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ON THE SECOND LARGEST EIGENVALUE OF A MIXED GRAPH*

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Abstract

Let G be a mixed graph. We discuss the relation between the second largest eigenvalue $\lambda_2(G)$ and the second largest degree $d_2(G)$, and present a sufficient condition for $\lambda_2(G) \geq d_2(G)$.

Keywords: mixed graph, Laplacian eigenvalue, degree.

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1. Introduction

Let $G = (V, E)$ be a *mixed graph* with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$, which is obtained from an undirected graph by orienting some (possibly none or all) of its edges. Hence in a mixed graph some edges are oriented while others are not. We denote respectively by $\{u, v\}$ and (u, v) the unoriented edge and the oriented edge joining u and v; and for the oriented edge (u, v), we call u and v respectively the *head* and *tail* of the edge. It is important to stress that the mixed graphs are considered undirected graphs in terms of defining the degrees of

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vertices, path, cycle and connectedness, *etc.*. In addition, the mixed graphs throughout this paper contain no multi-edges or loops.

Denote by $d(v) = d_G(v)$ the degree of the vertex $v \in V(G)$. For each $e \in E(G)$, we define the *sign* of e and denote by $\operatorname{sgn} e = 1$ if e is unoriented and $\operatorname{sgn} e = -1$ if e is oriented. Set $a_{ij} = \operatorname{sgn} e$ if there exists an edge e joining v_i and v_j, and $a_{ij} = 0$, otherwise. Then the resulting matrix $A = (a_{ij})$ is called the *adjacency matrix* of G. The *incidence matrix* of G is an $n \times m$ matrix $M = M(G) = (m_{ij})$ whose entries are given by $m_{ij} = 1$ if e_j is an unoriented edge incident to v_i or e_j is an oriented edge with head v_i, $m_{ij} = -1$ if e_j is an oriented edge with tail v_i, and $m_{ij} = 0$, otherwise. The *Laplacian matrix* of G is defined as $L(G) = MM^T$ (see [1] or [15]), where M^T denotes the transpose of M. Obviously $L(G)$ is symmetric and positive semi-definite, and $L(G) = D(G) + A(G)$ (or see [15, Lemma 2.1]), where $D(G) = \operatorname{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$. Therefore the eigenvalues of $L(G)$ can be arranged as follows:

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G).$$

We briefly called the eigenvalues and eigenvectors of $L(G)$ as those of G, respectively. G is called *singular* (or *nonsingular*) if $L(G)$ is singular (or nonsingular).

Clearly if G is *all-oriented* (i.e., all edges of G are oriented), then $L(G)$ is consistent with the Laplacian matrix of a simple graph (note that when we define the Laplacian matrix of a simple graph we first give an orientation to each edge of the graph, and then obtain the Laplacian matrix via incidence matrix as above; see [13]). For simple graphs, there are a wealth of results involved with the relations between its spectrum and numerous graph invariants, such as connectivity, diameter, matching number, isoperimetric number, and expanding properties of a graph; see, e.g., [6, 9, 13, 14].

For the algebraic property of mixed graphs, Bapat *et al.* [1, 2] extent the definition of Laplacian matrix of simple graphs to that of mixed graphs, and generalized the classical Matrix-Tree theorem. In [15, 16], Zhang *et al.* gave some relations between a mixed graph and its line graph, and obtained some upper bounds for the largest eigenvalue and lower bounds for the second largest eigenvalue of mixed graphs. Fan [3] characterized mixed graphs which maximize or minimize the largest eigenvalue over all unicyclic mixed graphs. Also for unicyclic mixed graphs, Fan [4] gave a structural property of the eigenvectors corresponding to the least eigenvalue, and used the result to characterize those graphs with fixed number of vertices and with girth 3 which minimize the least eigenvalue. In addition, Fan [5] discussed the spectral perturbation of a mixed graph by adding an edge.

Since the matrix $L(G)$, as well as $A(G)$, is determined by the signs of edges of G, we simply concern ourselves with whether an edge is oriented or not, and do not care which one is the head and which one is the tail of an oriented edge in the following discussion. In this sense, the notion of *signed graphs* [11] instead of mixed graph will be more suitable for our discussion. A mixed graph G is called *quasi-bipartite* if it does not contain a nonsingular cycle, or equivalently, G contains no cycles with an odd number of unoriented edges (see [1, Lemma 1]). Denote by $\overrightarrow{G}$ the all-oriented graph obtained from G by arbitrarily orienting every unoriented edge of G (if one exists). Note that a *signature matrix* is a diagonal matrix with 1 or -1 along its diagonal.

Lemma 1.1 ([15, Lemma 2.2], [5, Lemma 5]). *Let G be a connected mixed graph. Then G is singular if and only if G is quasi-bipartite.*

Theorem 1.2 ([1, Theorem 4]). *Let G be a mixed graph. Then G is quasi-bipartite if and only if there exists a signature matrix D such that $D^T L(G) D = L(\overrightarrow{G})$.*

Denote by $d_1(G), d_2(G)$ respectively the largest and the second largest degree of vertices of G. If G is a simple connected graph containing at least 3 vertices and one edge, then

(i) (*Grone and Merris's bound* [8]) $\lambda_1(G) \geq d_1(G) + 1$,

(ii) (*Li and Pan's bound* [12]) $\lambda_2(G) \geq d_2(G)$.

In [16], Zhang and Luo show that (i) also holds for mixed graph; independently Hou *et al.* [11] also obtain this result with a little interpretation. We find that (ii) does not always hold for mixed graphs, and then give a sufficient condition for (ii) holding on mixed graphs. Our result implies Li and Pan's result in the case of the mixed graph G be all-oriented, in which case the Laplacian matrix is consistent with that of a simple graph.

2. Main Results

Let $G = (V, E)$ be a mixed graph with $V = \{v_1, v_2, \ldots, v_n\}$, and let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be a real vector. It will be convenient to adopt the following terminology from [7]: x is said to give a valuation of the vertices of V, that is, for each vertex v_i of V, we associate the value x_i, i.e., $x(v_i) = x_i$. Then λ is an eigenvalue of G with the corresponding eigenvector $x = (x_1, x_2, \ldots, x_n)$ if and only if $x \neq 0$ and

$$(2.1) \qquad [\lambda - d(v_i)]x(v_i) = \sum_{e=\{v_i,v_j\}\in E} (\operatorname{sgn} e)\, x(v_j), \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Lemma 2.1 ([16, Lemma 2.2])**.** *Let G be a mixed graph on n vertices and let e be an* (*oriented or unoriented*) *edge of G. Then*

$$\lambda_1(G) \geq \lambda_1(G-e) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \lambda_n(G-e).$$

Denote by $|S|$ the cardinality of a set S. We first introduce a mixed graph W as follows, which will play an important role in our discussion. Let X_1, X_2, Y_1, Y_2 and $\{v_1\}$, $\{v_2\}$ be pairwise disjoint vertex sets, where $|X_1| \geq |X_2| \geq 0$, $|Y_1| \geq 0$ and $|Y_2| \geq 0$. The graph $W = (V(W), E(W))$ is defined as in Figure 2.1, where the vertex set $V(W) = X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup \{v_1\} \cup \{v_2\}$, and the edge set

$$\begin{aligned} E(W) = \{&(v_1, u) | u \in X_1 \cup Y_1 \cup Y_2 \cup \{v_2\}\} \\ &\cup \{(v_2, u) | u \in X_2 \cup Y_2\} \cup \{\{v_2, u\} | u \in Y_1\}. \end{aligned}$$

Clearly, $d_1(W) = 1 + |X_1| + |Y_1| + |Y_2| \geq 1 + |X_2| + |Y_1| + |Y_2| = d_2(W)$. If W is exactly a cycle on 3 vertices (simply called a *triangle*) and is further nonsingular, or equivalently, $|X_1| = |X_2| = |Y_2| = 0$ and $|Y_1| = 1$, then $\lambda_2(W) = 1 < d_2(W) = 2$.

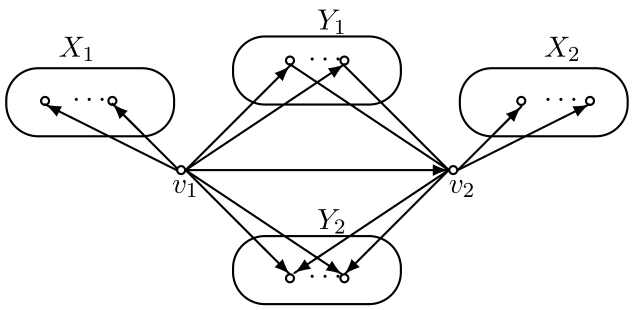

Figure 2.1

Lemma 2.2. *Let W be the graph of Figure 2.1 containing v_1, v_2 and at least one other vertex, except the case of W being a nonsingular triangle. If $[d_2(W)-2]|Y_1| - d_2(W)|Y_2| \leq 0$, then*

$$\lambda_2(W) \geq d_2(W).$$

In particular for the case of $|X_1| = |X_2|$, then $\lambda_2(W) \geq d_2(W)$ if and only if

$$[d_2(W)-2]|Y_1| - d_2(W)|Y_2| \leq 0.$$

Proof. Assume first that $|X_1| = |X_2|$. Then $d_1(W) = d_2(W) =: d_2$. The graph W has the possibility to be one of the following cases.

$$\begin{aligned}
&\text{(i)}\ \ |Y_1| \geq 1,\ |Y_2| \geq 1,\ |X_2| \geq 1; &&\text{(ii)}\ \ |Y_1| \geq 1,\ |Y_2| = 0,\ |X_2| \geq 1;\\
&\text{(iii)}\ \ |Y_1| = 0,\ |Y_2| \geq 1,\ |X_2| \geq 1; &&\text{(iv)}\ \ |Y_1| = 0,\ |Y_2| = 0,\ |X_2| \geq 1;\\
&\text{(v)}\ \ |Y_1| \geq 1,\ |Y_2| \geq 1,\ |X_2| = 0; &&\text{(vi)}\ \ |Y_1| \geq 2,\ |Y_2| = 0,\ |X_2| = 0;\\
&\text{(vii)}\ \ |Y_1| = 0,\ |Y_2| \geq 1,\ |X_2| = 0.
\end{aligned}$$

Note that in case (vi) $|Y_1| = 1$ is not allowed otherwise W is a nonsingular triangle. Let λ ($\lambda \neq 1, \lambda \neq 2$) be an eigenvalue of W with the corresponding eigenvector x. We discuss the cases (i)–(vii) as follows.

Case (i). Since $\lambda \neq 1, 2$, by (2.1), $x(u) = x(v)$ for each pair u, v in X_i and each pair u, v in Y_i for $i = 1, 2$. Thus, let

$$x(u) =: y_1, \forall u \in X_1; \quad x(u) =: y_2, \forall u \in Y_1;$$

$$x(u) =: y_3, \forall u \in X_2; \quad x(u) =: y_4, \forall u \in Y_2.$$

Let $x(v_1) =: y_5$ and $x(v_2) =: y_6$. Then equations (2.1) for λ and the corresponding eigenvector x are equivalent to the following equations:

$$\text{(2.2)} \qquad \begin{cases}
(\lambda-1)y_1 = -y_5,\\
(\lambda-2)y_2 = -y_5 + y_6,\\
(\lambda-1)y_3 = -y_6,\\
(\lambda-2)y_4 = -y_5 - y_6,\\
(\lambda-d_2)y_5 = -|X_2|y_1 - |Y_1|y_2 - |Y_2|y_4 - y_6,\\
(\lambda-d_2)y_6 = |Y_1|y_2 - |X_2|y_3 - |Y_2|y_4 - y_5.
\end{cases}$$

We turn the equations (2.2) into the matrix equation $(\lambda I - B)y = 0$, where B is the coefficient matrix of above linear equations and $y = (y_1, \ldots, y_6)^T$. Noting that $|X_2| = d_2 - |Y_1| - |Y_2| - 1$, then the solutions of λ of (2.2) are exactly the roots of the following polynomial $\Phi_1(\lambda) = \det(\lambda I - B)$:

(2.3)

$$\Phi_1(\lambda) = \det \begin{bmatrix} \lambda - 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \lambda - 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & \lambda - 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda - 2 & 1 & 1 \\ d_2 - |Y_1| - |Y_2| - 1 & |Y_1| & 0 & |Y_2| & \lambda - d_2 & 1 \\ 0 & -|Y_1| & d_2 - |Y_1| - |Y_2| - 1 & |Y_2| & 1 & \lambda - d_2 \end{bmatrix}.$$

By an elementary calculation,

$$\begin{aligned} \Phi_1(\lambda) &=: \Phi_1(\lambda; d_2, |Y_1|, |Y_2|) \\ &= [-(4 + 2|Y_2|) + (6 - |Y_1| + |Y_2| + 2d_2)\lambda - (4 + d_2)\lambda^2 + \lambda^3] \\ &\quad \times [-2|Y_1| + (|Y_1| - |Y_2| + 2d_2)\lambda - (2 + d_2)\lambda^2 + \lambda^3]. \end{aligned}$$

Let

$$f(\lambda) = -(4 + 2|Y_2|) + (6 - |Y_1| + |Y_2| + 2d_2)\lambda - (4 + d_2)\lambda^2 + \lambda^3.$$

Then $f(0) = -(4 + 2|Y_2|) < 0$, $f(1) = d_2 - |Y_1| - |Y_2| - 1 = |X_2| > 0$, $f(2) = -2|Y_1| < 0$, $f(d_2 + 1) = -(1 + d_2)|Y_1| - (d_2 - 1)(d_2 - |Y_2| - 1) < 0$. So the largest root of $f(\lambda)$ is greater than $d_2 + 1$, and the second largest root lies in the open interval $(1, 2)$. Let

$$g(\lambda) = -2|Y_1| + (|Y_1| - |Y_2| + 2d_2)\lambda - (2 + d_2)\lambda^2 + \lambda^3.$$

Then $g(0) = -2|Y_1| < 0$, $g(1) = d_2 - |Y_1| - |Y_2| - 1 = |X_2| > 0$, $g(2) = -2|Y_2| < 0$, $g(d_2 + 1) = (d_2 + 1)(d_2 - |Y_2| - 1) + (d_2 - 1)|Y_1| > 0$. So the largest root $g(\lambda)$ lies in the open interval $(2, d_1 + 1)$.

By above discussion, $\lambda_2(W)$ is exactly the largest root of $g(\lambda)$. Note that

$$g(d_2) = (d_2 - 2)|Y_1| - d_2|Y_2|.$$

Hence $\lambda_2(W) \geq d_2$ if and only if $g(d_2) \leq 0$ if and only if

$$(d_2 - 2)|Y_1| - d_2|Y_2| \leq 0. \tag{2.4}$$

Case (ii). Similar to Case (i), we obtain a system of equations from equations in (2.2) just by deleting the 4th equation and letting $|Y_2| = 0$, and get a polynomial on λ, denoted by $\Phi_2(\lambda)$, which equals the principal minor of determinant (2.3) by deleting the 4th row and the 4th column. Then

$$\begin{aligned}\Phi_2(\lambda) &= \Phi_1(\lambda; d_2, |Y_1|, 0)/(\lambda - 2)\\ &= [-4 + (6 - |Y_1| + 2d_2)\lambda - (4 + d_2)\lambda^2 + \lambda^3](|Y_1| - d_2\lambda + \lambda^2).\end{aligned}$$

Since $|Y_1| \geq 1$, W contains at least one nonsingular cycle, and hence is nonsingular by Lemma 1.1. So W is positive definite and its eigenvalues are all positive. Therefore, the roots of $\Phi_2(\lambda)$ are all positive, which implies that the largest root of the polynomial $|Y_1| - d_2\lambda + \lambda^2$ on λ is less than d_2. So $\lambda_2(W) < d_2$ by the prior discussion on $f(\lambda)$ in Case (i) and the fact that $d_2 \geq 3$ in this case.

Case (iii). Similar to case (ii), we also obtain a group of equations from equations (2.2) by deleting the 2nd equation and letting $|Y_1| = 0$, and a polynomial on λ, denoted by $\Phi_3(\lambda)$, which equals the principal minor of determinant (2.3) by deleting the 2nd row and column. Then

$$\begin{aligned}\Phi_3(\lambda) &= \Phi_1(\lambda; d_2, 0, |Y_2|)/(\lambda - 2)\\ &= \lambda[(2d_2 - |Y_2|) - (2 + d_2)\lambda + \lambda^2][(2 + |Y_2|) - (2 + d_2)\lambda + \lambda^2].\end{aligned}$$

Then we get the largest and the second largest roots of $\Phi_3(\lambda)$ as follows:

$$\frac{(2 + d_2) + \sqrt{(d_2 - 2)^2 + 4|Y_2|}}{2}, \ \frac{(2 + d_2) + \sqrt{(d_2 + 2)^2 - 4(|Y_2| + 2)}}{2}.$$

Since $d_2 > |Y_2| + 1$ and $|Y_2| \geq 1$, one can see these two roots are both greater than d_2, and hence $\lambda_2(W) > d_2$.

Case (iv). We obtain four equations from (2.2) by deleting the 2nd and the 4th equations and letting $|Y_1| = 0$ and $|Y_2| = 0$, and $\Phi_4(\lambda)$ equal to

$\Phi_3(\lambda)/(\lambda-2)$ by taking $|Y_2|=0$. Then

$$\Phi_4(\lambda)=\lambda(\lambda-d_2)[2-(2+d_2)\lambda+\lambda^2].$$

It is easily seen that the largest and the second largest roots of $\Phi_4(\lambda)$ are $\frac{2+d_2+\sqrt{-4+4d_2+d_2^2}}{2}$ and d_2. So $\lambda_2(W)=d_2$ in this case.

Case (v). We obtain four equations from (2.2) by deleting the 1st and the 3rd equations and letting $|X_2|=0$, and

$$\begin{aligned}\Phi_5(\lambda)&=\Phi_1(\lambda;|Y_1|+|Y_2|+1,|Y_1|,|Y_2|)/(\lambda-1)^2\\&=[2|Y_1|-(2+|Y_1|+|Y_2|)\lambda+\lambda^2][(4+2|Y_2|)-(4+|Y_1|+|Y_2|)\lambda+\lambda^2].\end{aligned}$$

Then the largest and the second largest roots of $\Phi_5(\lambda)$ are:

$$\mu=\frac{(2+|Y_1|+|Y_2|)+\sqrt{(|Y_1|+|Y_2|)^2-4(|Y_1|-|Y_2|)+4}}{2}>2,$$
$$\nu=\frac{(4+|Y_1|+|Y_2|)+\sqrt{(|Y_1|+|Y_2|)^2+8|Y_1|}}{2}>|Y_1|+|Y_2|+2=d_2+1.$$

Hence $\lambda_2(W)\geq d_2$ if and only if $\mu\geq d_2=|Y_1|+|Y_2|+1$ if and only if

$$|Y_1|-|Y_2|-1\leq 0.$$

Note that in this case $d_2=|Y_1|+|Y_2|+1$ and $(d_2-2)|Y_1|-d_2|Y_2|=(|Y_1|-|Y_2|-1)(|Y_1|+|Y_2|)$. So $\lambda_2(W)\geq d_2$ if and only if (2.4) holds.

Case (vi). We obtain three equations from (2.2) by deleting the 1st, the 3rd and the 4th equations and letting $|X_2|=0$ and $|Y_2|=0$, and a polynomial on λ, denoted by $\Phi_6(\lambda)$, which equals $\Phi_5(\lambda)/(\lambda-2)$ by taking $|Y_2|=0$. Hence

$$\Phi_6(\lambda)=(\lambda-|Y_1|)[4-(4+|Y_1|)\lambda+\lambda^2].$$

Then the largest root of $\Phi_6(\lambda)$ is $\frac{(4+|Y_1|)+\sqrt{|Y_1|(8+|Y_1|)}}{2}>|Y_1|+2=d_2+1$, and the second largest root is $\max\{|Y_1|, \frac{(4+|Y_1|)-\sqrt{|Y_1|(8+|Y_1|)}}{2}\} = |Y_1| < d_2 = |Y_1|+1$. So in this case $\lambda_2(W) < d_2$.

Case (vii). Similar to case (vi), $\Phi_7(\lambda)$ equals $\Phi_5(\lambda)/(\lambda-2)$ by taking $|Y_1| = 0$, and hence

$$\Phi_7(\lambda) = \lambda(-2+\lambda-|Y_2|)^2.$$

Then the largest and the second largest roots are both $|Y_2|+2 = d_2+1 > d_2$, and hence $\lambda_2(W) > d_2$.

From the above discussion, neither (2.4) nor $\lambda_2(W) \geq d_2$ holds in the cases (ii) and (vi); both (2.4) and $\lambda_2(W) \geq d_2$ hold in the cases (iii), (iv) and (vii); and in the cases (i) and (v) $\lambda_2(W) \geq d_2$ holds if and only if (2.4) is true. This proves the second assertion of the lemma.

Next we consider the case of $|X_1| > |X_2|$. In this case by deleting $(|X_1| - |X_2|)$ pendant vertices adjacent to v_1, we then obtain a graph W' with same number of pendant vertices adjacent to v_1 and v_2. Except W' being a nonsingular triangle, if (2.4) holds, by Lemma 2.1 we have

$$\lambda_2(W) \geq \lambda_2(W') \geq d_2(W') = d_2(W).$$

If W' is a nonsingular triangle, then W is the graph obtained from W' by appending at least one pendant vertex to v_1. (Note that in this case W also holds (2.4).) If $|X_1| = 1$, by a little calculation, we find that $\lambda_2(W) = 2 = d_2(W)$. So if $|X_1| > 1$, by Lemma 2.1 we still have $\lambda_2(W) \geq 2 = d_2(W)$. The result follows. ■

Remark 1. In Lemma 2.2, in particular if $|Y_1| = 0$ in W (i.e., case (iii), or (iv), or (vii)), then W is all-oriented and holds $\lambda_2(W) \geq d_2(W)$ from [12, Lemma 4] by Li and Pan. Here we provide a uniform proof for mixed graphs. In addition, the proof of Lemma 4 in [12] omits the cases of $|Y_2| = 0$ and $|X_2| = 0$.

Let $G = (V, E)$ be a connected mixed graph with $d_1(G) = d(v_1)$ and $d_2(G) = d(v_2)$. If v_1 and v_2 are not adjacent, then the left-top 2×2 principal submatrix of $L(G)$ has two eigenvalues $d_1(G)$ and $d_2(G)$. By Cauchy-Poincare separation theorem (see [10, Theorem 4.3.15]), we have $\lambda_2(G) \geq d_2(G)$. Next we give the main result of this paper.

Theorem 2.3. *Let $G = (V, E)$ be a connected mixed graph on at least three vertices, except the case of G being a nonsingular triangle. Let v_1, v_2 be the vertices of G with $d_1(G) = d(v_1)$ and $d_2(G) = d(v_2)$. Let $\triangle_n, \triangle_s$ be respectively the numbers of nonsingular and singular triangles of G consisting of v_1, v_2 and one of their common adjacent vertices. If $[d_2(G) - 2]\triangle_n - d_2(G)\triangle_s \leq 0$, then*

$$\lambda_2(G) \geq d_2(G).$$

Proof. If v_1 and v_2 are not adjacent, then $\triangle_n = \triangle_s = 0$, and the result holds obviously by the discussion prior to this theorem. Now suppose that v_1 and v_2 are adjacent. Let Y be the set of vertices of G adjacent to both v_1 and v_2, and let X_1, X_2 be respectively the set of vertices adjacent to v_1 and v_2 both not within Y. Considering the subgraph of G induced by the vertices of $X_1 \cup X_2 \cup Y$, and deleting respectively the edges within X_1, X_2 and Y, we then obtain a subgraph H of G with the same underlying graph of W of Figure 2.1.

Let F be set of edges of H which joins v_2 and one vertex of Y. Then $H - F$ contains no cycles and hence is quasi-bipartite. By Theorem 1.2 there exists a signature matrix D such that $D^T L(H-F)D = L(\overrightarrow{H-F})$. Consider the matrix $D^T L(H)D =: L(^D H)$, where $^D H$ is a mixed graph with the same underlying graph and the same labels of vertices as H. For the mixed graph $^D H$, except edges of F, all edges are oriented (or of sign -1); and for the edges of F some are oriented and some are unoriented. Let Y_1, Y_2 denote respectively the set of vertices of Y which joins v_2 by an unoriented edge and by an oriented edge in the graph $^D H$. Then $^D H$, and hence H, has exactly $|Y_1|$ nonsingular (and $|Y_2|$ singular) triangles consisting of v_1, v_2 and one vertex of Y_1 (and one vertex of Y_2), since the signature matrix D does not change the singularity of each cycle of H. So $|Y_1| = \triangle_n$ and $|Y_2| = \triangle_s$.

Note that H (or $^D H$) preserves the largest and second largest degrees of G. Now applying Lemma 2.2 to the graph $^D H$, if $^D H$ is not a nonsingular triangle and $[d_2(^D H) - 2]\triangle_n - d_2(^D H)\triangle_s \leq 0$, then

$$\lambda_2(H) = \lambda_2(^D H) \geq d_2(^D H) = d_2(H).$$

By Lemma 2.1,

$$\lambda_2(G) \geq \lambda_2(H) \geq d_2(H) = d_2(G).$$

If ${}^{D}H$ (or H) is a nonsingular triangle, then $d_1(H) = d_2(H) = 2$ and hence $d_1(G) = d_2(G) = 2$, which implies that G itself is a nonsingular triangle. The result follows. ■

Remark 2. (1) By Lemma 2.2, we know that there exist mixed graphs G with $\lambda_2(G) \le d_2(G)$. In addition, as the graph of Figure 2.2 shows that our condition in Theorem 2.3 is sufficient but not necessary.

(2) In Theorem 2.3, if $\triangle_n = 0$, then $\lambda_2(G) \ge d_2(G)$. We know that G also has the possibilities to be nonsingular and to be singular. If G is singular, in particular G is all-oriented, then our result in this case is exactly that of Li and Pan's ([12, Theorem 4]).

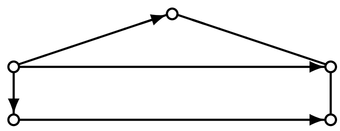

Figure 2.2. The graph G with $\lambda_2(G) \approx 3.61803 > d_2(G) = 3$.

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