Discussiones Mathematicae
Graph Theory 27 (2007) 389–400

MONOCHROMATIC KERNEL-PERFECTNESS OF SPECIAL CLASSES OF DIGRAPHS

Hortensia Galeana-Sánchez

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F. 04510, México

and

Luis Alberto Jiménez Ramírez

Facultad de Ciencias
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F. 04510, México

Abstract

In this paper, we introduce the concept of monochromatic kernel-perfect digraph, and we prove the following two results:

(1) If D is a digraph without monochromatic directed cycles, then D and each $\alpha_v, v \in V(D)$ are monochromatic kernel-perfect digraphs if and only if the composition over D of $(\alpha_v)_{v \in V(D)}$ is a monochromatic kernel-perfect digraph.

(2) D is a monochromatic kernel-perfect digraph if and only if for any $B \subseteq V(D)$, the duplication of D over B, D^B, is a monochromatic kernel-perfect digraph.

Keywords: kernel, kernel by monochromatic paths, composition, duplication.

2000 Mathematics Subject Classification: 05C20.

1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. Let $S_1, S_2 \subseteq V(D)$, an arc (u_1, u_2) of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of D induced by S_1. A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A *kernel* N of D is an independent set of vertices such that for each $z \in (V(D) - N)$ there exists a zN-arc in D. A digraph D is called a *kernel-perfect digraph* when every induced subdigraph of D has a kernel.

A digraph D is said to be an m-coloured digraph, if its arcs are coloured with m colours without loss of generality $\{1, 2, \ldots, m\}$. A directed path (or a directed cycle) is called *monochromatic* if all of its arcs are coloured alike.

A set $N \subseteq V(D)$ of vertices of D is said to be a *kernel by monochromatic paths* of the m-coloured digraph D, if it satisfies the two following properties, (1) N is independent by monochromatic paths; i.e., for any two different vertices $x, y \in N$, there is no monochromatic directed path between them, and (2) N is *absorbent* by monochromatic paths; i.e., for each $u \in (V(D) - N)$ there exists a uv-monochromatic directed path, for some $v \in N$.

In this paper, we prove that if D is a digraph without monochromatic directed cycles, then (i) D has a kernel by monochromatic paths if and only if any composition over D of a family of digraphs $(\alpha_v)_{v \in V(D)}$ each one of them having a kernel by monochromatic paths, has a kernel by monochromatic paths, and (ii) D has a kernel by monochromatic paths if and only if for any $B \subseteq V(D)$ the duplication of D over B, D^B, has a kernel by monochromatic paths.

As a consequence we obtain the two results mentioned in the abstract.

Clearly, D has a kernel if and only if the m-coloured digraph D, in which every two different arcs have different colours, has a kernel by monochromatic paths. Sufficient conditions for the existence of a kernel in a digraph have been investigated by several authors, namely Von Neumann and Morgenstern [16], Richardson [13], Duchet and Meyniel [5] and Galeana-Sánchez and Neumann-Lara [6]. The concept of a kernel is very useful in applications, and clearly, the concept of a kernel by monochromatic paths generalizes that of kernel. Sufficient conditions for the existence of kernels by monochromatic paths in m-coloured digraphs have also been investigated by several authors; see for example [7, 9, 10, 14, 15, 18].

Definition 1.1. Let D be an arc coloured digraph and $\alpha = (\alpha_v)_{v\in V(D)}$ a family of pairwise vertex disjoint arc coloured digraphs. We define the *composition of* α *over* D, denoted $\sigma(D,\alpha)$, by the following conditions:

(i) $V(\sigma(D,\alpha)) = \bigcup_{v\in V(D)} V(\alpha_v)$.

(ii) $A(\sigma(D,\alpha)) = \big(\bigcup_{v\in V(D)} A(\alpha_v)\big) \cup \{(x,y) \text{ coloured } i \,|\, x \in \alpha_u, y \in \alpha_v, (u,v) \in F(D) \text{ coloured } i\}$.

The composition of a family of graphs $\beta = (G_v)_{v\in V(G)}$ over a graph G was studied in [3] and its definition was extended to digraphs in [17]. The existence of kernels in the composition $\sigma(D,\alpha)$ of a family of digraphs $\alpha = (\alpha_v)_{v\in V(D)}$ over a digraph D was studied in [8], and the result was used to prove the existence of kernel-perfect digraphs with an arbitrarily large dichromatic number whose underlying graphs have no triangles.

In this paper, we study the existence of kernels by monchromatic paths in the composition $\sigma(D,\alpha)$ of a family of arc coloured digraphs $\alpha = (\alpha_v)_{v\in V(D)}$ over an arc coloured digraph D.

The duplication of a vertex of a graph was introduced in [4], and [11] gives the definition of the duplication of a subset of vertices of a graph as a generalization of the duplication of a vertex of a graph. This definition can be applied to arc coloured digraphs as follows:

Definition 1.2. Let D be an arc coloured digraph, B a proper subset of $V(D)$ and let B'_D a digraph isomorphic to $D[B]$ with $V(B'_D)\cap V(D) = \emptyset$. A vertex belonging to B'_D and corresponding to a vertex $x \in B$ will be denoted by x'. *The duplication of* D *over* B is the arc coloured digraph denoted D^B and defined as follows:

$$V(D^B) = V(D) \cup V(B'_D)$$

and

$$A(D^B) = A(D) \cup A(B'_D) \cup A_0 \cup A$$

in which $A_0 = \{(x',y) \text{ coloured } i \,|\, x' \in V(B'_D), y \in V(D) \text{ and } (x,y) \in A(D) \text{ coloured } i\}$. $A_1 = \{(y,x') \text{ coloured } i \,|\, y \in V(D), x' \in V(B'_D) \text{ and } (y,x) \in A(D) \text{ coloured } i\}$.

We will denote $B' = V(B'_D)$. A vertex $x' \in B'$ (resp., a subset $S' \subseteq B'$) we will call the copy of the vertex $x \in B$ (resp., the copy of the subset $S \subseteq B$).

The vertex x (resp., the subset S) will be named the original of the vertex x' (resp., of the subset S').

We will denote by Proy the function Proy: $V(\sigma(D,\alpha)) \to V(D)$ such that $\mathrm{Proy}(x) = v$ if and only if $x \in V(\alpha_v)$.

The existence of kernels in the duplication of a digraph D has been studied in [2]. In this paper, we study the existence of kernels by monochromatic paths in the duplication of an arc coloured digraph D over a proper subset of vertices of $V(D)$.

The composition and the duplication are two operations in digraphs which have been considerated several times, see for example [3, 12, 17], and they constitute a powerful tool in the construction of many examples and counterexamples in digraphs.

Also we consider an extension of the concept of kernel perfectness of a digraph and obtain a large variety of monochromatic kernel-perfect digraphs.

2. Kernels by Monochromatic Paths in the Composition over D, and in the Duplication of D over B

We start this section with a lemma which will be useful in the proof of Theorem 2.1. Its proof is easy and will be omitted.

Lemma 2.1. *Let D be a digraph and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of pairwise vertex disjoint digraphs. If $T = (x_0, x_1, \ldots, x_n)$ is a directed path in $\sigma(D,\alpha)$ such that $\{x_0, x_n\} \subseteq V(\alpha_v)$ for some $v \in V(D)$, then* $\mathrm{Proy}(T)$ *is a join of directed cycles of D or a single vertex of D.*

Theorem 2.1. *Let D be an arc coloured digraph which has no monochromatic directed cycle and $\alpha = (\alpha_v)_{v \in V(D)}$ a family of arc coloured pairwise vertex disjoint digraphs. A set $N^* \subseteq V(\sigma(D,\alpha))$ is a kernel by monochromatic paths of $\sigma(D,\alpha)$ if and only if there exists a kernel by monochromatic paths of D, say $N \subseteq V(D)$, such that $N^* = \bigcup_{v \in N} N_v$, in which N_v is a kernel by monochromatic paths of α_v.*

Proof. Let $N \subseteq V(D)$ be a kernel by monochromatic paths of D and N_v a kernel by monochromatic paths of α_v, $v \in N$. We will prove that $N^* = \bigcup_{v \in N} N_v$ is a kernel by monochromatic paths of $\sigma(D,\alpha)$.

(a) N^* is absorbent by monochromatic paths.

Let $z \in (V(\sigma(D,\alpha)) - N^*)$. There exists $v_0 \in V(D)$ such that $z \in V(\alpha_{v_0})$. When $v_0 \in N$, we have $N_{v_0} \subseteq N^*$, in which N_{v_0} is a kernel by monochromatic paths of α_{v_0}; and there exists a zN_{v_0}-monochromatic directed path (as $z \in (V(\alpha_0) - N_{v_0})$).

When $v_0 \notin N$, we have $v_0 \in (V(D) - N)$ and thus, there exists a monochromatic directed path contained in D, say $T = (v_0, v_1, \ldots, v_{n-1}, u)$ with $u \in N$ (because N is a kernel by monochromatic paths of D); since $z \in V(\alpha_0)$; taking $z_i \in V(\alpha_i)$ and $z_u \in N_u$, we have $T' = (z, z_1, z_2, \ldots, z_{n-1}, z_u)$; a zz_u-monochromatic directed path in $\alpha(D,\sigma)$ with $z_u \in N_u \subseteq N^*$.

(b) N^* is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist $x_0, x_n \in N^*$ and a x_0x_n-monochromatic directed path, say $T = (x_0, x_1, \ldots, x_n)$ contained in $\sigma(D,\alpha)$. We consider two possible cases:

Case (b.1). $\{x_0, x_n\} \subseteq V(\alpha_v)$, for some $v \in V(D)$.
When $T \subseteq \alpha_v$, we have that T is an x_0x_n-monochromatic directed path contained in α_v, with $\{x_0, x_n\} \subseteq N_v$, a contradiction.

When $T \nsubseteq \alpha_v$, we have from Lemma 2.1 that $\text{Proy}(T)$ is a join of monochromatic directed cycles contained in D, contradicting our hypothesis on D.

Case (b.2). $x_0 \in \alpha_v$ and $x_n \in \alpha_u$ with $u \neq v$.
In this case, it follows from the definition of N^* that $x_0 \in N_v$ and $x_n \in N_u$ with $\{u, v\} \subseteq N$. Since T is monochromatic, We have that $\text{Proy}(T)$ contains a vu-monochromatic path, which is contained in D, contradicting that N is a kernel by monochromatic paths of D. We conclude that N^* is a kernel by monochromatic paths.

Now let N^* be a kernel by monochromatic paths of $\sigma(D,\alpha)$. We will prove that $N = \{v \in V(D) \mid N^* \cap \alpha_v \neq \emptyset\}$ is a kernel by monochromatic paths of D and $N^* \cap V(\alpha_v) = N_v$ is a kernel by monochromatic paths of α_v, for each $v \in N$.

N is absorbent by monochromatic paths.

Let $v \in (V(D) - N)$ and $z_0 \in V(\alpha_v)$; since $v \notin N$ we have that $z_0 \notin N^*$; thus there exists a monochromatic directed path $T = (z_0, \ldots, z_n)$ with $z_n \in N^*$; now, $z_n \in V(\alpha_u)$ for some $u \in V(D)$; moreover, from the definition of N we have $u \in N$ and then $\text{Proy}(T)$ contains a vu-monochromatic directed path with $u \in N$.

N is independent by monochromatic paths.

We proceed by contradiction, suppose that there exist $v_0, v_n \in N$ and a v_0v_n-monochromatic directed path $T = (v_0, v_1, \ldots, v_n)$ contained in D. Since $v_0, v_n \in N$ there exist $z_0 \in V(\alpha_{v_0}) \cap N^*$ and $z_n \in V(\alpha_{v_n}) \cap N^*$; now taking any vertex $z_i \in V(\alpha_{v_i})$ for each $1 \leq i \leq n-1$; we have from the definition of $\sigma(D,\alpha)$ that $(z_0, z_1, \ldots, z_n)$ is a z_0z_n-monochromatic directed path with $\{z_0, z_n\} \subseteq N^*$, a contradiction.

Now; let $v \in V(D)$ be such that $N^* \cap V(\alpha_v) \neq \emptyset$. We will prove that $N_v = N^* \cap V(\alpha_v)$ is a kernel by monochromatic paths of α_v.

N_v is independent by monochromatic paths.

We proceed by contradiction. Suppose that there exist $u, x \in N_v$, $u \neq x$, and a monochromatic directed path T between them, with $T \subseteq \alpha_v$; clearly, $T \subseteq \sigma(D,\alpha)$ and $\{u, x\} \subseteq N^*$, a contradiction (as N^* is independent by monochromatic paths in $\sigma(D,\alpha)$).

N is absorbent by monochromatic paths.

Let $u \in (V(\alpha_v) - N)$ clearly $u \in (V(\sigma(D,\alpha)) - N^*)$; thus there exists $z \in N^*$ and a uz-monochromatic directed path $T \subseteq \sigma(D,\alpha)$. Let $T = (u = u_0, u_1, \ldots, u_n = z)$, we will prove that $T \subseteq \alpha_v$. When $u_n = z \in V(\alpha_v)$; it follows from Lemma 2.1 that $\mathrm{Proy}(T)$ is a single vertex i.e., $T \subseteq \alpha_v$; otherwise D contains a monochromatic directed cycle, contradicting our hypothesis on D. When $u_n \notin V(\alpha_v)$; we have $u_n \in \alpha_w$ for some $w \in V(D)$ and $w \in N^*$. Now take $x \in N_v^*$ (recall $N^* \cap \alpha_v \neq \emptyset$); it follows from the definition of $\alpha(D,\alpha)$ that $(x, u_1, u_2, \ldots, u_n = z)$ is a xz-monochromatic directed path in $\sigma(D,\alpha)$ with $x \neq z$, $x, z \in N^*$, a contradiction. ■

Lemma 2.2. *Let D be an arc coloured digraph, $B \subset V(D)$; D^B the duplication of D over B; and $\psi\colon D[B] \to B'_D$ the isomorphism defined by the duplication (i.e., $\psi(x) = x'$ for any $x \in V(D[B])$); and denote by ϕ the function defined as follows: $\phi\colon D \to D^B - D[B]$*

$$\phi(x) = \begin{cases} x & \text{if } x \notin B, \\ \phi(x) = x' & \text{if } x \in B\,. \end{cases}$$

Then ϕ is an isomorphism such that (x, y) is coloured in i if and only if $(\phi(x), \phi(y))$ is coloured in i; in particular $T \subseteq D$ is a monochromatic directed path if and only if $\phi(T) \subseteq D^B - D[B]$ is a monochromatic directed path.

This Lemma is a direct consequence of the definition of ϕ and the definition of the duplication of D over B.

Theorem 2.2. *Let D be an arc coloured digraph which has no monochromatic directed cycles; $B \subset V(D)$ and D^B the duplication of D over B.*

D has a kernel by monochromatic paths if and only if D^B has a kernel by monochromatic paths.

Proof. Let D, B and D^B be as in the hypothesis and suppose that D has a kernel by monochromatic paths, say N.

We consider two possible cases:

Case 1. $N \cap B = \emptyset$.
In this case, we will prove that N is a kernel by monochromatic paths of D^B.

N is independent by monochromatic paths in D^B.
Let $x, y \in N$; $x \neq y$ and assume for a contradiction that there exists an xy-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in D^B.

When $V(T) \cap B = \emptyset$, we have $T \subseteq D^B - D[B]$, and from Lemma 2.1 $\phi^{-1}(T)$ is an xy-monochromatic directed path contained in D, contradicting that N is independent by monochromatic paths. When $V(T) \cap B \neq \emptyset$, we denote $I = V(T) \cap B$; say $I = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, $i_1 < i_2 < \cdots < i_k$, we also denote by $T(I) = (x_0, \ldots, x_{i_1 - 1}, x'_{i_1}, x_{i_1+1}, \ldots, x_{i_2-1}, x'_{i_2}, \ldots, x_n = y)$ (the succesion obtained from T by substituting x_{i_j}, for x'_{i_j} in T, for each $j \in \{1, \ldots, k\}$). It follows from the definition of D^B that $T(I)$ is a monochromatic directed path contained in $D^B - D[B]$; and from Lemma 2.2 $\phi^{-1}(T(I))$ is an xy-monochromatic directed path contained in D, a contradiction.

N is absorbent by monochromatic paths in D^B.

Let $z \in (V(D^B) - N)$. If $z \notin B'_D$, then $z \in (V(D) - N)$ and there exists a zN-monochromatic directed path, say T, with $T \subseteq D \subseteq D^B$.

If $z \in V(B'_D)$, then there exists $y \in B$ such that $z = y' \in V(B'_D)$; we have $y \notin N$ because $N \cap B = \emptyset$; thus there exists a yN-monochromatic directed path, say $T = (y, x_1, \ldots, x_n)$ and then from definition of D^B we have that $T' = (y', x_1, \ldots, x_n)$ is a zN-monochromatic directed path in D^B.

Case 2. $N \cap B \neq \emptyset$.
Let $Z = N \cap B$, and denote by $Z' = \{z' \in B'_D \,|\, z \in Z\}$.

We will prove that $N^* = N \cup Z'$ is a kernel by monochromatic paths of D^B.

N^* is independent by monochromatic paths.

Let $x, y \in N^*$, $x \neq y$, and assume for a contradiction that there exists an xy-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in D^B. Here we consider several possible cases:

Case 2.a. $x, y \in N$.
Let $I' = V(T) \cap V(B'_D) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ and denote by y_{i_j} the original of x_{i_j} (i.e., $y_{i_j} = \psi^{-1}(x_{i_j})$. Now let T' be the succession obtained from T by substituting each x_{i_j} for y_{i_j}. It follows from the definition of ψ and from the definition of D^B that T' contains an xy-monochromatic directed path contained in D, with $x, y \in N$, a contradiction.

Case 2.b. $x \in N$, $y \in Z'$ and $x \notin B$. In this case, we proceed as in Case 2.a to get a contradiction.

Case 2.c. $x \in N \cap B$, $y \in Z'$.
When x is the original vertex of y, taking the succession T' defined in Case 2.a we have that T' contains a monochromatic directed cycle, contradicting our hypothesis on D; as $T' \subseteq D$.

When x is not the original vertex of y; taking again the succession T' defined in Case 2.a, we have that T' contains an xz-monochromatic directed path, in which z is the original vertex of y and $x, z \in N$ with $x \neq z$, contradicting that N is independent by monochromatic paths.

Case 2.d. $x, y \in Z'$.
Let $\overline{x}$ (resp., $\overline{y}$) be the original vertex of x (resp., y); clearly, in this case T' (defined in Case 2.a) contains an $\overline{x}\overline{y}$-monochromatic directed path which is contained in D; with $\overline{x} \neq \overline{y}$, $\overline{x}, \overline{y} \in N$, a contradiction. So, we conclude that N^* is independent by monochromatic paths.

Now we prove that N^* is absorbent by monochromatic paths.

Let $z \in (V(D^B) - N^*)$. When $z \in B'$, we have $z = y'$ in which $y \in B$ is the original vertex of z. Since N is a kernel by monochromatic paths of D; there exists a yN-monochromatic directed path in D, say, $T = (y = x_0, x_1, \ldots, x_n)$; thus $T' = (y' = z, x_1, \ldots, x_n)$ is a zN^*-monochromatic directed path contained in D^B. When $z \notin B'$, we have $z \in (V(D) - N)$ and there exists a zN-monochromatic directed path contained in D; say, T. Clearly, T is a zN^*-monochromatic directed path contained in D^B.

We conclude that N^* is a kernel by monochromatic paths of D^B. Now suppose that D^B has a kernel by monochromatic paths and let N^* be a kernel by monochromatic paths of D^B. We will prove that D has a kernel by monochromatic paths.

Let Z be such that $Z' = N^* \cap V(B'_D)$ in which Z' is defined by the process introduced in the construction of B'_D, when $Z' = \emptyset$ we define $Z = \emptyset$. Denote by $N = (N^* - Z') \cup Z$. We will show that N is a kernel by monochromatic paths of D.

N is independent by monochromatic paths in D.

Assume by contradiction that there exist $x, y \in N$; $x \neq y$; and an xy-monochromatic directed path $T = (x = x_0, x_1, \ldots, x_n = y)$ contained in D.

Let $\overline{x}$ and $\overline{y}$ be defined as follows: $\overline{x} = x$ if $x \in (N^* - Z')$ and $\overline{x}$ is the copy of x if $x \in Z$, $\overline{y} = y$ if $y \in (N^* - Z^*)$ and $\overline{y}$ is the copy of y if $y \in Z$. Clearly, $T' = (\overline{x}, x_1, \ldots, x_{n-1}, \overline{y})$ is a monochromatic directed path in D^B with $\overline{x} \neq \overline{y}$ and $\overline{x}, \overline{y} \in N^*$, a contradiction.

N is absorbent by monochromatic paths in D.

Let $z \in (V(D) - N)$, then from the definition of N, we have $z \in (V(D^B) - N)$, thus there exists a zN^*-monochromatic directed path, say $T = (z = x_0, x_1, \ldots, x_n)$ contained in D^B. Let $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} = V(T) \cap V(B'_D)$; y_{i_j} the original vertex of x_{i_j} and T' the succession obtained from T by substituting x_{i_j} for y_{i_j} for each $1 \leq j \leq k$ in T. Clearly, T' contains a zN-monochromatic directed path, and $T' \subseteq D$.

We conclude that N is absorbent by monochromatic paths. ∎

3. Monochromatic Kernel Perfectness of Composition and Duplication

The following definition is a generalization of the concept of kernel perfectness of a digraph.

Definition 3.1. Let D be an arc coloured digraph, D is said to be a monochromatic kernel perfect digraph whenever for every nonempty subset B of vertices of D, the digraph $D[B]$ has a kernel by monochromatic paths.

Theorem 3.1. *Let D be an arc coloured digraph which has no monochromatic directed cycle and $\alpha = (\alpha_v)_{v \in V(D)}$ a family in which the α_v are mutually disjoint arc coloured digraphs.*

D and each α_v, $v \in V(D)$ are monochromatic kernel perfect digraphs if and only if $\sigma(D, \alpha)$ is a monochromatic kernel perfect digraph.

Proof. Theorem 3.1 follows directly from Theorem 2.1 and the two following assertions: (1) The disjoint union of monochromatic kernel perfect digraphs is also a monochromatic kernel perfect digraph. (2) Every connected induced subdigraph of $\sigma(D, \alpha)$ has the form $\sigma(D', \alpha')$ for a suitable D' and $\alpha' = (\alpha'_v)_{v \in V(D)}$ (actually D' is an induced subdigraph of D and α'_v is an induced subdigraph of α_v for each $v \in V(D')$). ■

Theorem 3.2. *Let D be an arc coloured digraph which has no monochromatic directed cycle, $B \subset V(D)$ and D^B the duplication of D over B. Then D is a monochromatic kernel perfect digraph if and only if D^B is a monochromatic kernel perfect digraph.*

Proof. Clearly, an arc coloured digraph D is a monochromatic kernel perfect digraph if and only if each induced subdigraph of D is a monochromatic kernel perfect digraph. Thus if D^B is a monochromatic kernel perfect digraph, then D is a monochromatic kernel perfect digraph.

Now suppose that D is a monochromatic kernel perfect digraph and let $A \subseteq V(D^B)$. We will prove that $D^B[A]$ has a kernel by monochromatic paths. Here we consider two possible cases:

Case 1. $A \cap V(B'_D) = \emptyset$.
In this case, $A \subseteq V(D^B - V(B'_D))$ and $D^B[A] \cong D[A]$ and since $D[A]$ has a kernel by monochromatic paths; it follows that $D^B[A]$ has a kernel by monochromatic paths.

Case 2. $A \cap V(B'_D) \neq \emptyset$.
Let $C' = \{x' \in V(D^B) \,|\, x' \in A \cap V(B'_D)\}$ and $E = A - C'$ be, thus $A = C' \cup E$.

Case 2.1. $E \cap C = \emptyset$. (In Which $C = \psi^{-1}(C')$).
In this case, we have $D^B[E \cup C'] \cong D^B[E \cup C] \cong D[E \cup C]$ and then $D^B[A] \cong D[E \cup C]$ has a kernel by monochromatic paths.

Case 2.2. $E \cap C \neq \emptyset$.
It follows from the hypothesis that $D[E \cup C]$ has a kernel by monochromatic paths, say N.

When $N \cap B = \emptyset$ it follows as in Case 1 of the proof of Theorem 2.2 that N is a kernel by monochromatic paths of $D^C[E \cup C]$ (the duplication of $D[E \cup C]$ over C); therefore N is independent by monochromatic paths in $D^B[E \cup C']$. Since N is absorbent by monochromatic paths in $D[E \cup C]$ it follows that N is absorbent by monochromatic paths in $D^B[E \cup C']$.

(Clearly, to each monochromatic directed path in $D[E \cup C]$, say T there corresponds an unique monochromatic directed path in $D^B[E \cup C']$, T' obtained from T by substituting each vertex x in $V(T) \cap (C - E)$ for its copy x' in C').

When $N \cap B \neq \emptyset$, we denote by $Z = N \cap B$; we have proved in Case 2 of the proof of Theorem 2.2 that $N \cup Z'$ is a kernel by monochromatic paths of $D^C[E \cup C]$ (the duplication of $D[E \cup C]$ over C). So $N \cup Z'$ is independent by monochromatic paths in $D^B[E \cup C']$. Now, let $z \in (V(D^B[E \cup C']) - (N \cup Z'))$; clearly, $z \in (V(D^C[E \cup C] - (N \cup Z')))$ and then there exists a $z \in (N \cup Z')$-monochromatic directed path, say $T = (z = x_0, x_1, \ldots, x_n)$; if $T \cap (C - E) = \{x_{i_1}, \ldots, x_{i_k}\}$ then let T' be the succession obtained from T by substituting each x_{i_j}, $1 \leq j \leq k$ for its copy x'_{i_j} in C'. Since D has no monochromatic directed cycles we have $x'_{i_j} \notin V(T)$ for each $1 \leq j \leq k$. Therefore from the definition of D^B we have that T' is a monochromatic directed path contained in $D^B[E \cup C']$ from z to $(N \cup Z')$. We conclude that $N \cup Z'$ is a kernel by monochromatic paths of $D^B[E \cup C']$. ∎

Acknowledgement

The authors wish to thank the anonymous referees for many suggestions which helped improve the final version of this paper.

References

[1] C. Berge, Graphs (North-Holland, Amsterdam, 1985).

[2] M. Blidia, P. Duchet, H. Jacob, F. Maffray and H. Meyniel, *Some operations preserving the existence of kernels*, Discrete Math. **205** (1999) 211–216.

[3] M. Borowiecki and A. Szelecka, *One-factorizations of the generalized Cartesian product and of the X-join of regular graphs*, Discuss. Math. Graph Theory **13** (1993) 15–19.

[4] M. Burlet and J. Uhry, *Parity Graphs*, Annals of Discrete Math. **21** (1984) 253–277.

[5] P. Duchet and H. Meyniel, *A note on kernel-critical graphs*, Discrete Math. **33** (1981) 103–105.

[6] H. Galeana-Sánchez and V. Neumann-Lara, *On kernels and semikernels of digraphs*, Discrete Math. **48** (1984) 67–76.

[7] H. Galeana-Sánchez, *On monochromatic paths and monochromatics cycles in edge coloured tournaments*, Discrete Math. **156** (1996) 103–110.

[8] H. Galeana-Sánchez and V. Neumann-Lara, *On the dichromatic number in kernel theory*, Math. Slovaca **48** (1998) 213–219.

[9] H. Galeana-Sánchez and R. Rojas-Monroy, *On monochromatic paths and monochromatic* 4*-cycles in edge-coloured bipartite tournaments*, Discrete Math. **285** (2004) 313–318.

[10] G. Hahn, P. Ille and R.E. Woodrow, *Absorbing sets in arc-coloured tournaments*, Discrete Math. **283** (2004) 93–99.

[11] M. Kucharska, *On* (k,ℓ)*-kernels of orientations of special graphs*, Ars Combinatoria **60** (2001) 137–147.

[12] M. Kucharska, *On* (k,ℓ)*-kernel perfectness of special classes of digraphs*, Discussiones Mathematicae Graph Theory **25** (2005) 103–119.

[13] M. Richardson, *Solutions of irreflexive relations*, Ann. Math. **58** (1953) 573.

[14] S. Minggang, *On monochromatic paths in* m*-coloured tournaments*, J. Combin. Theory (B) **45** (1988) 108–111.

[15] B. Sands, N. Sauer and R. Woodrow, *On monochromatic paths in edge-coloured digraphs*, J. Combin. Theory (B) **33** (1982) 271–275.

[16] J. von Neumann and O. Morgenstern, Theory of games and economic behavior (Princeton University Press, Princeton, 1944).

[17] A. Włoch and I. Włoch, *On* (k,ℓ)*-kernels in generalized products*, Discrete Math. **164** (1997) 295–301.

[18] I. Włoch, *On kernels by monochromatic paths in the corona of digraphs*, preprint.

Received 17 January 2006
Revised 11 June 2007
Accepted 11 June 2007