# ON THE SECOND LARGEST EIGENVALUE OF A MIXED GRAPH* 

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#### Abstract

Let $G$ be a mixed graph. We discuss the relation between the second largest eigenvalue $\lambda_{2}(G)$ and the second largest degree $d_{2}(G)$, and present a sufficient condition for $\lambda_{2}(G) \geq d_{2}(G)$.


Keywords: mixed graph, Laplacian eigenvalue, degree.
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## 1. Introduction

Let $G=(V, E)$ be a mixed graph with vertex set $V=V(G)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and edge set $E=E(G)=\left\{e_{1}, e_{2}, \ldots, e_{m}\right\}$, which is obtained from an undirected graph by orienting some (possibly none or all) of its edges. Hence in a mixed graph some edges are oriented while others are not. We denote respectively by $\{u, v\}$ and $(u, v)$ the unoriented edge and the oriented edge joining $u$ and $v$; and for the oriented edge $(u, v)$, we call $u$ and $v$ respectively the head and tail of the edge. It is important to stress that the mixed graphs are considered undirected graphs in terms of defining the degrees of

[^0]vertices, path, cycle and connectedness, etc.. In addition, the mixed graphs throughout this paper contain no multi-edges or loops.

Denote by $d(v)=d_{G}(v)$ the degree of the vertex $v \in V(G)$. For each $e \in E(G)$, we define the sign of $e$ and denote by $\operatorname{sgn} e=1$ if $e$ is unoriented and $\operatorname{sgn} e=-1$ if $e$ is oriented. Set $a_{i j}=\operatorname{sgn} e$ if there exists an edge $e$ joining $v_{i}$ and $v_{j}$, and $a_{i j}=0$, otherwise. Then the resulting matrix $A=\left(a_{i j}\right)$ is called the adjacency matrix of $G$. The incidence matrix of $G$ is an $n \times m$ matrix $M=M(G)=\left(m_{i j}\right)$ whose entries are given by $m_{i j}=1$ if $e_{j}$ is an unoriented edge incident to $v_{i}$ or $e_{j}$ is an oriented edge with head $v_{i}, m_{i j}=-1$ if $e_{j}$ is an oriented edge with tail $v_{i}$, and $m_{i j}=0$, otherwise. The Laplacian matrix of $G$ is defined as $L(G)=M M^{T}$ (see [1] or [15]), where $M^{T}$ denotes the transpose of $M$. Obviously $L(G)$ is symmetric and positive semi-definite, and $L(G)=D(G)+A(G)$ (or see [15, Lemma 2.1]), where $D(G)=\operatorname{diag}\left\{d\left(v_{1}\right), d\left(v_{2}\right), \ldots, d\left(v_{n}\right)\right\}$. Therefore the eigenvalues of $L(G)$ can be arranged as follows:

$$
\lambda_{1}(G) \geq \lambda_{2}(G) \geq \cdots \geq \lambda_{n}(G) .
$$

We briefly called the eigenvalues and eigenvectors of $L(G)$ as those of $G$, respectively. $G$ is called singular (or nonsingular) if $L(G)$ is singular (or nonsingular).

Clearly if $G$ is all-oriented (i.e., all edges of $G$ are oriented), then $L(G)$ is consistent with the Laplacian matrix of a simple graph (note that when we define the Laplacian matrix of a simple graph we first give an orientation to each edge of the graph, and then obtain the Laplacian matrix via incidence matrix as above; see [13]). For simple graphs, there are a wealth of results involved with the relations between its spectrum and numerous graph invariants, such as connectivity, diameter, matching number, isoperimetric number, and expanding properties of a graph; see, e.g., $[6,9,13,14]$.

For the algebraic property of mixed graphs, Bapat et al. [1, 2] extent the definition of Laplacian matrix of simple graphs to that of mixed graphs, and generalized the classical Matrix-Tree theorem. In [15, 16], Zhang et al. gave some relations between a mixed graph and its line graph, and obtained some upper bounds for the largest eigenvalue and lower bounds for the second largest eigenvalue of mixed graphs. Fan [3] characterized mixed graphs which maximize or minimize the largest eigenvalue over all unicyclic mixed graphs. Also for unicyclic mixed graphs, Fan [4] gave a structural property of the eigenvectors corresponding to the least eigenvalue, and used the result to characterize those graphs with fixed number of vertices and with
girth 3 which minimize the least eigenvalue. In addition, Fan [5] discussed the spectral perturbation of a mixed graph by adding an edge.

Since the matrix $L(G)$, as well as $A(G)$, is determined by the signs of edges of $G$, we simply concern ourselves with whether an edge is oriented or not, and do not care which one is the head and which one is the tail of an oriented edge in the following discussion. In this sense, the notion of signed graphs [11] instead of mixed graph will be more suitable for our discussion. A mixed graph $G$ is called quasi-bipartite if it does not contain a nonsingular cycle, or equivalently, $G$ contains no cycles with an odd number of unoriented edges (see [1, Lemma 1]). Denote by $\vec{G}$ the all-oriented graph obtained from $G$ by arbitrarily orienting every unoriented edge of $G$ (if one exists). Note that a signature matrix is a diagonal matrix with 1 or -1 along its diagonal.

Lemma 1.1 ([15, Lemma 2.2], [5, Lemma 5]). Let $G$ be a connected mixed graph. Then $G$ is singular if and only if $G$ is quasi-bipartite.

Theorem 1.2 ([1, Theorem 4]). Let $G$ be a mixed graph. Then $G$ is quasi-bipartite if and only if there exists a signature matrix $D$ such that $D^{T} L(G) D=L(\vec{G})$.

Denote by $d_{1}(G), d_{2}(G)$ respectively the largest and the second largest degree of vertices of $G$. If $G$ is a simple connected graph containing at least 3 vertices and one edge, then
(i) (Grone and Merris's bound [8]) $\lambda_{1}(G) \geq d_{1}(G)+1$,
(ii) (Li and Pan's bound [12]) $\lambda_{2}(G) \geq d_{2}(G)$.

In [16], Zhang and Luo show that (i) also holds for mixed graph; independently Hou et al. [11] also obtain this result with a little interpretation. We find that (ii) does not always hold for mixed graphs, and then give a sufficient condition for (ii) holding on mixed graphs. Our result implies Li and Pan's result in the case of the mixed graph $G$ be all-oriented, in which case the Laplacian matrix is consistent with that of a simple graph.

## 2. Main Results

Let $G=(V, E)$ be a mixed graph with $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$, and let $x=$ $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$ be a real vector. It will be convenient to adopt the
following terminology from [7]: $x$ is said to give a valuation of the vertices of $V$, that is, for each vertex $v_{i}$ of $V$, we associate the value $x_{i}$, i.e., $x\left(v_{i}\right)=x_{i}$. Then $\lambda$ is an eigenvalue of $G$ with the corresponding eigenvector $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ if and only if $x \neq 0$ and

$$
\begin{equation*}
\left[\lambda-d\left(v_{i}\right)\right] x\left(v_{i}\right)=\sum_{e=\left\{v_{i}, v_{j}\right\} \in E}(\operatorname{sgn} e) x\left(v_{j}\right), \quad \text { for } \quad i=1,2, \ldots, n \tag{2.1}
\end{equation*}
$$

Lemma 2.1 ([16, Lemma 2.2]). Let $G$ be a mixed graph on $n$ vertices and let $e$ be an (oriented or unoriented) edge of $G$. Then

$$
\lambda_{1}(G) \geq \lambda_{1}(G-e) \geq \lambda_{2}(G) \geq \cdots \geq \lambda_{n}(G) \geq \lambda_{n}(G-e)
$$

Denote by $|S|$ the cardinality of a set $S$. We first introduce a mixed graph $W$ as follows, which will play an important role in our discussion. Let $X_{1}, X_{2}$, $Y_{1}, Y_{2}$ and $\left\{v_{1}\right\},\left\{v_{2}\right\}$ be pairwise disjoint vertex sets, where $\left|X_{1}\right| \geq\left|X_{2}\right| \geq 0$, $\left|Y_{1}\right| \geq 0$ and $\left|Y_{2}\right| \geq 0$. The graph $W=(V(W), E(W))$ is defined as in Figure 2.1, where the vertex set $V(W)=X_{1} \cup X_{2} \cup Y_{1} \cup Y_{2} \cup\left\{v_{1}\right\} \cup\left\{v_{2}\right\}$, and the edge set

$$
\begin{aligned}
E(W)= & \left\{\left(v_{1}, u\right) \mid u \in X_{1} \cup Y_{1} \cup Y_{2} \cup\left\{v_{2}\right\}\right\} \\
& \cup\left\{\left(v_{2}, u\right) \mid u \in X_{2} \cup Y_{2}\right\} \cup\left\{\left\{v_{2}, u\right\} \mid u \in Y_{1}\right\}
\end{aligned}
$$

Clearly, $d_{1}(W)=1+\left|X_{1}\right|+\left|Y_{1}\right|+\left|Y_{2}\right| \geq 1+\left|X_{2}\right|+\left|Y_{1}\right|+\left|Y_{2}\right|=d_{2}(W)$. If $W$ is exactly a cycle on 3 vertices (simply called a triangle) and is further nonsingular, or equivalently, $\left|X_{1}\right|=\left|X_{2}\right|=\left|Y_{2}\right|=0$ and $\left|Y_{1}\right|=1$, then $\lambda_{2}(W)=1<d_{2}(W)=2$.


Figure 2.1

Lemma 2.2. Let $W$ be the graph of Figure 2.1 containing $v_{1}, v_{2}$ and at least one other vertex, except the case of $W$ being a nonsingular triangle. If $\left[d_{2}(W)-2\right]\left|Y_{1}\right|-d_{2}(W)\left|Y_{2}\right| \leq 0$, then

$$
\lambda_{2}(W) \geq d_{2}(W)
$$

In particular for the case of $\left|X_{1}\right|=\left|X_{2}\right|$, then $\lambda_{2}(W) \geq d_{2}(W)$ if and only if

$$
\left[d_{2}(W)-2\right]\left|Y_{1}\right|-d_{2}(W)\left|Y_{2}\right| \leq 0
$$

Proof. Assume first that $\left|X_{1}\right|=\left|X_{2}\right|$. Then $d_{1}(W)=d_{2}(W)=: d_{2}$. The graph $W$ has the possibility to be one of the following cases.
(i) $\left|Y_{1}\right| \geq 1,\left|Y_{2}\right| \geq 1,\left|X_{2}\right| \geq 1 ; ~(i i) ~\left|Y_{1}\right| \geq 1,\left|Y_{2}\right|=0,\left|X_{2}\right| \geq 1$;
(iii) $\left|Y_{1}\right|=0,\left|Y_{2}\right| \geq 1,\left|X_{2}\right| \geq 1$; (iv) $\left|Y_{1}\right|=0,\left|Y_{2}\right|=0,\left|X_{2}\right| \geq 1$;
(v) $\left|Y_{1}\right| \geq 1,\left|Y_{2}\right| \geq 1,\left|X_{2}\right|=0 ;$ (vi) $\left|Y_{1}\right| \geq 2,\left|Y_{2}\right|=0,\left|X_{2}\right|=0$;
(vii) $\left|Y_{1}\right|=0,\left|Y_{2}\right| \geq 1,\left|X_{2}\right|=0$.

Note that in case (vi) $\left|Y_{1}\right|=1$ is not allowed otherwise $W$ is a nonsingular triangle. Let $\lambda(\lambda \neq 1, \lambda \neq 2)$ be an eigenvalue of $W$ with the corresponding eigenvector $x$. We discuss the cases (i)-(vii) as follows.

Case (i). Since $\lambda \neq 1,2$, by (2.1), $x(u)=x(v)$ for each pair $u, v$ in $X_{i}$ and each pair $u, v$ in $Y_{i}$ for $i=1,2$. Thus, let

$$
\begin{array}{ll}
x(u)=: y_{1}, \forall u \in X_{1} ; & x(u)=: y_{2}, \forall u \in Y_{1} ; \\
x(u)=: y_{3}, \forall u \in X_{2} ; & x(u)=: y_{4}, \forall u \in Y_{2} .
\end{array}
$$

Let $x\left(v_{1}\right)=: y_{5}$ and $x\left(v_{2}\right)=: y_{6}$. Then equations (2.1) for $\lambda$ and the corresponding eigenvector $x$ are equivalent to the following equations:

$$
\left\{\begin{array}{l}
(\lambda-1) y_{1}=-y_{5},  \tag{2.2}\\
(\lambda-2) y_{2}=-y_{5}+y_{6}, \\
(\lambda-1) y_{3}=-y_{6} \\
(\lambda-2) y_{4}=-y_{5}-y_{6}, \\
\left(\lambda-d_{2}\right) y_{5}=-\left|X_{2}\right| y_{1}-\left|Y_{1}\right| y_{2}-\left|Y_{2}\right| y_{4}-y_{6} \\
\left(\lambda-d_{2}\right) y_{6}=\left|Y_{1}\right| y_{2}-\left|X_{2}\right| y_{3}-\left|Y_{2}\right| y_{4}-y_{5}
\end{array}\right.
$$

We turn the equations (2.2) into the matrix equation $(\lambda I-B) y=0$, where $B$ is the coefficient matrix of above linear equations and $y=\left(y_{1}, \ldots, y_{6}\right)^{T}$. Noting that $\left|X_{2}\right|=d_{2}-\left|Y_{1}\right|-\left|Y_{2}\right|-1$, then the solutions of $\lambda$ of (2.2) are exactly the roots of the following polynomial $\Phi_{1}(\lambda)=\operatorname{det}(\lambda I-B)$ :

$$
\Phi_{1}(\lambda)=\operatorname{det}\left[\begin{array}{cccccc}
\lambda-1 & 0 & 0 & 0 & 1 & 0  \tag{2.3}\\
0 & \lambda-2 & 0 & 0 & 1 & -1 \\
0 & 0 & \lambda-1 & 0 & 0 & 1 \\
0 & 0 & 0 & \lambda-2 & 1 & 1 \\
d_{2}-\left|Y_{1}\right|-\left|Y_{2}\right|-1 & \left|Y_{1}\right| & 0 & \left|Y_{2}\right| & \lambda-d_{2} & 1 \\
0 & -\left|Y_{1}\right| & d_{2}-\left|Y_{1}\right|-\left|Y_{2}\right|-1 & \left|Y_{2}\right| & 1 & \lambda-d_{2}
\end{array}\right] .
$$

By an elementary calculation,

$$
\begin{aligned}
\Phi_{1}(\lambda)= & \Phi_{1}\left(\lambda ; d_{2},\left|Y_{1}\right|,\left|Y_{2}\right|\right) \\
= & {\left[-\left(4+2\left|Y_{2}\right|\right)+\left(6-\left|Y_{1}\right|+\left|Y_{2}\right|+2 d_{2}\right) \lambda-\left(4+d_{2}\right) \lambda^{2}+\lambda^{3}\right] } \\
& \times\left[-2\left|Y_{1}\right|+\left(\left|Y_{1}\right|-\left|Y_{2}\right|+2 d_{2}\right) \lambda-\left(2+d_{2}\right) \lambda^{2}+\lambda^{3}\right] .
\end{aligned}
$$

Let

$$
f(\lambda)=-\left(4+2\left|Y_{2}\right|\right)+\left(6-\left|Y_{1}\right|+\left|Y_{2}\right|+2 d_{2}\right) \lambda-\left(4+d_{2}\right) \lambda^{2}+\lambda^{3}
$$

Then $f(0)=-\left(4+2\left|Y_{2}\right|\right)<0, f(1)=d_{2}-\left|Y_{1}\right|-\left|Y_{2}\right|-1=\left|X_{2}\right|>0$, $f(2)=-2\left|Y_{1}\right|<0, f\left(d_{2}+1\right)=-\left(1+d_{2}\right)\left|Y_{1}\right|-\left(d_{2}-1\right)\left(d_{2}-\left|Y_{2}\right|-1\right)<0$. So the largest root of $f(\lambda)$ is greater than $d_{2}+1$, and the second largest root lies in the open interval (1,2). Let

$$
g(\lambda)=-2\left|Y_{1}\right|+\left(\left|Y_{1}\right|-\left|Y_{2}\right|+2 d_{2}\right) \lambda-\left(2+d_{2}\right) \lambda^{2}+\lambda^{3} .
$$

Then $g(0)=-2\left|Y_{1}\right|<0, g(1)=d_{2}-\left|Y_{1}\right|-\left|Y_{2}\right|-1=\left|X_{2}\right|>0, g(2)=$ $-2\left|Y_{2}\right|<0, g\left(d_{2}+1\right)=\left(d_{2}+1\right)\left(d_{2}-\left|Y_{2}\right|-1\right)+\left(d_{2}-1\right)\left|Y_{1}\right|>0$. So the largest root $g(\lambda)$ lies in the open interval $\left(2, d_{1}+1\right)$.

By above discussion, $\lambda_{2}(W)$ is exactly the largest root of $g(\lambda)$. Note that

$$
g\left(d_{2}\right)=\left(d_{2}-2\right)\left|Y_{1}\right|-d_{2}\left|Y_{2}\right| .
$$

Hence $\lambda_{2}(W) \geq d_{2}$ if and only if $g\left(d_{2}\right) \leq 0$ if and only if

$$
\begin{equation*}
\left(d_{2}-2\right)\left|Y_{1}\right|-d_{2}\left|Y_{2}\right| \leq 0 . \tag{2.4}
\end{equation*}
$$

Case (ii). Similar to Case (i), we obtain a system of equations from equations in (2.2) just by deleting the 4th equation and letting $\left|Y_{2}\right|=0$, and get a polynomial on $\lambda$, denoted by $\Phi_{2}(\lambda)$, which equals the principal minor of determinant (2.3) by deleting the 4th row and the 4th column. Then

$$
\begin{aligned}
\Phi_{2}(\lambda) & =\Phi_{1}\left(\lambda ; d_{2},\left|Y_{1}\right|, 0\right) /(\lambda-2) \\
& =\left[-4+\left(6-\left|Y_{1}\right|+2 d_{2}\right) \lambda-\left(4+d_{2}\right) \lambda^{2}+\lambda^{3}\right]\left(\left|Y_{1}\right|-d_{2} \lambda+\lambda^{2}\right) .
\end{aligned}
$$

Since $\left|Y_{1}\right| \geq 1, W$ contains at least one nonsingular cycle, and hence is nonsingular by Lemma 1.1. So $W$ is positive definite and its eigenvalues are all positive. Therefore, the roots of $\Phi_{2}(\lambda)$ are all positive, which implies that the largest root of the polynomial $\left|Y_{1}\right|-d_{2} \lambda+\lambda^{2}$ on $\lambda$ is less than $d_{2}$. So $\lambda_{2}(W)<d_{2}$ by the prior discussion on $f(\lambda)$ in Case (i) and the fact that $d_{2} \geq 3$ in this case.

Case (iii). Similar to case (ii), we also obtain a group of equations from equations (2.2) by deleting the 2nd equation and letting $\left|Y_{1}\right|=0$, and a polynomial on $\lambda$, denoted by $\Phi_{3}(\lambda)$, which equals the principal minor of determinant (2.3) by deleting the 2 nd row and column. Then

$$
\begin{aligned}
\Phi_{3}(\lambda) & =\Phi_{1}\left(\lambda ; d_{2}, 0,\left|Y_{2}\right|\right) /(\lambda-2) \\
& =\lambda\left[\left(2 d_{2}-\left|Y_{2}\right|\right)-\left(2+d_{2}\right) \lambda+\lambda^{2}\right]\left[\left(2+\left|Y_{2}\right|\right)-\left(2+d_{2}\right) \lambda+\lambda^{2}\right] .
\end{aligned}
$$

Then we get the largest and the second largest roots of $\Phi_{3}(\lambda)$ as follows:

$$
\frac{\left(2+d_{2}\right)+\sqrt{\left(d_{2}-2\right)^{2}+4\left|Y_{2}\right|}}{2}, \frac{\left(2+d_{2}\right)+\sqrt{\left(d_{2}+2\right)^{2}-4\left(\left|Y_{2}\right|+2\right)}}{2}
$$

Since $d_{2}>\left|Y_{2}\right|+1$ and $\left|Y_{2}\right| \geq 1$, one can see these two roots are both greater than $d_{2}$, and hence $\lambda_{2}(W)>d_{2}$.

Case (iv). We obtain four equations from (2.2) by deleting the 2nd and the 4 th equations and letting $\left|Y_{1}\right|=0$ and $\left|Y_{2}\right|=0$, and $\Phi_{4}(\lambda)$ equal to
$\Phi_{3}(\lambda) /(\lambda-2)$ by taking $\left|Y_{2}\right|=0$. Then

$$
\Phi_{4}(\lambda)=\lambda\left(\lambda-d_{2}\right)\left[2-\left(2+d_{2}\right) \lambda+\lambda^{2}\right]
$$

It is easily seen that the largest and the second largest roots of $\Phi_{4}(\lambda)$ are $\frac{2+d_{2}+\sqrt{-4+4 d_{2}+d_{2}^{2}}}{2}$ and $d_{2}$. So $\lambda_{2}(W)=d_{2}$ in this case.

Case (v). We obtain four equations from (2.2) by deleting the 1st and the 3 rd equations and letting $\left|X_{2}\right|=0$, and

$$
\begin{aligned}
\Phi_{5}(\lambda) & =\Phi_{1}\left(\lambda ;\left|Y_{1}\right|+\left|Y_{2}\right|+1,\left|Y_{1}\right|,\left|Y_{2}\right|\right) /(\lambda-1)^{2} \\
& =\left[2\left|Y_{1}\right|-\left(2+\left|Y_{1}\right|+\left|Y_{2}\right|\right) \lambda+\lambda^{2}\right]\left[\left(4+2\left|Y_{2}\right|\right)-\left(4+\left|Y_{1}\right|+\left|Y_{2}\right|\right) \lambda+\lambda^{2}\right]
\end{aligned}
$$

Then the largest and the second largest roots of $\Phi_{5}(\lambda)$ are:

$$
\begin{aligned}
& \mu=\frac{\left(2+\left|Y_{1}\right|+\left|Y_{2}\right|\right)+\sqrt{\left(\left|Y_{1}\right|+\left|Y_{2}\right|\right)^{2}-4\left(\left|Y_{1}\right|-\left|Y_{2}\right|\right)+4}}{2}>2 \\
& \nu=\frac{\left(4+\left|Y_{1}\right|+\left|Y_{2}\right|\right)+\sqrt{\left(\left|Y_{1}\right|+\left|Y_{2}\right|\right)^{2}+8\left|Y_{1}\right|}}{2}>\left|Y_{1}\right|+\left|Y_{2}\right|+2=d_{2}+1
\end{aligned}
$$

Hence $\lambda_{2}(W) \geq d_{2}$ if and only if $\mu \geq d_{2}=\left|Y_{1}\right|+\left|Y_{2}\right|+1$ if and only if

$$
\left|Y_{1}\right|-\left|Y_{2}\right|-1 \leq 0
$$

Note that in this case $d_{2}=\left|Y_{1}\right|+\left|Y_{2}\right|+1$ and $\left(d_{2}-2\right)\left|Y_{1}\right|-d_{2}\left|Y_{2}\right|=$ $\left(\left|Y_{1}\right|-\left|Y_{2}\right|-1\right)\left(\left|Y_{1}\right|+\left|Y_{2}\right|\right)$. So $\lambda_{2}(W) \geq d_{2}$ if and only if (2.4) holds.

Case (vi). We obtain three equations from (2.2) by deleting the 1st, the 3rd and the 4th equations and letting $\left|X_{2}\right|=0$ and $\left|Y_{2}\right|=0$, and a polynomial on $\lambda$, denoted by $\Phi_{6}(\lambda)$, which equals $\Phi_{5}(\lambda) /(\lambda-2)$ by taking $\left|Y_{2}\right|=0$. Hence

$$
\Phi_{6}(\lambda)=\left(\lambda-\left|Y_{1}\right|\right)\left[4-\left(4+\left|Y_{1}\right|\right) \lambda+\lambda^{2}\right]
$$

Then the largest root of $\Phi_{6}(\lambda)$ is $\frac{\left(4+\left|Y_{1}\right|\right)+\sqrt{\left|Y_{1}\right|\left(8+\left|Y_{1}\right|\right)}}{2}>\left|Y_{1}\right|+2=d_{2}+1$,
and the second largest root is $\max \left\{\left|Y_{1}\right|, \frac{\left(4+\left|Y_{1}\right|\right)-\sqrt{\left|Y_{1}\right|\left(8+\left|Y_{1}\right|\right)}}{2}\right\}=\left|Y_{1}\right|<d_{2}=$ $\left|Y_{1}\right|+1$. So in this case $\lambda_{2}(W)<d_{2}$.

Case (vii). Similar to case (vi), $\Phi_{7}(\lambda)$ equals $\Phi_{5}(\lambda) /(\lambda-2)$ by taking $\left|Y_{1}\right|=0$, and hence

$$
\Phi_{7}(\lambda)=\lambda\left(-2+\lambda-\left|Y_{2}\right|\right)^{2}
$$

Then the largest and the second largest roots are both $\left|Y_{2}\right|+2=d_{2}+1>d_{2}$, and hence $\lambda_{2}(W)>d_{2}$.

From the above discussion, neither (2.4) nor $\lambda_{2}(W) \geq d_{2}$ holds in the cases (ii) and (vi); both (2.4) and $\lambda_{2}(W) \geq d_{2}$ hold in the cases (iii), (iv) and (vii); and in the cases (i) and (v) $\lambda_{2}(W) \geq d_{2}$ holds if and only if (2.4) is true. This proves the second assertion of the lemma.

Next we consider the case of $\left|X_{1}\right|>\left|X_{2}\right|$. In this case by deleting $\left(\left|X_{1}\right|-\left|X_{2}\right|\right)$ pendant vertices adjacent to $v_{1}$, we then obtain a graph $W^{\prime}$ with same number of pendant vertices adjacent to $v_{1}$ and $v_{2}$. Except $W^{\prime}$ being a nonsingular triangle, if (2.4) holds, by Lemma 2.1 we have

$$
\lambda_{2}(W) \geq \lambda_{2}\left(W^{\prime}\right) \geq d_{2}\left(W^{\prime}\right)=d_{2}(W)
$$

If $W^{\prime}$ is a nonsingular triangle, then $W$ is the graph obtained from $W^{\prime}$ by appending at least one pendant vertex to $v_{1}$. (Note that in this case $W$ also holds (2.4).) If $\left|X_{1}\right|=1$, by a little calculation, we find that $\lambda_{2}(W)=2=$ $d_{2}(W)$. So if $\left|X_{1}\right|>1$, by Lemma 2.1 we still have $\lambda_{2}(W) \geq 2=d_{2}(W)$. The result follows.

Remark 1. In Lemma 2.2, in particular if $\left|Y_{1}\right|=0$ in $W$ (i.e., case (iii), or (iv), or (vii)), then $W$ is all-oriented and holds $\lambda_{2}(W) \geq d_{2}(W)$ from [12, Lemma 4] by Li and Pan. Here we provide a uniform proof for mixed graphs. In addition, the proof of Lemma 4 in [12] omits the cases of $\left|Y_{2}\right|=0$ and $\left|X_{2}\right|=0$.

Let $G=(V, E)$ be a connected mixed graph with $d_{1}(G)=d\left(v_{1}\right)$ and $d_{2}(G)=d\left(v_{2}\right)$. If $v_{1}$ and $v_{2}$ are not adjacent, then the left-top $2 \times 2$ principal submatrix of $L(G)$ has two eigenvalues $d_{1}(G)$ and $d_{2}(G)$. By Cauchy-Poincare separation theorem (see [10, Theorem 4.3.15]), we have $\lambda_{2}(G) \geq d_{2}(G)$. Next we give the main result of this paper.

Theorem 2.3. Let $G=(V, E)$ be a connected mixed graph on at least three vertices, except the case of $G$ being a nonsingular triangle. Let $v_{1}, v_{2}$ be the vertices of $G$ with $d_{1}(G)=d\left(v_{1}\right)$ and $d_{2}(G)=d\left(v_{2}\right)$. Let $\triangle_{n}, \triangle_{s}$ be respectively the numbers of nonsingular and singular triangles of $G$ consisting of $v_{1}, v_{2}$ and one of their common adjacent vertices. If $\left[d_{2}(G)-2\right] \triangle_{n}-$ $d_{2}(G) \triangle_{s} \leq 0$, then

$$
\lambda_{2}(G) \geq d_{2}(G)
$$

Proof. If $v_{1}$ and $v_{2}$ are not adjacent, then $\triangle_{n}=\triangle_{s}=0$, and the result holds obviously by the discussion prior to this theorem. Now suppose that $v_{1}$ and $v_{2}$ are adjacent. Let $Y$ be the set of vertices of $G$ adjacent to both $v_{1}$ and $v_{2}$, and let $X_{1}, X_{2}$ be respectively the set of vertices adjacent to $v_{1}$ and $v_{2}$ both not within $Y$. Considering the subgraph of $G$ induced by the vertices of $X_{1} \cup X_{2} \cup Y$, and deleting respectively the edges within $X_{1}, X_{2}$ and $Y$, we then obtain a subgraph $H$ of $G$ with the same underlying graph of $W$ of Figure 2.1.

Let $F$ be set of edges of $H$ which joins $v_{2}$ and one vertex of $Y$. Then $H-F$ contains no cycles and hence is quasi-bipartite. By Theorem 1.2 there exists a signature matrix $D$ such that $D^{T} L(H-F) D=L(\overrightarrow{H-F})$. Consider the matrix $D^{T} L(H) D=: L\left({ }^{D} H\right)$, where ${ }^{D} H$ is a mixed graph with the same underlying graph and the same labels of vertices as $H$. For the mixed graph ${ }^{D} H$, except edges of $F$, all edges are oriented (or of sign -1 ); and for the edges of $F$ some are oriented and some are unoriented. Let $Y_{1}, Y_{2}$ denote respectively the set of vertices of $Y$ which joins $v_{2}$ by an unoriented edge and by an oriented edge in the graph ${ }^{D} H$. Then ${ }^{D} H$, and hence $H$, has exactly $\left|Y_{1}\right|$ nonsingular (and $\left|Y_{2}\right|$ singular) triangles consisting of $v_{1}, v_{2}$ and one vertex of $Y_{1}$ (and one vertex of $Y_{2}$ ), since the signature matrix $D$ does not change the singularity of each cycle of $H$. So $\left|Y_{1}\right|=\triangle_{n}$ and $\left|Y_{2}\right|=\triangle_{s}$.

Note that $H$ (or ${ }^{D} H$ ) preserves the largest and second largest degrees of $G$. Now applying Lemma 2.2 to the graph ${ }^{D} H$, if ${ }^{D} H$ is not a nonsingular triangle and $\left[d_{2}\left({ }^{D} H\right)-2\right] \triangle_{n}-d_{2}\left({ }^{D} H\right) \triangle_{s} \leq 0$, then

$$
\lambda_{2}(H)=\lambda_{2}\left({ }^{D} H\right) \geq d_{2}\left({ }^{D} H\right)=d_{2}(H)
$$

By Lemma 2.1,

$$
\lambda_{2}(G) \geq \lambda_{2}(H) \geq d_{2}(H)=d_{2}(G)
$$

If ${ }^{D} H$ (or $H$ ) is a nonsingular triangle, then $d_{1}(H)=d_{2}(H)=2$ and hence $d_{1}(G)=d_{2}(G)=2$, which implies that $G$ itself is a nonsingular triangle. The result follows.

Remark 2. (1) By Lemma 2.2, we know that there exist mixed graphs $G$ with $\lambda_{2}(G) \leq d_{2}(G)$. In addition, as the graph of Figure 2.2 shows that our condition in Theorem 2.3 is sufficient but not necessary.
(2) In Theorem 2.3, if $\triangle_{n}=0$, then $\lambda_{2}(G) \geq d_{2}(G)$. We know that $G$ also has the possibilities to be nonsingular and to be singular. If $G$ is singular, in particular $G$ is all-oriented, then our result in this case is exactly that of Li and Pan's ([12, Theorem 4]).


Figure 2.2. The graph $G$ with $\lambda_{2}(G) \approx 3.61803>d_{2}(G)=3$.

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