arguments that do not work for list colourings. In the present paper we use alternative arguments that do work for list colourings. We claimed in [3] that we had used the embedding only to prove that $\mathrm{mad}(G) \leqslant 2\frac{1}{2}$, and that our proof therefore showed that $\chi''(G) = \Delta + 1$ for every graph G with maximum degree $\Delta \geqslant 3$ such that $\mathrm{mad}(G) \leqslant 2\frac{1}{2}$. But we overlooked the fact that we had made implicit (and in one place explicit) use of the lower bound on the girth, so that in fact we had proved the result only when $\Delta \geqslant 3$, $\mathrm{mad}(G) \leqslant 2\frac{1}{2}$ and $g \geqslant 10$. It is not difficult to fill in the missing case $g < 10$, but in any case it follows from Corollary 2.1.

3. The Proof for $\mathrm{mad}(G) < 2\frac{1}{2}$

It is well known that every multigraph with maximum degree 2 is totally 4-choosable, but there is no need for us to assume this result. Throughout the proof of Theorem 2, $G_{\min}$ will denote a multigraph with maximum degree *at most* 3, and Λ will denote an assignment of a list of four colours to every element of $G_{\min}$ that is not constant on any copy of C_4^+ in $G_{\min}$ and such that $G_{\min}$ has no total Λ-colouring, but every proper submultigraph of $G_{\min}$ has a total Λ-colouring. We will prove in this section that $\mathrm{mad}(G_{\min}) \geqslant 2\frac{1}{2}$ and in the next section that $\mathrm{mad}(G_{\min}) \neq 2\frac{1}{2}$, and this will suffice to prove Theorem 2. If $G_{\min}$ consists of two vertices joined by three parallel edges then certainly $\mathrm{mad}(G_{\min}) > 2\frac{1}{2}$, and so we may suppose that this is not the case. Then it is easy to see that every vertex of $G_{\min}$ has at least two distinct neighbours. We will prove various further statements about $G_{\min}$. It will be convenient to prove the following basic lemma first.

Lemma 1. *Suppose that three pairwise adjacent elements x, y and z are given lists $L(x)$, $L(y)$ and $L(z)$ of colours such that $|L(x)| = |L(y)| = |L(z)| = 2$, and the lists are not all equal. Then there are two different* *colourings $\lambda_x, \lambda_y, \lambda_z$ and μ_x, μ_y, μ_z of x, y, z from these lists such that one of the following holds:*

(i) $\lambda_x = \mu_x$ *and* $\lambda_y \neq \mu_y$;

(ii) $\lambda_y = \mu_y$ *and* $\lambda_x \neq \mu_x$;

(iii) $\lambda_x = \mu_y$ *and* $\lambda_y = \mu_x$;

(iv) $\lambda_x = \mu_y$ *and* $\lambda_y \neq \mu_x$.

Proof. It is well known, and easy to see, that there is a colouring $\lambda_x, \lambda_y, \lambda_z$ of x, y, z from these lists: start by assigning a colour c that is not in the list of every element, and end by colouring an element that does not have c in its list. If two of the three lists are equal, then we can obtain another colouring μ_x, μ_y, μ_z by interchanging the colours of two elements; then (i), (ii) or (iii) holds, according as the two elements with equal lists are y and z, or x and z, or x and y. So assume that no two lists are equal. If $|L(x) \cap L(y)| = 1$, say $L(x) = \{a, b\}$ and $L(y) = \{a, c\}$, then $L(z) \neq \{a, b\}$ or $\{a, c\}$, and so the colourings (a, c) and (b, a) for x, y both extend to z and are related as in (iv). Finally, if $L(x) \cap L(y) = \emptyset$, say $L(x) = \{a, b\}$ and $L(y) = \{c, d\}$, then at least three of the four possible colourings (a, c), (a, d), (b, c) and (b, d) for x, y can be extended to z, and among any three of these there are two that are related as in (i). This completes the proof of Lemma 1. ■

Lemma 2. (a) *No triangle in $G_{\min}$ contains two 2-vertices.*
(b) *No 2-vertex in $G_{\min}$ is adjacent to two 2-vertices.*
(c) *There is no path $uvwx$ in $G_{\min}$ such that u, v and x are all 2-vertices.*

Proof. (a) Suppose uvw is a triangle and u, v are 2-vertices. By hypothesis, $G_{\min} - \{u, v\}$ has a total colouring from its lists. For each uncoloured element z of $G_{\min}$, let $L(z)$ denote the subset of colours from $\Lambda(z)$ that are not used on any neighbour of z. Then $|L(z)| \geqslant 2$ if $z = uw$ or vw, $|L(z)| \geqslant 3$ if $z = u$ or v, and $|L(uv)| = 4$. If we try to colour the elements in the order

$$uw, vw, v, u, uv, \tag{1}$$

then it is only with uv that we may fail. If we can give the same colour to uw and v, then we will succeed even with uv. But if $L(uw) \cap L(v) = \emptyset$ then either $L(uw)$ or $L(v)$ contains a colour not in $L(uv)$; so colour the elements in the order (1), using a colour not in $L(uv)$ at the first available opportunity. Then in all cases the elements can be coloured. This gives a total Λ-colouring of $G_{\min}$, which contradicts the definition of $G_{\min}$, and this contradiction proves (a).

(b) Suppose u, v, w are 2-vertices forming a path uvw. We can totally colour $G_{\min} - uv$ from its lists and then erase the colours on u and v. Now there are two colours available for each of u, uv and v, and the only problem arises if they are the same two colours in each case. But the edge vw has only two restrictions on its colour, and it has a list of four colours, and so by recolouring vw if necessary we can complete the colouring and obtain the required contradiction.

(c) Suppose there is such a path $uvwx$ in $G_{\min}$. Totally colour $G_{\min} - v$ from its lists and then erase the colours on all the elements in the path. For each uncoloured element z, let $L(z)$ denote the subset of colours from $\Lambda(z)$ that are not used on any neighbour of z. Then the uncoloured elements are u, uv, v, vw, w, wx, x, and they have lists of size at least $2, 3, 4, 3, 2, 2, 2$ respectively. Moreover, the lists of w, wx and x are not all equal, since these elements were coloured in the colouring of $G_{\min} - v$, and so by Lemma 1 there are two different colourings $\lambda_w, \lambda_{wx}, \lambda_x$ and μ_w, μ_{wx}, μ_x of w, wx, x such that the ordered pairs $(\lambda_w, \lambda_{wx})$ and (μ_w, μ_{wx}) are different. Choose colours $\lambda_{vw} \in L(vw) \setminus \{\lambda_w, \lambda_{wx}\}$ and $\mu_{vw} \in L(vw) \setminus \{\mu_w, \mu_{wx}\}$. If we assign colours $\lambda_{vw}, \lambda_w, \lambda_{wx}, \lambda_x$ to vw, w, wx, x, then there remain two possible colours for each of u, uv and v, and the only problem is if they are the same two colours in each case. For this to happen, it must be that

$$L(uv) \setminus L(u) = \{\lambda_{vw}\} \quad \text{and} \quad L(v) \setminus L(uv) = \{\lambda_w\}. \tag{2}$$

If it is possible to change the colour of vw without changing any other colour, then do so, and the problem is avoided. If not, then $L(vw) = \{\lambda_{vw}, \lambda_w, \lambda_{wx}\}$. We may suppose by the same argument that (2) holds with λ_{vw} and λ_w replaced by μ_{vw} and μ_w, and that $L(vw) = \{\mu_{vw}, \mu_w, \mu_{wx}\}$. But then $\lambda_{vw} = \mu_{vw}$, $\lambda_w = \mu_w$, and $\lambda_{wx} = \mu_{wx}$. This contradiction completes the proof of Lemma 2. ■

To progress further, we will need the powerful technique of Alon and Tarsi [1]. By a *subflow* of a digraph $D = (V, A)$ we will mean a subset of the arc-set A that forms a subdigraph F of D in which every vertex v has indegree $d_F^-(v)$ equal to its outdegree $d_F^+(v)$. (This is what Alon and Tarsi call an *Eulerian* subdigraph.) A subflow is *even* or *odd* according as it has an even or an odd number of arcs; of course, the empty subset of A is an even subflow. Alon and Tarsi [1] proved that if a graph G has an orientation that forms a digraph D in which the number of even subflows is different from the number of odd subflows, and if every vertex v of G is given a list $L(v)$ of at least $d_D^+(v)+1$ colours, then the vertices can be properly coloured from these lists. We will use this result in proving the following lemmas.

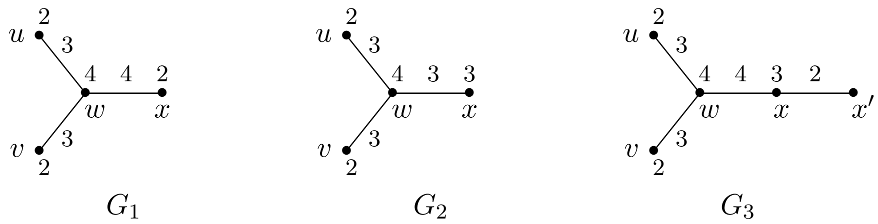


Figure 1

Lemma 3. *If each element of the graphs in Figure* 1 *is given a list of colours of the size indicated against it, then each graph can be totally coloured from these lists.* (*In* G_3, *the vertex* x' *is not given a list and is not required to be coloured.*)

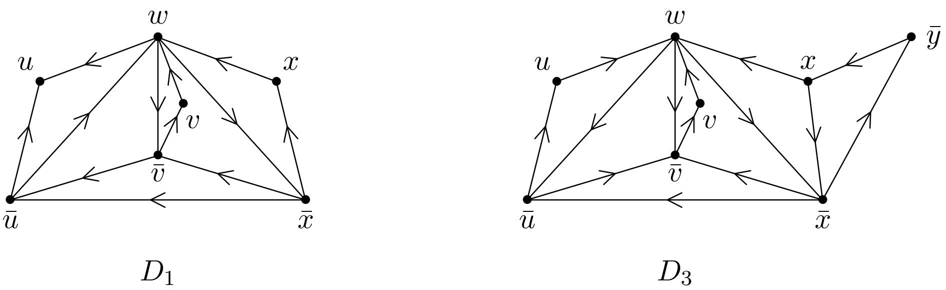


Figure 2

Proof. The digraph D_1 in Figure 2 is an orientation of the total graph $T(G_1)$ of the multigraph G_1 in Figure 1, with the edges wu, wv, wx represented by vertices $\bar{u}, \bar{v}, \bar{x}$ respectively. Each vertex in D_1 has outdegree less than the number of colours available to it. It remains to prove that the number of even subflows of D_1 is different from the number of odd subflows. A computer search shows that there are 6 even subflows and 5 odd subflows. However, we can obtain the result without using a computer. Note that D_1

contains two arc-disjoint 3-cycles $C_1\colon w\bar{v}vw$ and $C_2\colon w\bar{x}xw$. The subflows that contain none of the arcs of C_1 are in a natural $1:1$ correspondence with the subflows that contain all of these arcs, with each corresponding pair comprising one even and one odd subflow; thus it suffices to consider the subflows that contain either one or two arcs of C_1. Of these, the ones that contain none of the arcs of C_2 pair off in a similar way with those that contain all of these arcs, and so it suffices to consider the subflows that contain either one or two arcs of C_1 and either one or two arcs of C_2. It is not difficult to see that there is exactly one such subflow, forming a 4-cycle $w\bar{x}\bar{v}vw$. Since the number of even subflows is different from the number of odd subflows, it follows from the theorem of Alon and Tarsi that the vertices of D_1 can be properly coloured from their lists, so that the elements of G_1 can be totally coloured from their lists.

Let $D_2 := D_3 - \bar{y}$, where D_3 is shown in Figure 2. Then D_2 is an orientation of $T(G_2)$, where G_2 is shown in Figure 1, and each vertex in D_2 has outdegree less than the number of colours available to it. The computer finds that there are again 6 even subflows and 5 odd subflows; however, we can again obtain the result without using a computer. There are two arc-disjoint 3-cycles $C_1\colon w\bar{u}uw$ and $C_2\colon w\bar{v}vw$. As in the proof for D_1, it suffices to consider subflows that contain either one or two arcs of C_1 and either one or two arcs of C_2. It is not difficult to see that there is exactly one such subflow, forming a 4-cycle $w\bar{u}\bar{v}vw$. Since the number of even subflows is different from the number of odd subflows, it follows that the elements of G_2 can be totally coloured from their lists.

Finally, D_3 is an orientation of $T(G_3)$ with the vertex x' removed, and with the edge xx' represented by the vertex $\bar{y}$, and each vertex has outdegree less than the number of colours available to it. This time the computer finds that there are 14 even subflows and 13 odd subflows; however, once again, we can obtain the result without using a computer. There are three arc-disjoint 3-cycles, C_1 and C_2 as in D_2, and $C_3\colon x\bar{x}\bar{y}x$. The only subflow that contains either one or two arcs of each of these three 3-cycles is the union of the two 4-cycles $w\bar{u}\bar{v}vw$ and $w\bar{x}\bar{y}xw$. It follows as before that the elements of G_3 (other than x') can be totally coloured from their lists. ∎

Lemma 4. *If each element of the graph G_4 in Figure* 3 *is given a list of colours of the size indicated against it, then the elements can be totally coloured from these lists.* (*The vertices x and y are not given lists and are not required to be coloured.*)

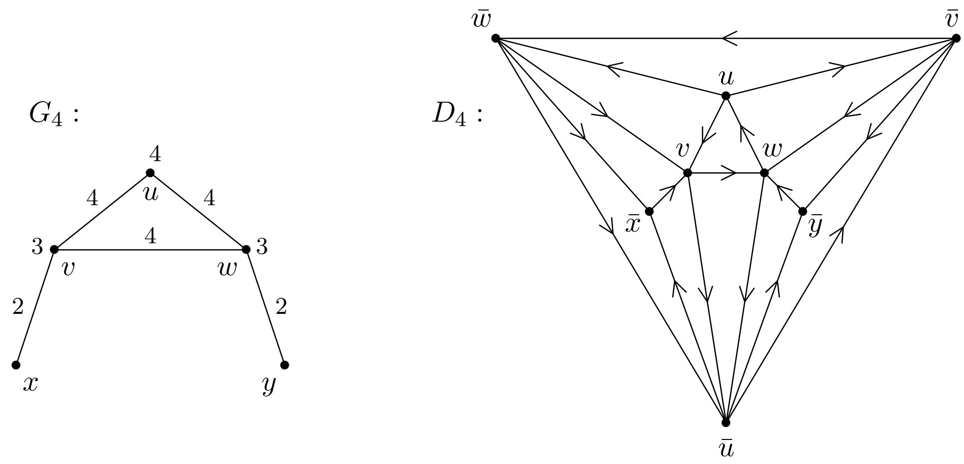


Figure 3

Proof. The digraph D_4 in Figure 3 is an orientation of the total graph $T(G_4)$ with the vertices x and y removed, and with the edges $uv, uw, vw,$ vx, wy represented by vertices $\bar{w}, \bar{v}, \bar{u}, \bar{x}, \bar{y}$ respectively. Each vertex in D_4 has outdegree one less than the number of colours available to it. It remains to prove that the number of even subflows of D_4 is different from the number of odd subflows. The computer finds that there are 64 even subflows and 62 odd subflows. Unfortunately, I cannot find a convincing way of demonstrating this difference without using a computer. There are four edge-disjoint directed triangles in D_4, namely $uvwu$, $\bar{u}\bar{v}\bar{w}\bar{u}$, $v\bar{u}\bar{x}v$ and $w\bar{u}\bar{y}w$, and the computer finds that there are 14 subflows that contain either one or two arcs of each of these four 3-cycles, 8 of which are even and 6 odd. Replacing the cycle $uvwu$ by $u\bar{v}wu$ gives 12 relevant subflows, of which 7 are even and 5 odd; but this is still too many to check reliably without using a computer. We rely on the computer result. ■

Lemma 5. *$G_{\min}$ does not contain a chordless cycle $C\colon u_1v_1u_2v_2\dots u_kv_ku_1$ of even length such that $v_1, v_2, \dots, v_k$ all have degree* 2.

Proof. Suppose it does. Then $k \geqslant 2$, since, as we observed at the start of the proof, every vertex of $G_{\min}$ has at least two distinct neighbours. For $i = 1, ..., k$, denote the edge $v_{i-1}u_i$ by e_i and the edge u_iv_i by f_i. (Subscripts should be interpreted modulo k throughout this proof.) Choose a total colouring of $G_{\min} - v_1$ from its lists, uncolour all the elements in the cycle, and for each uncoloured element z of $G_{\min}$ let $L(z)$ denote the set of colours from $\Lambda(z)$ that are not used on any neighbour of z. Then, for each i, $|L(u_i)| \geqslant 2$, $|L(z)| \geqslant 3$ if $z = e_i$ or f_i, and $|L(v_i)| = 4$. Let D be the orientation of $T(C)$ in which the arcs are, for each i, $\overrightarrow{e_iu_i}$, $\overrightarrow{u_if_i}$, $\overrightarrow{f_iv_i}$, $\overrightarrow{v_ie_{i+1}}$, $\overrightarrow{v_iu_i}$, $\overrightarrow{v_iu_{i+1}}$, $\overrightarrow{e_if_i}$ and $\overrightarrow{f_ie_{i+1}}$ (see Figure 4 for the case $k = 3$). Note that each vertex z has outdegree one less than the lower bound given above for the number of colours available to z. We distinguish two subflows: for each i, F_1 contains the arcs e_if_i, f_iv_i and v_ie_{i+1}, and F_2 contains the arcs e_iu_i, u_if_i and f_ie_{i+1}. Clearly F_1 and F_2 have the same parity, which is the same as that of k. We will show that all the other subflows pair off, each pair comprising one even and one odd.

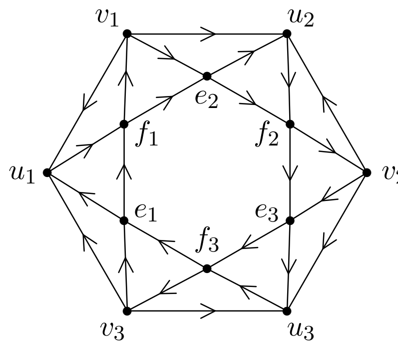


Figure 4

There are k arc-disjoint 3-cycles of the form $u_if_iv_iu_i$. By an argument introduced in Lemma 3, the subflows that contain all or none of the arcs of any of these 3-cycles all pair off, and so it suffices to consider subflows that contain one or two arcs of each 3-cycle. The arcs v_iu_i cannot occur in any such subflow, and can now be ignored. So for each i, the subflows that we are considering must contain at least one of the arcs u_if_i and f_iv_i.

For each i, the subflows that contain the arc $v_{i-1}u_i$ (and, therefore, neither of the arcs $v_{i-1}e_i$ and e_iu_i) pair off with those containing the arcs $v_{i-1}e_i$ and e_iu_i (and, therefore, not containing $v_{i-1}u_i$). So it suffices to consider subflows that contain exactly one of the arcs $v_{i-1}e_i$ and e_iu_i, and hence exactly one of the arcs $f_{i-1}e_i$ and e_if_i, and that do not contain $v_{i-1}u_i$. Note that such a subflow cannot contain both of the arcs $e_{i-1}f_{i-1}$ and $f_{i-1}e_i$; for if it did, and given that it contains at least one of the arcs $u_{i-1}f_{i-1}$ and $f_{i-1}v_{i-1}$ by the previous paragraph, then it must contain $f_{i-1}v_{i-1}$ and hence either $v_{i-1}u_i$ or both of $v_{i-1}e_i$ and e_iu_i, both of which we have already ruled out of consideration.

It follows from the previous paragraph that it suffices to consider subflows that contain either all the arcs e_if_i (and no arc f_ie_{i+1}), or all the arcs f_ie_{i+1} (and no arc e_if_i). But there is exactly one subflow of each type, namely F_1 and F_2 respectively. Since these have the same parity, the number of even subflows is not equal to the number of odd subflows. It follows from the theorem of Alon and Tarsi that all the uncoloured elements of $G_{\min}$ can be coloured from their lists, and this contradiction completes the proof of Lemma 5. ■

Lemma 6. *$G_{\min}$ does not contain a cycle $C\colon u_1v_1u_2v_2\dots u_kv_ku_1$ of even length such that $v_1, v_2, \dots, v_k$ all have degree 2.*

Proof. Suppose it does. By Lemma 5 we may assume that C has a chord, say u_1u_h, and we may choose this chord so that there is no chord $u_iu_j \neq u_1u_h$ such that $1 \leqslant i < j \leqslant h$. Choose a total colouring of $G_{\min} - u_1u_h$ from its lists, and uncolour all elements in the segment $v_ku_1 \dots u_hv_h$ of C; let the graph formed by the uncoloured elements of $G_{\min}$ be H. If $h = 2$ and we recolour v_k and v_h with the colours they had before, then regardless of whether $k = 2$ (when $v_k = v_h$) or $k \geqslant 3$ (when $v_k \neq v_h$), this colouring can be extended to the whole of H, and hence to the whole of $G_{\min}$, by Lemma 4. This contradiction shows that $h \geqslant 3$. By the same argument we may also assume that C has no chord of the form u_hu_{h+1}, so that $v_k \neq v_h$.

Let g denote the chord u_1u_h, and let the edges of C be labelled as in Lemma 5, so that in particular $v_ku_1 = e_1$, $u_1v_1 = f_1$, $v_{h-1}u_h = e_h$, and $u_hv_h = f_h$ (see Figure 5(a) for the case $h = 3$). For each uncoloured element z of $G_{\min}$, let $L(z)$ denote the set of colours from $\Lambda(z)$ that are not used on any neighbour of z. Then $|L(z)| \geqslant 2$ if $z \in \{v_k, v_h\}$, $|L(z)| \geqslant 3$ if $z \in \{e_1, f_h\}$, $|L(z)| \geqslant 4$ if $z \in \{g, u_1, f_1, e_h, u_h\}$, and for each element z in the segment $v_1 \dots v_{h-1}$ of C, $|L(z)|$ is at least as large as the lower bound given in Lemma 5. Let D be the orientation of $T(H)$ in which the arcs with both endvertices in the set $\{v_k, e_1, u_1, f_1, g, e_h, u_h, f_h, v_h\}$ are oriented as in Figure 5(b), and all other arcs are oriented in the same way as in Lemma 5 (Figure 5(b) shows the case $h = 3$). Note that each vertex has outdegree less than the number of colours available to it.

We wish to prove that the number of even subflows of D is different from the number of odd subflows. In doing this, it suffices to consider subflows

F with the following four properties, which follow from arguments used in the previous lemma.

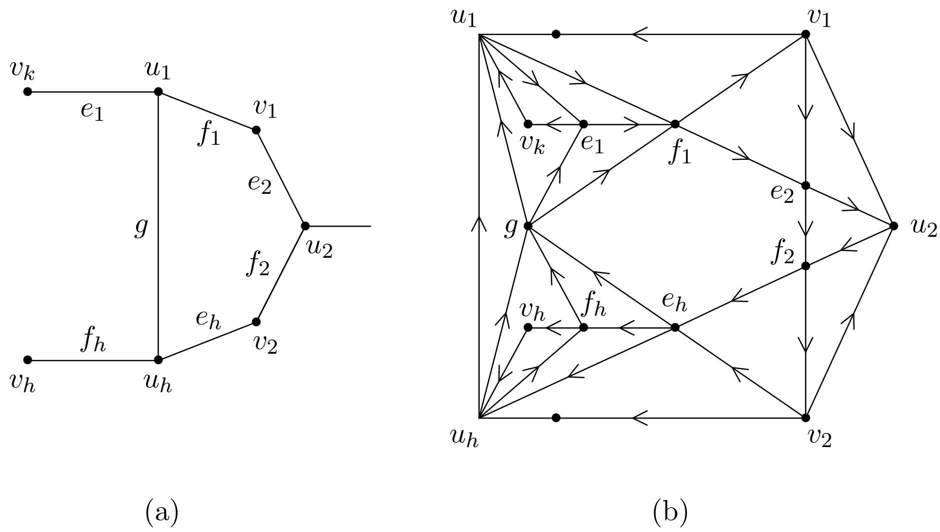


Figure 5

P1. For each i $(2 \leqslant i \leqslant h-1)$, F does not contain either of the arcs $v_{i-1}u_i$ and v_iu_i, it contains exactly one of the arcs $v_{i-1}e_i$ and e_iu_i, and it contains at least one of the arcs u_if_i and f_iv_i.

P2. F contains exactly one or two of the arcs u_1f_1, f_1v_1 and v_1u_1.

P3. Either F contains arc u_1e_1 but neither of arcs e_1v_k and v_ku_1, or else F contains arcs e_1v_k and v_ku_1 but not arc u_1e_1.

P4. Either F contains arc u_hf_h but neither of arcs f_hv_h and v_hu_h, or else F contains arcs f_hv_h and v_hu_h but not arc u_hf_h.

It is not difficult to see that a subflow F that satisfies P1–P3 must satisfy the following:

• If F contains both of the arcs f_1v_1 and f_1e_2, then F contains all five of the arcs

(3) $$gf_1,\ f_1v_1,\ v_1u_1,\ u_1e_1,\ e_1f_1,$$

and no other arc incident with u_1, e_1 or f_1; in this case we say that F is of *type* 0 at 2. If F is not of type 0 at 2, then F contains exactly one of the arcs f_1v_1 and f_1e_2, and cannot contain arc v_1u_1.

- For each i ($2 \leqslant i \leqslant h-1$), exactly one of the following holds:
  (i) F contains all of the arcs $f_{i-1}e_i$, e_iu_i and u_if_i and none of the arcs $f_{i-1}v_{i-1}$, $v_{i-1}e_i$ and e_if_i (we say that F is of *type* 1 *at* i);
  (ii) F contains all of the arcs $f_{i-1}v_{i-1}$, $v_{i-1}e_i$ and e_if_i and none of the arcs $f_{i-1}e_i$, e_iu_i and u_if_i (we say that F is of *type* 2 *at* i);
  (iii) $i=2$ and F is of type 0 at i, in which case F contains arcs f_1e_2, e_2u_2 and u_2f_2 (as in type 1), as well as the five arcs listed in (3).
- If F is of type 2 at i ($2 \leqslant i \leqslant h-2$) then F is of type 2 at $i+1$.

It follows that F contains a path of $3(h-2)$ edges from f_1 to f_{h-1}, and there are $h-1$ routes that this path can take: for some j ($1 \leqslant j \leqslant h-1$) F is of type 0 or 1 at i for all i such that $2 \leqslant i \leqslant j$, and of type 2 at i for all i such that $j+1 \leqslant i \leqslant h-1$.

Suppose first that F is of type 1 at $h-1$ and, therefore, of type 0 or 1 at 2. Then F enters f_{h-1} along arc $u_{h-1}f_{h-1}$. Since the subflows that contain $f_{h-1}e_h$ but not $f_{h-1}v_{h-1}$ or $v_{h-1}e_h$ pair off with those that contain $f_{h-1}v_{h-1}$ and $v_{h-1}e_h$ but not $f_{h-1}e_h$, we may assume that F contains $f_{h-1}v_{h-1}$ but not $v_{h-1}e_h$. It follows that F cannot contain any arc entering or leaving e_h, and so by P4 it contains arcs $v_{h-1}u_h$, u_hf_h and f_hg. By P2 and P3, we see that if F is of type 1 at 2 then F is a directed cycle comprising the path just described from f_1 to g, with the addition of the path $ge_1v_ku_1f_1$ of four arcs, while if F is of type 0 at 2 then it consists of the same path from f_1 to g with the addition of the five arcs listed in (3). These two possibilities for F have different parities, and so cancel each other out.

Suppose now that F is of type 2 at $h-1$; then there are $h-2$ routes that F can take between f_1 and f_{h-1}, but all of them enter f_{h-1} along arc $e_{h-1}f_{h-1}$, so that, by P1, F must contain arc $f_{h-1}v_{h-1}$. Since the subflows that contain $v_{h-1}u_h$ but not $v_{h-1}e_h$ or e_hu_h pair off with those that contain $v_{h-1}e_h$ and e_hu_h but not $v_{h-1}u_h$, we may suppose that F contains $v_{h-1}e_h$ but not e_hu_h. By P4 it must therefore contain arcs e_hf_h, f_hv_h and v_hu_h, but not u_hf_h or, therefore, f_hg or e_hg. If it contains arc u_hg then, as in the previous paragraph, there is one possibility for F if it has type 0 at 2 and one possibility if it has type 1 at 2, and these two possibilities have different parities and so cancel each other out; if however F has type 2 at 2, then by P2 and P3 F is a cycle comprising a path from f_1 to g that is completed by one of the paths $gu_1e_1f_1$ and $ge_1v_ku_1f_1$; and these two cycles have lengths of different parities, and so cancel each other out. The only other possibility is that F contains arc u_hu_1, in which case, by P2 and P3, F is of type 2 at 2 and is a cycle that is completed by the path $u_1e_1f_1$. Since there is only one route that F can take between f_1 and f_{h-1} if it is of type 2 at 2, it follows that the number of subflows with the parity of this subflow F (which is the same as the parity of h) is one more than the number of subflows with the opposite parity, so that the numbers of even and odd subflows are not the same. This gives a contradiction in the same way as in Lemma 5, and this completes the proof of Lemma 6. ■

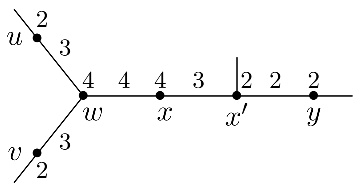


Figure 6

Lemma 7. *Suppose w is a 3-vertex of $G_{\min}$ with neighbours u, v, x, all of degree 2, and let the other neighbours of u, v, x be u', v', x' respectively, necessarily of degree 3 by Lemma 2(c). Then none of u', v', x' has another neighbour of degree 2.*

Proof. Suppose one of them does, say x' has a neighbour y of degree 2, as in Figure 6. Note that $y \neq u$ or v by Lemma 6. Choose a total colouring of $G_{\min} - x$ from its lists, and erase the colours on all elements of the paths uwv and $wxx'y$. For each uncoloured element z of $G_{\min}$, let $L(z)$ comprise those colours from $\Lambda(z)$ that are not used on any neighbour of z. Then $|L(z)|$ is at least as large as indicated beside each element in Figure 6, and there is no loss of generality in assuming that $|L(z)|$ has exactly this size in each case. Moreover, the lists of x', $x'y$ and y are not all equal, since these elements were coloured in the colouring of $G_{\min} - x$, and so by Lemma 1 there are two different colourings $\lambda_{x'}, \lambda_{x'y}, \lambda_y$ and $\mu_{x'}, \mu_{x'y}, \mu_y$ of these three elements such that one of the following holds:

(i) $\lambda_{x'} = \mu_{x'}$ and $\lambda_{x'y} \neq \mu_{x'y}$;

(ii) $\lambda_{x'y} = \mu_{x'y}$ and $\lambda_{x'} \neq \mu_{x'}$;

(iii) $\lambda_{x'} = \mu_{x'y}$ and $\lambda_{x'y} = \mu_{x'}$;

(iv) $\lambda_{x'} = \mu_{x'y}$ and $\lambda_{x'y} \neq \mu_{x'}$.

We will consider each possibility in turn.

Suppose first that (i) holds. Let us replace the lists of x and xx' by $L(x) \setminus \{\lambda_{x'}\}$ and $L(xx') \setminus \{\lambda_{x'}\}$ respectively. Then the sizes of the lists of all elements except $x', x'y, y$ are at least as large as indicated on the graph G_3 in Figure 1, and these elements can be coloured from their lists by Lemma 3. This colouring can then be extended to the elements $x', x'y, y$ by using one of the two possible colourings, since at least one of $\lambda_{x'y}$ and $\mu_{x'y}$ is different from the colour that has been given to xx'.

Before considering (ii)–(iv), note that if $L(xx')$ does not contain all the (two or three) colours in the set $\{\lambda_{x'}, \lambda_{x'y}, \mu_{x'}, \mu_{x'y}\}$, then we can colour $x', x'y, y$ first in such a way that at least one of x' and $x'y$ is given a colour not in $L(xx')$; the task of extending this colouring to the remaining elements is then that of colouring the graph G_3 in Figure 1 from lists of the size indicated, which can be done by Lemma 3. So we may suppose from now on that $L(xx')$ contains all these colours.

Suppose now that (ii) holds. Then $L(xx') = \{\lambda_{x'}, \mu_{x'}, \lambda_{x'y}\}$. Replace the list of x by $L(x) \setminus \{\lambda_{x'}, \mu_{x'}\}$. Then the sizes of the lists of all elements except $xx', x', x'y, y$ are at least as large as indicated on the graph G_1 in Figure 1, and these elements can be coloured from their lists by Lemma 3. This colouring can then be extended to the remaining elements by giving $xx', x', x'y, y$ the colours $\mu_{x'}, \lambda_{x'}, \lambda_{x'y}, \lambda_y$ if wx has colour $\lambda_{x'}$, and $\lambda_{x'}, \mu_{x'}, \mu_{x'y}, \mu_y$ otherwise.

Suppose next that (iii) holds. Then $L(xx') = \{\lambda_{x'}, \mu_{x'}, c\}$, for some colour c. Replace the lists of wx and x by $L(wx) \setminus \{c\}$ and $L(x) \setminus \{c\}$ respectively. Then the sizes of the lists of all elements except $xx', x', x'y, y$ are at least as large as indicated on the graph G_2 in Figure 1, and these elements can be coloured from their lists by Lemma 3. This colouring can then be extended to the elements $xx', x', x'y, y$ by colouring xx' with c, and choosing one of the two colourings for $x', x'y, y$ so that x' does not have the same colour as x.

Finally, suppose that (iv) holds. Then $L(xx') = \{\lambda_{x'}, \mu_{x'}, \lambda_{x'y}\}$. If $L(xx') \not\subset L(x)$, then give $xx', x', x'y, y$ colours $\mu_{x'}, \lambda_{x'}, \lambda_{x'y}, \lambda_y$ if the colour not in $L(x)$ is $\lambda_{x'}$ or $\mu_{x'}$, and colours $\lambda_{x'y}, \mu_{x'}, \mu_{x'y}, \mu_y$ if it is $\lambda_{x'y}$; the task of colouring the remaining elements is then that of colouring the graph G_2 in Figure 1 from lists of the size indicated, which can be done by Lemma 3. So we may suppose that $L(xx') \subset L(x)$, say $L(x) = L(xx') \cup \{c\}$. Give w a colour λ_w not in $L(u) \cup \{c\}$, then colour v, vw, uw, u, wx in that order, which is possible since each of these elements has a spare colour at the time it is coloured. Now give x a colour $\lambda_x \in L(x) \setminus \{\lambda_w, \lambda_{wx}, \mu_{x'}\}$ (where λ_{wx} is the colour given to wx), choosing $\lambda_x = c$ if $\lambda_{wx} \neq c$. This colouring can then be extended to the remaining elements by giving $xx', x', x'y, y$ the colours $\mu_{x'}, \lambda_{x'}, \lambda_{x'y}, \lambda_y$ if whichever of λ_{wx}, λ_x is not c is equal to $\lambda_{x'y}$, and colours $\lambda_{x'y}, \mu_{x'}, \mu_{x'y}, \mu_y$, otherwise.

In every case, the colouring can be extended to all elements of $G_{\min}$, and this contradiction completes the proof of Lemma 7. ∎

The following result proves Theorem 2 when $\mathrm{mad}(G_{\min}) < 2\frac{1}{2}$.

Theorem 4. *Define $G_{\min}$ as in the first paragraph of Section 3. Then $\mathrm{mad}(G_{\min}) \geqslant 2\frac{1}{2}$. Moreover, if $\mathrm{mad}(G_{\min}) = 2\frac{1}{2}$ then $G_{\min}$ has the following form: the submultigraph G_3 of $G_{\min}$ induced by its 3-vertices is a union of disjoint cycles—call them D-cycles—and the vertices of G_3 are connected in pairs by paths of three edges lying outside G_3—call them D-paths—whose internal vertices both have degree 2.*

Proof. Let G_{23} be the bipartite subgraph of $G_{\min}$ comprising $V(G_{\min})$ and all edges of $G_{\min}$ that join a 2-vertex to a 3-vertex. By Lemma 6, G_{23} is a forest. By Lemma 2(b), no 2-vertex of $G_{\min}$ is an isolated vertex in G_{23}. By parts (a) and (c) of Lemma 2, a component of G_{23} that is a nontrivial path either consists of a single edge joining a 2-vertex and a 3-vertex of $G_{\min}$, or else ends in two 3-vertices of $G_{\min}$. In view of Lemma 7, there are thus three different types of component in G_{23}:

(i) a component containing three 2-vertices and four 3-vertices of $G_{\min}$, consisting of a 3-vertex adjacent to three 2-vertices which in turn are adjacent to three more 3-vertices, necessarily distinct by Lemma 6;

(ii) a component that is a path of even length, say $2k$, possibly trivial (i.e., $k = 0$), containing k 2-vertices and $k+1$ 3-vertices of $G_{\min}$;

(iii) a component consisting of a single edge joining a 2-vertex and a 3-vertex of $G_{\min}$.

It follows that $G_{\min}$ has at least as many 3-vertices as 2-vertices, and so $\mathrm{mad}(G_{\min}) \geqslant 2\frac{1}{2}$. Moreover, if $\mathrm{mad}(G_{\min}) = 2\frac{1}{2}$ then there are no components of types (i) and (ii), and so $G_{\min} \setminus E(G_{23})$ consists of the disjoint union of a 2-regular multigraph induced by the 3-vertices of $G_{\min}$ and a 1-regular graph induced by the 2-vertices of $G_{\min}$; thus $G_{\min}$ has the form described in the theorem. ∎

Corollary 4.1. *Let G be a multigraph with maximum degree 3 such that $\mathrm{mad}(G) < 2\frac{1}{2}$. Then G is totally 4-choosable.*

Proof. If not, then G has a minimal non-totally-4-choosable submultigraph $G_{\min}$ with maximum degree at most 3, and $G_{\min}$ contains no copy of C_4^+ since $\mathrm{mad}(G) < 2\frac{1}{2}$. By Theorem 4, $\mathrm{mad}(G_{\min}) \geqslant 2\frac{1}{2}$, which is a contradiction. ∎

4. EXTENSION TO $\mathrm{mad}(G) = 2\frac{1}{2}$

We start by examining a D-path.

Lemma 8. *Let $P\colon uxyv$ be a path, and suppose that every element z of P is given a list $L(z)$ of four colours, and that u, ux, vy, v are then coloured with colours $\lambda_u, \lambda_{ux}, \lambda_{vy}, \lambda_v$ from their lists, where $\lambda_u \neq \lambda_{ux}$ and $\lambda_{vy} \neq \lambda_v$. Then this colouring can be extended to a total L-colouring of P unless the lists of x, xy, y, and the colours assigned to u, ux, vy, v, match one of the rows in Table* 1.

Type	$L(x)$	$L(xy)$	$L(y)$	Bad colourings of u, ux, vy, v	
1	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	μ,ν,μ,ν	$(\mu,\nu \in \{a,b,c,d\},\ \mu \neq \nu)$
2	$\{a,b,c,d\}$	$\{a,b,c,d\}$	$\{a,b,d,e\}$	μ,c,μ,e	$(\mu \in \{a,b,d\})$
3	$\{a,b,c,e\}$	$\{a,b,c,d\}$	$\{a,b,c,d\}$	e,ν,d,ν	$(\nu \in \{a,b,c\})$
4	$\{a,b,c,e\}$	$\{a,b,c,d\}$	$\{a,b,d,e\}$	e,c,d,e	
5	$\{a,b,c,e\}$	$\{a,b,c,d\}$	$\{a,b,d,f\}$	e,c,d,f	

Table 1

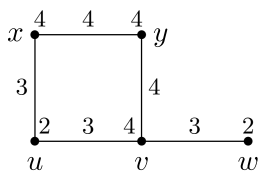
Proof. Let $L'(x) := L(x) \setminus \{\lambda_u, \lambda_{ux}\}$, $L'(xy) := L(xy) \setminus \{\lambda_{ux}, \lambda_{vy}\}$ and $L'(y) := L(y) \setminus \{\lambda_{vy}, \lambda_v\}$. Then x, xy, y can be coloured from these lists unless they are identical lists of two colours, say $L'(x) = L'(xy) = L'(y) = \{a,b\}$, which implies $\lambda_{ux} \neq \lambda_{vy}$, say $\lambda_{ux} = c$ and $\lambda_{vy} = d$, and $L(x) = \{a,b,c,\lambda_u\}$, $L(xy) = \{a,b,c,d\}$, and $L(y) = \{a,b,d,\lambda_v\}$. Evidently $\lambda_u \notin \{a,b,c\}$ and $\lambda_v \notin \{a,b,d\}$. So the essentially different possibilities for (λ_u, λ_v) are then (d,c), (d,e), (e,c), (e,e) and (e,f), where distinct letters represent distinct colours, giving the lists shown in Table 1. In each case, if an assignment of colours $\mu_u, \mu_{ux}, \mu_{vy}, \mu_v$ to u, ux, vy, v cannot be extended to x, xy and y, that is,

$$L(x) \setminus \{\mu_u, \mu_{ux}\} = L(xy) \setminus \{\mu_{ux}, \mu_{vy}\} = L(y) \setminus \{\mu_{vy}, \mu_v\} = X, \text{ say,}$$

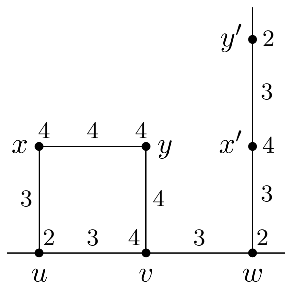
where $|X| = 2$, then this 'bad colouring' must match the pattern shown in the last column of Table 1. ■

In what follows, if we say, for example, that an element w is given the unique colour in $L(uv) \setminus L(u)$, then this implies that there is a unique colour in $L(uv) \setminus L(u)$, and this colour is given to w. But if we say that w is *not* given the unique colour in $L(uv) \setminus L(u)$, then this should not be taken to imply that there is such a unique colour, but only that *if* there is one then it is not used to colour w.

Lemma 9. *Suppose that each element of the graph in Figure* 7(a) *is given a list of colours of the size indicated against it, and* f_1, f_2 *is a pair of distinct 'forbidden' colours. Then the graph can be totally coloured from these lists in such a way that if* vw *is coloured with* f_1 *then* w *is not coloured with* f_2.



(a)



(b)

Figure 7

Proof. We will carry out the colouring in the following way. We will first colour vw and w (or in one case just vw) so that the colouring can be extended to u, uv and v; colouring ux and vy is now trivial, after which we will colour x, xy and y. The possible problems arise in colouring u, uv, v and in colouring x, xy, y. Colouring u, uv, v is not difficult in itself: the only time it cannot be done is if we have given vw the unique colour in $L(uv) \setminus L(u)$ and w the unique colour in $L(v) \setminus L(uv)$, since then, and only then, the three elements u, uv, v must all be coloured with the same two colours; this can only happen if $L(u) \subset L(uv) \subset L(v)$, and it is easily avoided by recolouring vw or w. However, the problem is not just to colour u, uv and v, but to colour them in such a way that the resulting colouring of u, ux, vy and v

can be extended to x, xy and y. This is automatically possible unless $L(x)$, $L(xy)$ and $L(y)$ match one of the rows in Table 1. There are two cases to consider.

Case 1. $L(x) = L(xy) = L(y)$.
Then the bad colourings for u, ux, vy, v are of the form μ, ν, μ, ν, as in row 1 of Table 1. Let $L(x) = \{a^*, b^*, c^*, d^*\}$, say, so that we can use the unstarred letters a, b, c, d to denote colours without implying that they belong to $L(x)$. Colour vw and w with colours $d \in L(vw) \setminus \{f_1\}$ and $e \in L(w) \setminus \{d\}$ in such a way that this colouring can be extended to u, uv and v. (This will hold if d is not the unique colour in $L(uv) \setminus L(u)$ or e is not the unique colour in $L(v) \setminus L(uv)$.) For each uncoloured element z, let $L'(z)$ denote the set of colours that are now available for use on z. Then $L'(u)$, $L'(uv)$ and $L'(v)$ are not identical sets of two colours. If there is a colour λ in one of these sets that is not in $L'(vy)$, then u, uv, v can be coloured so that λ is used on one of them. If λ is used on u then vy cannot be given the same colour as u, and if λ is used on uv or v then there is a choice of at least two colours for vy and we can colour vy differently from u; in either case the colouring will extend to x, xy and y. A similar remark applies with $L(ux)$ in place of $L'(vy)$. We may also suppose that $|L'(vy)| = 3$, since if $|L'(vy)| = 4$ then after colouring u, uv, v we can colour vy differently from u. Thus if this colouring of vw and w does *not* extend to the other elements, then

$$L(u) \cup L'(uv) \cup L'(v) = L(ux) = L'(vy) = \{a, b, c\},$$

say, where possibly d or $e \in \{a, b, c\}$. However, this means that $L(vy) = \{a, b, c, d\}$, so that these four colours are all distinct, and d is the unique colour in $L(vy) \setminus L(ux)$. Without loss of generality, $L(u) = \{a, b\}$, where possibly $f_1 = b$ but $f_1 \neq a$.

If $L(vw)$ contains a colour d' that is neither f_1 nor d nor the unique colour in $L(uv) \setminus L(u)$, then we can colour vw with d' and w with any colour $e' \in L(w) \setminus \{d'\}$, and this colouring will extend to u, uv, v, then to ux, vy, and then to x, xy, y, since d' is not the unique colour in $L(vy) \setminus L(ux)$. We may therefore assume that $L(uv) = \{a, b, c\}$ and $L(vw) = \{c, d, f_1\}$. Thus $f_1 \notin \{c, d\}$.

If $L(w) \setminus \{c\}$ contains a colour μ that is not the unique colour in $L(v) \setminus L(uv)$ then we can colour vw, w with c, μ and this colouring will extend to all the remaining elements. Thus we may assume that $L(v) = \{a, b, c, g\}$

and $L(w) = \{c, g\}$, where possibly $g = d$ or e. Now we can colour ux, u, uv with c, b, a and vy, v, vw, w with

$$\begin{array}{ll} b, g, d, c & \text{if } g \neq d, \\ c, d, f_1, c & \text{if } g = d \text{ and } c \neq f_2, \\ d, c, f_1, d & \text{if } g = d \text{ and } c = f_2. \end{array}$$

In each case either vy or v has a colour that is not in $\{a, b, c\}$ and so is different from the colour of u or ux respectively, and so this colouring can be extended to the remaining elements.

Case 2. The lists of x, xy and y are not all equal.
Suppose first that there is a colour $\lambda \in L(vw) \setminus L(w)$ such that $\lambda \neq f_1$. Colour vw with λ and define $L'(z) := L(z) \setminus \{\lambda\}$ if $z \in \{uv, v, vy\}$ and $L'(z) := L(z)$ for all other uncoloured elements. Then u, uv and v now have lists of sizes (at least) 2, 2 and 3. For each of the two colours in $L(u)$, we can colour u with that colour, then colour uv and v, followed by ux, vy and w. The two colourings so obtained differ on u, and so at least one of them can be extended to x, xy and y unless the lists of those three vertices conform to the pattern in row 2 of Table 1. In that case, if no colouring extends, then it must be the case that $L'(u) = L'(uv) = \{a, b\}$, say (where a, b are not necessarily the same as in Table 1), and $L'(ux) = \{a, b, c\}$ and $L'(v) = \{a, b, e\}$ where c, e is the forbidden pair of colours on ux and v (as in row 2 of Table 1), and $L'(vy) = \{a, b, e\}$, so that vy has to have the same colour as u in both of these colourings. Then $L(v) = L(vy) = \{a, b, \lambda, e\}$ and $L(uv) = \{a, b, \lambda\}$, so that λ is the unique element in $L(uv) \setminus L(u)$, and $\lambda \notin \{a, b, e\}$ but possibly $\lambda = c$. There is a colour $a' \in L(vw) \setminus \{\lambda, f_1\}$, where (by interchanging a and b if necessary) we may suppose that $a' \neq b$, but possibly $a' \in \{a, c, e\}$. Colour vw with a' and w with a colour $b' \in L(w) \setminus \{a'\}$. If $a' = e$ then colour u, uv, v with a, b, λ if $b' = b$ and with a, λ, b otherwise; this colouring can be extended to the remaining elements since v does not have colour e. If $a' \neq e$ then colour ux, u, uv with b, a, λ, and vy, v with b, e if $b' = b$ or with e, b otherwise; this colouring can be extended to the remaining elements since ux does not have colour c.

In view of this, we may assume that f_1 is the unique colour in $L(vw) \setminus L(w)$, say $L(vw) = \{a, b, c\}$ and $L(w) = \{a, b\}$, where $(f_1, f_2) = (c, a)$ (and a, b, c have no connection with Table 1). Then the non-forbidden colourings available for vw, w are $(g_1, g_2) = (b, a)$, (a, b) and (c, b). At least one of these has the property that g_1 is not the unique colour in $L(uv) \setminus L(u)$ and

g_2 is not the unique colour in $L(v) \setminus L(uv)$ (since if b is the unique colour in $L(v) \setminus L(uv)$ then b cannot also be the unique colour in $L(uv) \setminus L(u)$). If we colour vw and w with that pair, then each of u, uv and v has a usable list of at least two colours, and u and uv do not have equal lists of two colours, and uv and v do not have equal lists of two colours. Thus not all the ways of colouring u, uv, v from these lists use the same colour on v, and not all of them use the same colour on u, and so whatever type the lists of x, xy, y conform to in Table 1 (apart from type 1, which was dealt with in Case 1), at least one of these colourings can be extended to all the remaining elements. ■

From now on we assume that $\text{mad}(G_{\min}) = 2\frac{1}{2}$, so that $G_{\min}$ has the structure described in Theorem 4.

Lemma 10. $G_{\min}$ *does not contain a* D*-path* $uxyv$ *such that* u *and* v *are joined by exactly one edge.*

Proof. Suppose it does. Let w be the vertex (other than u) that is adjacent to v in the D-cycle containing the edge uv, and let the D-path incident with w be $wx'y'w'$. By hypothesis, $G_{\min} - x$ has a total colouring from its lists. Uncolour every element shown in Figure 7(b), and for every uncoloured element z let $L(z)$ be the set of colours from $\Lambda(z)$ that can now be used on z. Then $|L(z)|$ is at least as large as indicated beside each element z in Figure 7(b), and there is no loss of generality in supposing that $|L(z)|$ has exactly this size, in each case. Let f_3 be the unique colour in $L(x'y') \setminus L(y')$ if there is one, otherwise (if $|L(x'y') \setminus L(y')| > 1$) let f_3 be an arbitrary colour. Let f_2 be the unique colour in $L(x') \setminus L(x'y')$ if there is one, otherwise let f_2 be an arbitrary colour. Let f_1 be the unique colour in $L(wx') \setminus \{f_2, f_3\}$ if there is one, otherwise let f_1 be an arbitrary colour. By Lemma 9, all elements in Figure 7(a) (regarded as a subgraph of Figure 7(b)) can be coloured from their lists in such a way that if w is coloured with f_2 then vw is not coloured with f_1. Colour them thus, and then colour wx' in such a way that if w is coloured with f_2 then wx' is not coloured with f_3, which is possible by the definition of f_1. Now there are at least two colours available to each of x', $x'y'$ and y', and by the definition of f_2 and f_3, it is not exactly the same two colours in each case. Thus the colouring can be extended to these elements. So all elements of $G_{\min}$ can be totally coloured from their lists, and this contradiction completes the proof of Lemma 10. ■

We will complete the proof of Theorem 2 by describing how to colour $G_{\min}$. Let the D-cycles of $G_{\min}$ be $D_1, \ldots, D_t$. A D-path that connects vertices in

D_r and D_s will be called *forwards from* D_r and *backwards from* D_s if $r < s$, while a D-path connecting two vertices in D_r will be called *internal to* D_r. The vertices of D_r that are connected by internal D-paths of type 1 (i.e., with type-1 lists as shown in Table 1) will be called *essential* vertices, and all other vertices of D_r are *inessential* vertices.

For each D-cycle D_r ($r = 1, \ldots, t$) in turn, we will carry out the following four steps:

Step 1. Colour the essential vertices of D_r in such a way that the colouring can be extended to all elements of the type-1 internal D-paths.

Step 2. Colour the inessential vertices of D_r.

Step 3. Colour the edges of D_r.

Step 4. Colour all remaining uncoloured elements of any D-paths that are internal to D_r or backwards from D_r, and colour the first edge of every D-path that is forwards from D_r.

If Step 1 is carried out appropriately then the other steps are straightforward, as we now describe. In Step 2, we must colour the inessential vertices of the cycle D_r, each of which has a list of four colours; for reasons we are about to explain, there may be one colour in each list that we must not use, but this causes no problem, since a cycle is clearly 3-choosable. To ensure that the middle three elements of an internal D-path $P : uxyv$, not of type 1, can be coloured in Step 4, we will colour the inessential vertices u and v so that v does not have colour e if P is of type 2 in Table 1, and u does not have colour e if P is of type 3, 4 or 5. (If P does not have any of the five types, then there is no need for any restriction on the colours of u and v.) If $P : uxyv$ is a backwards D-path with $u \in D_r$, then at this point v and vy are already coloured; to ensure that the middle three elements of P can be coloured in Step 4, there is at most one colour that we must avoid for u, which is the colour of vy if P has type 1 or 2, and colour e if P has type 3, 4 or 5. If u is the endvertex of a forwards D-path, then no restriction is needed on the colour of u. In every case there is at most one colour in the list of each inessential vertex that we cannot use, and so Step 2 is easily completed.

In Step 3 we must colour the edges of D_r, each of which now has a usable list of at least two colours. The only problem is if D_r has odd length and every edge has the *same* list of two colours. In carrying out Steps 1 and 2, we must ensure that there is enough flexibility to change the colouring so that this does not happen; it suffices if we can change the colours of some of the vertices of D_r while leaving unchanged the colours of the endvertices of at least one edge of D_r. Assuming that Step 3 can be completed, Step 4 is now straightforward: every edge not in D_r but incident with a vertex of D_r can be coloured with a colour from its list that is different from the colours of its three coloured neighbours, and the three middle elements of every internal and backwards D-path can now be coloured because of the way the endvertices were coloured in Steps 1 and 2.

If there are no essential vertices in D_r, then Step 1 is unnecessary and we proceed immediately to Step 2. In this case every vertex of D_r has an effective list of at least three colours. If the vertices are $v_1, \ldots, v_n$ in order around D_r, then they are easily coloured in this order, with colours $c_1, \ldots, c_n$, say. If Step 3 fails because n is odd and every edge now has the same usable list of two colours, then uncolour v_n, recolour v_{n-1} with a colour different from c_{n-1}, and then recolour v_n. This will change the usable lists of some but not all of the edges of D_r, and so avoid the problem, except possibly when $n = 3$, when all three edge-lists may have changed. But the usable list of v_2v_3 will not change if we have merely interchanged the colours of v_2 and v_3, and if we cannot interchange these two colours then at least one of these vertices has list different from $\{c_1, c_2, c_3\}$, and then we can change the colour of that vertex without changing the colour of either of its neighbours. Thus in every case the vertices can be coloured in such a way that the edges can be coloured in Step 3, and Step 4 is then straightforward as we described above. Thus from now on we may assume that D_r has at least one pair of essential vertices.

In particular, we may assume that D_r has at least three vertices, since otherwise $G_{\min}$ consists of a type-1 D-path whose endvertices are connected by two parallel edges, i.e., $G_{\min} \cong C_4^+$. In this case it is straightforward to prove (using Table 1) that $G_{\min}$ is totally Λ-colourable unless every element of $G_{\min}$ has the same list, which is explicitly disallowed in the definition of Λ.

Before describing how to carry out Step 1, we need some notation. Let H^+ be the subgraph of $G_{\min}$ induced by the vertices of D_r and all type-1 D-paths that are internal to D_r; by the previous paragraph, H^+ is a (simple) graph. Form H from H^+ by suppressing all the inessential vertices; that is, each segment $u_1u_2, \ldots, u_k$ of D_r such that u_1 and u_k are essential vertices and $\{u_2, \ldots, u_{k-1}\}$ is a nonempty set of inessential vertices is replaced by a single edge u_1u_k, which we call a *virtual edge* of H; every other edge of H

is a *real edge*. Finally, form K from H by contracting the edges of all the D-paths in H. Then K is a 4-regular pseudograph, that is, it may contain loops as well as parallel edges. (A loop can arise if a virtual edge of H joins the endvertices of a D-path.) So every vertex u of K corresponds to a D-path $P(u) : u^{(1)}x^{(1)}x^{(2)}u^{(2)}$ of H. The three lists $\Lambda(x^{(1)})$, $\Lambda(x^{(1)}x^{(2)})$ and $\Lambda(x^{(2)})$ are all equal, since $P(u)$ is a type-1 D-path by construction; let $\check{\Lambda}(u)$ denote this common list. We must ensure that, for each vertex $u \in V(K)$, either $u^{(1)}$ and $u^{(2)}$ are coloured with the same colour, or else one of them is given a colour not in $\check{\Lambda}(u)$; then this colouring can be extended to the three middle elements of $P(u)$, no matter how the end edges of $P(u)$ are coloured. Let $k := |V(K)|$. There are several cases to consider.

Case 1. For some vertex u of K, either $\Lambda(u^{(1)})$ or $\Lambda(u^{(2)})$, w.l.o.g. $\Lambda(u^{(1)})$, is different from $\check{\Lambda}(u)$.
Colour $u^{(1)}$ with a colour in $\Lambda(u^{(1)}) \setminus \check{\Lambda}(u)$, and form K' from K by deleting the two edges corresponding to edges incident with $u^{(1)}$ in H; clearly K' is connected, since H^+ is 2-connected and so $H - u^{(1)}$ is connected. Order the vertices of K' as $u_1, \ldots, u_k$, where $u_k = u$, in such a way that, for each j, the subgraph of K' induced by $u_j, \ldots, u_k$ is connected. (For example, let T_1 be a spanning tree of K'. For $i = 1, 2, \ldots, k-1$ let u_i be an endvertex of T_i that is different from u, and set $T_{i+1} := T_i - u_i$.) For $i = 1, \ldots, k-1$ in turn, we will colour the two vertices $u_i^{(1)}$ and $u_i^{(2)}$ of H. When we come to colour these vertices, they are together adjacent to at most three vertices that have already been coloured, since at least one of them is adjacent to $u_j^{(1)}$ or $u_j^{(2)}$ for some $j \in \{i+1, \ldots, k-1\}$, or to $u_k^{(2)}$. Say $u_i^{(1)}$ has at most one coloured neighbour and $u_i^{(2)}$ has at most two coloured neighbours, so that if $X_i^{(1)}$ and $X_i^{(2)}$ are the sets of colours that we can use on $u_i^{(1)}$ and $u_i^{(2)}$ respectively, then $|X_i^{(1)}| \geqslant 3$ and $|X_i^{(2)}| \geqslant 2$. If $X_i^{(1)}$ or $X_i^{(2)}$ contains a colour not in $\check{\Lambda}(u_i)$, colour one of $u_i^{(1)}$ and $u_i^{(2)}$ with such a colour, and colour the other one with any colour from its list. If however $X_i^{(1)}$ and $X_i^{(2)}$ are both subsets of $\check{\Lambda}(u_i)$, then they have a colour in common, and so we colour $u_i^{(1)}$ and $u_i^{(2)}$ the same. In all cases, the remaining elements of $P(u_i)$ can be coloured in Step 4, whatever colours are used in Steps 2 and 3. Finally, $u_k^{(2)}$ can be coloured with any colour from its list that is not used on either of its neighbours in D_r. The choice of colour for $u_k^{(2)}$ can thus be left until after Step 2 is completed, in order to ensure that not every edge of D_r has the same list of two colours. This completes the discussion of Case 1.

In view of Case 1, we may assume from now on that $\Lambda(u^{(1)}) = \Lambda(u^{(2)}) = \check{\Lambda}(u)$, for every vertex u of K. The problem thus reduces to that of colouring the vertices of K properly from their lists, the colour assigned to a vertex u in K then being given to both $u^{(1)}$ and $u^{(2)}$ in D_r. Thus effectively we need the choosability analogue of Brooks's theorem, which was proved by both Vizing [8] and Erdős, Rubin and Taylor [5]. However, we must proceed with caution because of the need to find two different colourings in case the edge-lists in Step 3 are all equal. If there are no virtual edges in H, hence no inessential vertices in D_r, then this last problem does not arise, since then D_r has even length and it does not matter if its edge-lists are all equal; thus we need only one vertex-colouring of K in this case. Note that if there are two vertices of K that are joined only by a virtual edge, and not by a real edge as well, then they can be coloured with the same colour, since they do not correspond to adjacent vertices in D_r.

Case 2. Case 1 does not arise, and either $|V(K)| \leqslant 4$ or K is not a (simple) graph.
In this case we carry out Steps 1 and 2 simultaneously. Recall that every inessential vertex has an effective list of at least three colours, while the essential vertices, which correspond to vertices of K, have lists of four colours. Let the vertices of K be $u_1, \ldots, u_k$. In what follows it is to be assumed that whenever we colour a vertex u_i of K, then we automatically and simultaneously give the colour of u_i to the vertices $u_i^{(1)}$ and $u_i^{(2)}$ of H.

Suppose first that K has a loop, which is necessarily a virtual edge, by Lemma 10. Choose a loop, based at u_k, say, and let L be the corresponding subpath of D_r between $u_k^{(1)}$ and $u_k^{(2)}$; that is, L is the segment of D_r that is replaced in H by the virtual edge that becomes a loop in K. First colour the vertices of K; this can be done in such a way that the subgraph induced by the uncoloured vertices remains connected and u_k is coloured last (using a spanning tree, as in Case 1). Then colour all the inessential vertices of D_r except those in L; now at least one edge of D_r has two coloured endvertices, since there must be at least one vertex of D_r that is not in (or an endvertex of) L. Finally, colour the inessential vertices in L. There is a choice of colours for the first of these to be coloured, since if it has two coloured neighbours then they are $u_k^{(1)}$ and $u_k^{(2)}$, which have the same colour. Thus there are least two different possible colourings of this type, and at least one of these will work in Step 3.

Suppose now that K has no loop and $k \leqslant 4$; then $2 \leqslant k \leqslant 4$. If there are no virtual edges in H then we need only one vertex-colouring of K and we simply colour $u_1, \dots, u_k$ in order. So suppose there is at least one virtual edge. Label the vertices of H and K so that $u_1^{(1)}$ and $u_2^{(1)}$ are the endvertices of a virtual edge of H, corresponding to a subpath P of D_r. Colour first the inessential vertices in P, then colour u_1 (and $u_1^{(1)}$ and $u_1^{(2)}$) differently from the neighbour v_1 of $u_1^{(1)}$ in P, and u_2 (and $u_2^{(1)}$ and $u_2^{(2)}$) differently from both u_1 and the neighbour of $u_2^{(1)}$ in P, before colouring u_3 and u_4 if they exist, followed by any remaining inessential vertices. Note that there are two choices of colour for u_2, and so the colour of u_2 can be changed without changing the colour of either endvertex of the edge $u_1^{(1)}v_1$; thus at least one colouring constructed in this way will work in Step 3.

Finally, suppose that $k \geqslant 5$ and K has a pair of parallel edges, between u_{k-1} and u_k, say. First colour $u_1, \dots, u_{k-2}$, then colour all inessential vertices that are not in virtual edges between u_{k-1} and u_k. Since $k \geqslant 5$, at least one edge of D_r now has two coloured endvertices. Now u_{k-1} and u_k each have at least two possible colours, since each has only two edges going to vertices different from the other; thus the colouring of these two vertices can be completed in two different ways, and at least one of these will work in Step 3. This completes the discussion of Case 2.

In view of Case 2, we may assume from now on that K is a 4-regular (simple) graph. We say that $\check{\Lambda}$ is *constant* on a set $X \subseteq V(K)$ if $\check{\Lambda}(u) = \check{\Lambda}(v)$ for all $u, v \in X$.

Case 3. Cases 1 and 2 do not arise, and K is not 2-connected.
From the way in which K is constructed, it is clear that if u is a cutvertex of K then $K - u$ has two components, and u is joined by two edges to each of them. Let B_0 be an endblock of K with cutvertex z_0, and let $A_0 := K - (B_0 - z_0)$, so that $K = A_0 \cup B_0$ and $A_0 \cap B_0 = \{z_0\}$. We consider two subcases.

Subcase 3.1. $\check{\Lambda}$ is constant on B_0.
Colour the vertices of A_0, by keeping the graph induced by the uncoloured vertices connected as before, ending with z_0. Independently, colour the vertices of a copy of B_0, ending with z_0. Since $\check{\Lambda}$ is constant on B_0, the colours in this second colouring can be permuted (in more than one way) so as to agree with the colour previously assigned to z_0. Thus there are at least two different colourings of K, which agree on all vertices of A_0 but are different on $B_0 - z_0$, and at least one of these must work in Step 3.

Subcase 3.2. $\check{\Lambda}$ is not constant on B_0.
Let u, v be adjacent vertices of B_0 that have different lists, where $u \neq z_0$, and colour u with a colour in $\check{\Lambda}(u) \setminus \check{\Lambda}(v)$. Note that $B_0 - u$ is connected, since B_0 is 2-connected. Colour the vertices of $A_0 - z_0$ (as in Subcase 3.1, but leaving z_0 uncoloured), and then colour all vertices of $B_0 - u$ (including z_0), keeping the graph induced by the uncoloured vertices connected and ending with v; this is possible because although v has four coloured neighbours at the time we colour it, one of them, u, has a colour not in $\check{\Lambda}(v)$. There is a choice of colours for the first vertex of $B_0 - \{u, z_0\}$ to be coloured, since it has at most two coloured neighbours at the time it is coloured. Thus we can obtain two different colourings in this way, which agree on all vertices of $A_0 - z_0$ but diverge thereafter, and at least one of these must work in Step 3. This completes the discussion of Case 3.

Case 4. Cases 1–3 do not arise, and $\check{\Lambda}$ is not constant on K.
By Case 3, K is now 2-connected. Let u, v be adjacent vertices such that $\check{\Lambda}(u) \neq \check{\Lambda}(v)$. Note that $K - u$ and $K - v$ are connected. If there are no virtual edges then we need only one vertex-colouring of K, which we can construct by giving u a colour in $\check{\Lambda}(u) \setminus \check{\Lambda}(v)$ and then colouring all remaining vertices, ending with v. So assume there is at least one virtual edge. Note that K has no cut-edges, and so every edge belongs to a cycle, and a cycle containing edge uv must contain another edge $u'v'$ such that $\check{\Lambda}(u') \neq \check{\Lambda}(v')$; thus by relabelling u', v' as u, v if necessary, we may assume that uv is not the unique virtual edge in K. Let $P : u_0, \ldots, u_l$ be a shortest path in K such that $u_0 u_1$ is the edge uv and $u_{l-1} u_l$ is a virtual edge different from uv. Colour u_0 with a colour in $\check{\Lambda}(u_0) \setminus \check{\Lambda}(u_1)$, then colour all the vertices of $K - P$ (for example, by extending $P - u_0$ to a spanning tree of $K - u_0$ and using the same idea as in Case 1), and then colour $u_l, \ldots, u_1$ in turn. When u_{l-1} is coloured, there are at least two different possible colours for it, since there is no real edge between u_{l-1} and u_l and so these two vertices can be given the same colour if necessary, and u_{l-2} is either uncoloured or (if $l = 2$) has been given a colour not in $\check{\Lambda}(u_{l-1})$; thus we can obtain two different colourings in this way, which differ only on some or all of the vertices $u_1, \ldots, u_{l-1}$. Note that u_l is adjacent to at most two of these vertices (u_{l-1} and possibly u_{l-2}), since otherwise P would not be a shortest path, and so there is an edge (in fact, at least two edges) each of whose endvertices has the same colour in both of these colourings. Thus at least one of these colourings must work in Step 3.

Case 5. Cases 1–4 do not arise, and K is not isomorphic to K_5.

By Case 4, $\check{\Lambda}$ is now assumed to be constant on K. Following [7], the idea is to find a path uvw of three vertices such that u and w are nonadjacent and $K - \{u, w\}$ is connected; to colour u and w with the same colour; and then to colour all remaining vertices, keeping the graph induced by the uncoloured vertices connected as usual, and ending with v, which can be coloured because it has two neighbours with the same colour. If there are no virtual edges in K, hence no inessential vertices in D_r, then one such colouring suffices, and we have just shown how to construct it (assuming that the path uvw can be found, which we prove below). So assume that there is at least one virtual edge. In this case we will prove below that the path uvw can be chosen in such a way that there is a virtual edge different from both uv and vw. So let $P : u_1, \ldots, u_l$ be a shortest path in $K - \{u, w\}$ such that u_1 is adjacent to both u and w and there is a virtual edge $u_l u_{l+1}$ different from both $u_1 u$ and $u_1 w$; then $u_{l+1} \notin \{u_1, \ldots, u_l\}$, but possibly $u_{l+1} \in \{u, w\}$ when $l \neq 1$. If $l \geqslant 3$ then u_2 is not adjacent to both u and w, since otherwise P would not be a shortest such path, and so u_2 has a neighbour $z \notin V(P) \cup \{u, w\}$. But if $l \leqslant 2$ then there is certainly a vertex z of K such that $z \notin V(P) \cup \{u, w\}$, since $|V(K)| \geqslant 5$. Either way, z has at most three neighbours in P, since otherwise P would not be a shortest path, and so there is an edge yz of K that has neither end in P.

Colour u and w with the same colour, then colour all remaining vertices not in P (using the same idea as in Case 4), and then colour $u_l, \ldots, u_1$ in turn. When we come to colour u_l, the endvertices of the edge yz have both already been coloured, and there is a choice of at least two possible colours for u_l. (This is true even if $l = 1$, when u_1 has two neighbours with the same colour and can also be given the same colour as u_2.) Thus we obtain two different colourings, at least one of which must work in Step 3.

It remains to show how we can find the path uvw. Let pq be a virtual edge if there is one, or any edge otherwise, and let x be any vertex of K different from p, q. Let y and z be two nonadjacent neighbours of x, which must exist since K is 4-regular and not K_5, and suppose without loss of generality that $z \notin \{p, q\}$. If $K - z$ is 2-connected, then $K - \{y, z\}$ is connected, and yxz will do for uvw. If however $K - z$ is not 2-connected, then let B_1, B_2 be two endblocks of $K - z$, with cutvertices z_1, z_2 respectively, and note that z is adjacent to at least one vertex $v_1 \in B_1 - z_1$ and to at least one vertex $v_2 \in B_2 - z_2$; then $v_1 z v_2$ will do for uvw. In each case, $K - \{u, w\}$ is connected, and the edge pq (which is virtual if any edge is) is not equal to uv or vw, and so K can be coloured as described in the previous two paragraphs.

Case 6. Cases 1–5 do not arise.
Then $K \cong K_5$ and $\check{\Lambda}$ is constant on K. If there is a virtual edge in K, then colour the vertices of K with different colours except that the endvertices of some virtual edge are given the same colour. In D_r, the inessential vertex or vertices in this virtual edge can then be coloured in at least two different ways, at least one of which will work in Step 3. So we may assume that there are no virtual edges in K, and hence no inessential vertices in D_r.

Colour four of the vertices of K, leaving the fifth vertex, u, uncoloured. This corresponds to colouring eight of the ten vertices of D_r. In D_r, $u^{(1)}$ is adjacent to vertices with two different colours, say a and b, and $u^{(2)}$ is adjacent to vertices with the other two colours, say c and d, where $\Lambda(u^{(1)}) = \Lambda(u^{(2)}) = \check{\Lambda}(u) = \{a, b, c, d\}$. So colour $u^{(1)}$ with c and $u^{(2)}$ with a, then colour the ten edges of D_r, and then colour the edges $u^{(1)}x^{(1)}$ and $u^{(2)}x^{(2)}$. We can extend this colouring to the three internal elements of $P(u)$ (and then extend it to the other internal D-paths, each of which joins two vertices of the same colour) unless $u^{(1)}x^{(1)}$ has the same colour, a, as $u^{(2)}$, and $u^{(2)}x^{(2)}$ has the same colour, c, as $u^{(1)}$. Suppose this happens. Let the vertices of D_r be $u_0, \ldots, u_9$ in order, where $u_1 = u^{(1)}$, and relabel $x^{(1)}$ as x_1. Let u_0, u_1, u_2 have colours a, c, b respectively. Note that $a \in \Lambda(u_1x_1)$. If $b \notin \Lambda(u_1x_1)$ then we could have avoided the problem altogether by colouring $u^{(2)}$ initially with b instead of a; so we may assume that $b \in \Lambda(u_1x_1)$. If we cannot now avoid the problem by changing the colour of u_1x_1 to b, then u_0u_1 must have colour b. If we cannot avoid the problem by changing the colour of u_1 to d, then u_1u_2 must have colour d. If we cannot avoid the problem by changing the colour of u_1x_1 to anything other than a, then $\Lambda(u_1x_1)$ must equal $\{a, b, c, d\}$. If we cannot avoid the problem by changing the colour of u_1u_2 and then colouring u_1x_1 with d, then $\Lambda(u_1u_2)$ must equal $\{a', b, c, d\}$, where a' (which could equal a, but is evidently not equal to c) is the colour of u_2u_3. But then we can avoid the problem by interchanging the colours of u_1 and u_1u_2, colouring u_1 with d and u_1u_2 with c. Now this colouring extends to all elements of the D-path $P(u)$, and of course it extends to all the other D-paths as well since we have not changed the colours of any vertices of D_r other than u_1.

We have thus shown that, in all cases, $G_{\min}$ can be totally Λ-coloured, which contradicts the definition of $G_{\min}$ and Λ. We have obtained this contradiction by assuming that $\mathrm{mad}(G_{\min}) = 2\frac{1}{2}$ and $G_{\min}$ has the structure described in the statement of Theorem 4, and so this contradiction completes the proof of Theorem 2.

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