

## VERTEX-DOMINATING CYCLES IN 2-CONNECTED BIPARTITE GRAPHS

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### Abstract

A cycle  $C$  is a vertex-dominating cycle if every vertex is adjacent to some vertex of  $C$ . Bondy and Fan [4] showed that if  $G$  is a 2-connected graph with  $\delta(G) \geq \frac{1}{3}(|V(G)| - 4)$ , then  $G$  has a vertex-dominating cycle. In this paper, we prove that if  $G$  is a 2-connected bipartite graph with partite sets  $V_1$  and  $V_2$  such that  $\delta(G) \geq \frac{1}{3}(\max\{|V_1|, |V_2|\} + 1)$ , then  $G$  has a vertex-dominating cycle.

**Keywords:** vertex-dominating cycle, dominating cycle, bipartite graph.

**2000 Mathematics Subject Classification:** 05C38, 05C45.

### 1. INTRODUCTION

In this paper, we only consider finite undirected graphs without loops or multiple edges. We denote the degree of a vertex  $x$  in a graph  $G$  by  $d_G(x)$ . Let  $\delta(G)$  be the minimum degree of a graph  $G$ . We denote the number of components of  $G$  by  $\omega(G)$ . A connected graph  $G$  is defined to be  $t$ -tough if  $|S| \geq t \cdot \omega(G - S)$  for every cutset  $S$  of  $V(G)$ . The *toughness* of  $G$ , denoted by  $t(G)$ , is the maximum value of  $t$  for which  $G$  is  $t$ -tough (taking  $t(K_n) = \infty$  for all  $n \geq 1$ ). A set  $S$  of vertices in a graph  $G$  is said to be  $d$ -stable if the distance of each pair of distinct vertices in  $S$  is at least  $d$ .

In 1960, Ore introduced a degree sum condition for hamiltonian cycles.

**Theorem 1** (Ore [8]). *Let  $G$  be a graph on  $n \geq 3$  vertices. If  $d_G(x) + d_G(y) \geq n$  for any nonadjacent vertices  $x$  and  $y$ , then  $G$  is hamiltonian.*

It is observed that weaker conditions guarantee the existence of hamiltonian cycles by putting a further assumption on graphs. For example, Jung (1972) and Moon and Moser (1963) showed that weaker degree sum conditions guarantee hamiltonian cycles in 1-tough graphs and in bipartite graphs, respectively.

**Theorem 2** (Jung [6]). *Let  $G$  be a 1-tough graph of order  $n \geq 11$ . If  $d_G(x) + d_G(y) \geq n - 4$  for any nonadjacent vertices  $x$  and  $y$ , then  $G$  is hamiltonian.*

**Theorem 3** (Moon and Moser [7]). *Let  $G$  be a bipartite graph with partite sets  $V_1$  and  $V_2$ , where  $|V_1| = |V_2| = n$ . If  $d_G(x) + d_G(y) \geq n + 1$  for each pair of nonadjacent vertices  $x \in V_1$  and  $y \in V_2$ , then  $G$  is hamiltonian.*

A cycle  $C$  is a *dominating cycle* if every edge is incident with some vertex of  $C$ . A cycle  $C$  is called a *vertex-dominating cycle* if every vertex is adjacent to some vertex of  $C$ . A dominating cycle is can be consider as a generalization of a hamiltonian cycle, and a vertex-dominating cycle as a generalization of a dominating cycle. Therefore there may be weaker sufficient conditions for the existence of dominating cycles or vertex-dominating cycles which correspond to that for hamiltonicity.

Bondy (1980) and Bondy and Fan (1987) gave a degree sum condition for dominating cycles and vertex-dominating cycles, respectively.

**Theorem 4** (Bondy [3]). *Let  $G$  be a 2-connected graph on  $n$  vertices. If  $d_G(x) + d_G(y) + d_G(z) \geq n + 2$  for any independent set of three vertices  $x$ ,  $y$  and  $z$ , then any longest cycle is a dominating cycle.*

**Theorem 5** (Bondy and Fan [4]). *Let  $k \geq 2$  and let  $G$  be a  $k$ -connected graph on  $n$  vertices. If  $\sum_{x \in S} d_G(x) \geq n - 2k$  for every 3-stable set  $S$  of  $G$  of order  $k + 1$ , then  $G$  has a vertex-dominating cycle.*

Like hamiltonian cycles, some sufficient conditions for the existence of dominating cycles can be relaxed if we put a further assumption on a graph. In 1989, Bauer, Veldman, Morgana and Schmeichel showed the following result for 1-tough graphs.

**Theorem 6** (Bauer *et al.* [2]). *Let  $G$  be a 1-tough graph of order  $n$ . If  $d_G(x) + d_G(y) + d_G(z) \geq n$  for any independent set of three vertices  $x$ ,  $y$  and  $z$ , then any longest cycle in  $G$  is a dominating cycle.*

In 1984, Ash and Jackson gave a minimum degree condition for a bipartite graph.

**Theorem 7** (Ash and Jackson [1]). *Let  $G$  be a 2-connected bipartite graph with partite sets  $V_1$  and  $V_2$ , where  $\max\{|V_1|, |V_2|\} = n$ . If  $\delta(G) \geq (n+3)/3$ , then there exists a longest cycle which is a dominating cycle.*

In 2003, Saito and the author showed that Theorem 5 also admits a similar relaxation under an additional assumption on toughness.

**Theorem 8** (Saito and Yamashita [9]). *Let  $k \geq 2$  and  $G$  be a  $k$ -connected graph on  $n$  vertices with  $t(G) > k/(k+1)$ . If  $\sum_{x \in S} d_G(x) \geq n - 2k - 2$  for every 4-stable set  $S$  of order  $k+1$ , then  $G$  has a vertex-dominating cycle.*

In this paper, we give a minimum degree condition for a bipartite graph to have a vertex-dominating cycle.

**Theorem 9.** *Let  $G$  be a 2-connected bipartite graph with partite sets  $V_1$  and  $V_2$ , where  $\max\{|V_1|, |V_2|\} = n$ . If  $\delta(G) \geq (n+1)/3$ , then  $G$  has a vertex-dominating cycle.*

In Theorem 9, the degree condition is sharp in the following sense. Let  $m_i, n_i$  be positive integers, where  $1 \leq i \leq 3$ . The graph  $G$  is obtained from  $K_{m_1, n_1} \cup K_{m_2, n_2} \cup K_{m_3, n_3}$ , by adding new two vertices  $x$  and  $y$ , and joining both  $x$  and  $y$  to every vertex in three partite sets of order  $n_i$ . It is easy to see that  $G$  is a 2-connected bipartite graph with partite sets  $V_1$  and  $V_2$  and  $\delta(G) \leq \max\{|V_1|, |V_2|\}/3$ , but has no vertex-dominating cycle.

## 2. PROOF OF THEOREM 9

Before proving Theorem 9, we prepare some definitions and notations, and refer to Diestel [5] for terminology and notations not defined here. For a subgraph  $H$  of  $G$  and a vertex  $x \in V(G) - V(H)$ , we also denote  $N_H(x) := N_G(x) \cap V(H)$  and  $d_H(x) := |N_H(x)|$ . For  $X \subset V(G)$ ,  $N_G(X)$  denote the set of vertices in  $G - X$  which are adjacent to some vertex in  $X$ . Furthermore, for a subgraph  $H$  of  $G$  and  $X \subset V(G) - V(H)$ , we sometimes write  $N_H(X) := N_G(X) \cap V(H)$ . If there is no fear of confusion, we often identify a subgraph  $H$  of a graph  $G$  with its vertex set  $V(H)$ . For example, we often write  $G - H$  instead of  $G - V(H)$ .

We write a cycle  $C$  with a given orientation by  $\vec{C}$ . For  $x, y \in V(C)$ , we denote by  $C[x, y]$  a path from  $x$  to  $y$  on  $\vec{C}$ . The reverse sequence of  $C[x, y]$  is denoted by  $\overleftarrow{C}[y, x]$ . We define  $C(x, y) = C[x, y] - \{x\}$ ,  $C[x, y) = C[x, y] - \{y\}$  and  $C(x, y) = C[x, y] - \{x, y\}$ . For convenience, we consider  $C[x, x) = \emptyset$ . For  $x \in V(C)$ , we denote the successor and the predecessor of  $x$  on  $\vec{C}$  by  $x^+$  and  $x^-$ , respectively. A path  $P$  connecting  $x$  and  $y$  is denoted by  $P[x, y]$ . For a subgraph  $H$  of  $G$ , a path  $P[x, y]$  is called an  $H$ -path if  $P[x, y] \cap V(H) = \{x, y\}$  and  $E(H) \cap E(P) = \emptyset$ .

Let  $S$  and  $T$  be subsets of  $V(G)$ . Then  $S$  is said to *dominate*  $T$  if every vertex in  $T$  either belongs to  $S$  or has a neighbor in  $S$ . If  $S$  dominates  $V(G)$ , then  $S$  is called a *dominating set*.

We define the following sets  $\mathcal{F}_k$  and  $\mathcal{H}_k$  of graphs for each odd integer  $k \geq 5$ . Let  $l, b_1, b_2, \dots, b_l$  be integers with  $l \geq 3$  and  $b_i \geq (k+1)/2$  ( $1 \leq i \leq l$ ). Let  $\bigcup_{i=1}^l K_{(k-3)/2, b_i}$  denote the vertex-disjoint union of  $K_{(k-3)/2, b_i}$  for all  $i \in \{1, 2, \dots, l\}$ . Then the graph  $F_{k, b_1, \dots, b_l}$  is obtained from  $\bigcup_{i=1}^l K_{(k-3)/2, b_i}$  by adding two new vertices  $x$  and  $y$ , and joining both  $x$  and  $y$  to every vertex of  $\bigcup_{i=1}^l K_{(k-3)/2, b_i}$  whose degree in  $\bigcup_{i=1}^l K_{(k-3)/2, b_i}$  is  $(k-3)/2$ . Let  $\mathcal{F}_k$  be the set of all such graphs. To define  $\mathcal{H}_k$ , let  $m, c_1, \dots, c_m$  be integers at least  $(k+1)/2$ . The graph  $H_{k, c_1, \dots, c_m}$  is obtained from  $\bigcup_{i=1}^m K_{1, c_i}$  by adding  $(k-1)/2$  new vertices  $x_1, \dots, x_{(k-1)/2}$ , and joining each  $x_i$  to every vertex of  $\bigcup_{i=1}^m K_{1, c_i}$  whose degree in  $\bigcup_{i=1}^m K_{1, c_i}$  is 1. Let  $\mathcal{H}_k$  be the set of all such graphs.

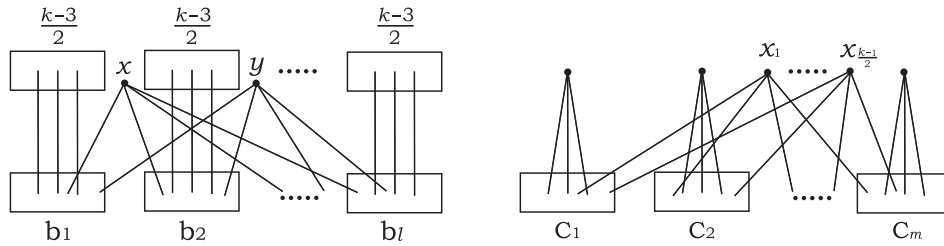


Figure 1.  $\mathcal{F}_k$  and  $\mathcal{H}_k$

To prove Theorem 9, we use the following result due to Wang.

**Theorem 10** (Wang [10]). *Let  $k \geq 2$  and let  $G$  be a 2-connected bipartite graph with partite sets  $V_1$  and  $V_2$ . If  $d_G(x) + d_G(y) \geq k + 1$  for every pair of nonadjacent vertices  $x$  and  $y$ , then  $G$  contains a cycle of length at least*

$\min\{2a, 2k\}$  where  $a = \min\{|V_1|, |V_2|\}$ , unless  $5 \leq k \leq a$ ,  $k$  is odd and  $G \in \mathcal{F}_k \cup \mathcal{H}_k$ .

**Proof of Theorem 9.** Suppose that  $G$  has no vertex-dominating cycle. Let  $C$  be a longest cycle in  $G$  such that  $\omega(G - C)$  is as small as possible, and let  $|V_1| = n_1$ ,  $|V_2| = n_2$  and  $n_1 \leq n_2$ .

**Claim 1.**  $|C| = \frac{2}{3}(2n_2 - 1)$  and  $|V_2 - C| = \frac{1}{3}(n_2 + 1)$ .

**Proof.** First suppose that  $G \in \mathcal{F}_k$ . Since  $\delta(G) = \frac{1}{2}(k + 1)$  and  $l \geq 3$ , we have

$$\frac{1}{3}(n_2 + 1) = \frac{1}{3} \left( \sum_{i=1}^l b_i + 1 \right) \geq \frac{1}{3} \left( \frac{l(k+1)}{2} + 1 \right) = \frac{l}{3} \delta(G) + \frac{1}{3} > \delta(G).$$

This contradicts the degree condition. Hence  $G \notin \mathcal{F}_k$ . Next suppose that  $G \in \mathcal{H}_k$ . Since  $\delta(G) = \frac{1}{2}(k + 1)$  and  $m \geq 3$ , we get

$$\frac{1}{3}(n_2 + 1) = \frac{1}{3} \left( \sum_{i=1}^m c_i + 1 \right) \geq \frac{1}{3} \left( \frac{m(k+1)}{2} + 1 \right) = \frac{m}{3} \delta(G) + \frac{1}{3} > \delta(G),$$

a contradiction. Therefore  $G \notin \mathcal{H}_k$ .

Since  $d_G(x) + d_G(y) \geq \frac{2}{3}(n_2 + 1) = \frac{1}{3}(2n_2 - 1) + 1$  for any  $x, y \in V(G)$ , we obtain  $|C| \geq \min\{2n_1, \frac{2}{3}(2n_2 - 1)\}$  by Theorem 10. Suppose that  $|C| \geq 2n_1$ . Then  $V_1 \subset V(C)$ . Since  $G$  is 2-connected,  $N_C(v_2) \neq \emptyset$  for any  $v_2 \in V_2 - C$ . Hence  $C$  is a vertex-dominating cycle, a contradiction. Suppose that  $|C| > \frac{2}{3}(2n_2 - 1)$ . Then  $|V_1 - C| \leq |V_2 - C| < n_2 - \frac{1}{3}(2n_2 - 1) = \frac{1}{3}(n_2 + 1)$ . Since  $\delta(G) \geq \frac{1}{3}(n_2 + 1)$ ,  $N_C(v) \neq \emptyset$  for any  $v \in V(G - C)$ , that is,  $C$  is a vertex-dominating cycle, a contradiction. Thus we obtain  $|C| = \frac{2}{3}(2n_2 - 1)$  and  $|V_2 - C| = \frac{1}{3}(n_2 + 1)$ . ■

Note that  $\frac{2}{3}(2n_2 - 1)$  and  $\frac{1}{3}(n_2 + 1)$  are integers. We shall partition  $V_i - C$  ( $i = 1, 2$ ) into three subsets as follows:

$$X_i := \{x_i \in V_i - C : N_C(x_i) \neq \emptyset, N_{G-C}(x_i) \neq \emptyset\},$$

$$Y_i := \{y_i \in V_i - C : N_{G-C}(y_i) = \emptyset\} \quad \text{and}$$

$$Z_i := \{z_i \in V_i - C : N_C(z_i) = \emptyset\}.$$

**Claim 2.** For any  $x_2 \in X_2$ ,  $|N_C(x_2)| \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|) \geq |Y_1|$ .

**Proof.** By the degree condition, for any  $x_2 \in X_2$ ,  $|N_C(x_2)| \geq \delta(G) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|)$ . Moreover, it follows from Claim 1 that  $|N_C(x_2)| \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2 + 1) - (\frac{1}{3}(n_2 + 1) - |Y_1|) \geq |Y_1|$ . ■

**Claim 3.** Let  $z_i \in Z_i$ . Then  $N_G(z_i) = V_{3-i} - C$  and  $|V_{3-i} - C| = \frac{1}{3}(n_2 + 1)$ .

**Proof.** Suppose that  $z_i \in Z_i$ . By Claim 1 and the definition of  $Z_i$ ,  $\frac{1}{3}(n_2 + 1) \geq |V_{3-i} - C| \geq d_G(z_i) \geq \frac{1}{3}(n_2 + 1)$ . This implies  $|V_{3-i} - C| = d_G(z_i) = \frac{1}{3}(n_2 + 1)$ , and so  $N_G(z_i) = V_{3-i} - C$  and  $|V_{3-i} - C| = \frac{1}{3}(n_2 + 1)$ . ■

**Claim 4.**  $Z_1$  or  $Z_2$  is non-empty. If  $Z_2$  is not empty, then  $|V_1| = |V_2|$  and  $Y_1$  is empty.

**Proof.** If  $Z_1 = \emptyset$  and  $Z_2 = \emptyset$ , then  $C$  is a vertex-dominating cycle. Hence  $Z_1 \neq \emptyset$  or  $Z_2 \neq \emptyset$ . If  $Z_2 \neq \emptyset$  then, by Claims 1 and 3,  $|V_1 - C| = |V_2 - C| = \frac{1}{3}(n_2 + 1)$ , that is,  $|V_1| = |V_2|$ . By Claim 3 and the definition of  $Y_i$ , we have  $Y_1 = \emptyset$ . ■

In view of Claim 4 and the symmetry, we may assume in the rest of the proof that  $Z_1$  is non-empty and consequently  $Y_2$  is empty.

If  $X_2 = \emptyset$ , let  $x_a, x_b \in X_1$ ; otherwise let  $x_a \in X_1 \cup X_2$  and  $x_b \in X_2$ . By Claims 3 and 4,  $X_1 \cup X_2 \cup Z_1 \cup Z_2$  is contained in a component of  $G - C$ . Hence there exists a path  $P_0[x_a, x_b]$  in  $G - C$ . We can choose  $x_a, x_b$  such that (i)  $a \in N_C(x_a)$  and  $b \in N_C(x_b)$  ( $a \neq b$ ) are as close as possible on  $C$ , and (ii)  $|P_0|$  is as large as possible, subject to (i). Let  $C_0 = x_b C[b, a] P_0[x_a, x_b]$ ,  $U_i := C(b, a) \cap V_i$  and  $U'_i := C(a, b) \cap V_i$ . We give an orientation on  $C$  such that  $|C(a, b)| \leq |C(b, a)|$ . By the choice of  $x_a$  and  $x_b$ , we have

$$(1) \quad |C(a, b)| \leq \frac{1}{2}|C| - 1 = \frac{1}{3}(2n_2 - 1) - 1 = 2 \left( \frac{1}{3}(n_2 + 1) - 1 \right).$$

**Claim 5.**  $C[b, a]$  dominates  $X_1 \cup X_2 \cup Y_1 \cup U_1$ .

**Proof.** By the choice of  $x_a$  and  $x_b$ ,  $N_G(x) \cap C(a, b) = \emptyset$  for any  $x \in X_1 \cup X_2$ . Hence  $N_G(x) \cap C[b, a] \neq \emptyset$  for any  $x \in X_1 \cup X_2$ , and so  $C[b, a]$  dominates  $X_1$  and  $X_2$ . It follows from (1) that  $|U_2| \leq \frac{1}{3}(n_2 + 1) - 1$ . Therefore  $N_G(y_1) \cap C[b, a] \neq \emptyset$  for any  $y_1 \in Y_1$ . Moreover, by the choice of  $x_a$  and  $x_b$ ,  $N_G(U_1) \cap X_2 = \emptyset$ , and so  $N_G(u_1) \cap C[b, a] \neq \emptyset$  for any  $u_1 \in U_1$ . Hence  $C[b, a]$  dominates  $Y_1$  and  $U_1$ . ■

Case 1.  $|C(a, b)|$  is even.

Then  $x_a \in X_1$  and  $x_b \in X_2$ . By Claim 3,  $\{x_a, x_b\}$  dominates  $Z_1$  and  $Z_2$ . Hence if  $C_0$  dominates  $U_2$  then by Claim 5,  $C_0$  is a vertex-dominating cycle. Thus, we may assume that  $C_0$  does not dominate  $U_2$ , that is, there exists  $u_2 \in U_2$  such that  $N_G(u_2) \subset U_1 \cup Y_1$ . By the degree condition, we have

$$(2) \quad \frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq |U_1| + |Y_1| \leq \frac{1}{2}|C(a, b)| + |Y_1|,$$

and by Claim 1,

$$(3) \quad |C| = \frac{3}{2}(2n_2 - 1) \leq 2|C(a, b)| + 4|Y_1| - 2.$$

By combining (1) and (2), we have  $|Y_1| \geq 1$ . Assume that  $|Y_1| \geq 2$ . Since  $u_2 \neq b^-$ ,  $|C(a, b)| \geq 4$ . It follows from Claim 2 and (3) that

$$\begin{aligned} & (|N_C(X_2)| + 1)(|C(a, b)| + 1) - |C| \\ & \geq (|Y_1| + 1)(|C(a, b)| + 1) - (2|C(a, b)| + 4|Y_1| - 2) \\ & = (|Y_1| - 1)(|C(a, b)| - 3) > 0, \end{aligned}$$

and so  $(|N_C(X_2)| + 1)(|C(a, b)| + 1) > |C|$ . On the other hand, by the choice of  $x_a$  and  $x_b$ ,  $C - N_C(\{x_a\} \cup X_2)$  consists of at least  $|N_C(X_2)| + 1$  paths of order at least  $|C(a, b)|$ . This implies  $|C| \geq (|N_C(X_2)| + 1)(|C(a, b)| + 1)$ . Thus we get a contradiction.

Hence  $|Y_1| = 1$ , say  $y_1 \in Y_1$ . By (1) and (2),  $|C(a, b)| = |C(b, a)| = 2(\frac{1}{3}(n_2 + 1) - 1)$ . Therefore  $N_C(X_1 \cup X_2) = \{a, b\}$ , and so  $\{a, b\}$  dominates  $X_1$  and  $X_2$ . By using the same argument as the proof of Claim 5,  $C[a, b]$  dominates  $U'_1$  and  $Y_1$ . Hence there exists  $u'_2 \in U'_2$  such that  $N_G(u'_2) \subset U'_1 \cup Y_1$ , otherwise  $x_a C[a, b] x_b P_0 x_a$  is a vertex-dominating cycle. Since  $|U_1| = |U'_1| = \frac{1}{3}(n_2 + 1) - 1$ , we see that  $y_1 \in N_G(u_2)$  and  $y_1 \in N_G(u'_2)$ .

Let  $v'_2 \in C(a, u'_2]$  and  $v_2 \in C(b, u_2]$  such that (i)  $y_1 \in N_G(v_2)$  and  $y_1 \in N_G(v'_2)$  and (ii)  $C(a, v'_2] \cup C(b, v_2]$  is inclusion-minimal, subject to (i). By the existence of the  $C$ -path  $v_2 y_1 v'_2$ , there exists a  $C$ -path  $P_1[w_2, w'_2]$  joining  $C(b, v_2]$  and  $C(a, v'_2]$ . Choose  $P_1$  such that  $C(a, w'_2] \cup C(b, w_2]$  is inclusion-minimal. By the choice of  $v'_2$  and  $P_1$ ,  $N(w) \cap (Y_1 \cup C(b, w_2)) = \emptyset$  for any  $w \in C(a, w'_2)$ . Thus, since  $|C(a, w'_2)| \leq |C(a, b)| \leq 2(\frac{1}{3}(n_2 + 1) - 1)$ ,  $N(w) \cap (C[w'_2, b] \cup C[w_2, a]) \neq \emptyset$  for any  $w \in C(a, w'_2)$ . Hence  $C[w'_2, b] \cup C[w_2, a]$  dominates  $C(a, w'_2)$ . Similarly,  $C[w'_2, b] \cup C[w_2, a]$  dominates  $C(b, w_2)$ . Moreover, since  $u_2 \in C[b, w'_2] \cup C[w_2, a]$ ,  $C[w'_2, b] \cup C[w_2, a]$  dominates  $Y_1$ . Hence

$x_a \overleftarrow{C}[a, w_2]P_1[w_2, w'_2]C(w'_2, b)P_0[x_b, x_a]$  is a vertex-dominating cycle. This completes the proof of Case 1.

*Case 2.*  $|C(a, b)|$  is odd.

Note that  $x_a \in X_i$  and  $x_b \in X_i$  for  $i = 1$  or  $i = 2$ .

*Case 2.1.*  $Z_2 = \emptyset$ .

Then  $X_2 \neq \emptyset$  and  $|X_2| = \frac{1}{3}(n_2 + 1)$ , otherwise  $C$  is a hamiltonian cycle by Claim 4. By the choice of  $x_a$  and  $x_b$ , note that  $x_a, x_b \in X_2$ . By Claim 3,  $\{x_a, x_b\}$  dominates  $Z_1$ . Hence there exists  $u_2 \in U_2$  such that  $N_G(u_2) \subset U_1 \cup Y_1$ , otherwise  $C_0$  is a vertex-dominating cycle. Since  $u_2 \neq a^+, b^-$ , we have

$$(4) \quad |C(a, b)| \geq 5.$$

Since  $a^+, b^- \in V_2$  and  $|C(a, b)|$  is odd,

$$(5) \quad \frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq |U_1| + |Y_1| \leq \frac{1}{2}(|C(a, b)| - 1) + |Y_1|,$$

and by Claim 1,

$$(6) \quad |C| = \frac{2}{3}(2n_2 - 1) \leq 2|C(a, b)| + 4|Y_1| - 4.$$

By (1) and (5), we have  $|Y_1| \geq 2$ . Since  $C - N_C(X_2)$  has at least  $|N_C(X_2)|$  paths of order at least  $|C(a, b)|$ , we have  $|C| \geq |N_C(X_2)|(|C(a, b)| + 1)$ . Assume that  $|Y_1| \geq 4$ . It follows from Claim 2, (4) and (6) that

$$\begin{aligned} & |N_C(X_2)|(|C(a, b)| + 1) - |C| \\ & \geq |Y_1|(|C(a, b)| + 1) - (2|C(a, b)| + 4|Y_1| - 4) \\ & = (|Y_1| - 2)(|C(a, b)| - 3) - 2 > 0, \end{aligned}$$

a contradiction. Therefore  $|Y_1| = 2$  or  $|Y_1| = 3$ .

**Claim 6.** (i)  $X_1 = \emptyset$ ,

(ii)  $|Z_1| = \frac{1}{3}(n_2 + 1) - |Y_1|$  and

(iii)  $N_C(X_2) = N_C(x_2)$  for any  $x_2 \in X_2$ .



**Proof.** First, suppose that  $X_1 \neq \emptyset$ , say  $x_1 \in X_1$ . Since  $C - N_C(\{x_1\} \cup X_2)$  has at least  $|N_C(X_2)| + 1$  paths of order at least  $|C(a, b)|$ ,  $|C| \geq |N_C(\{x_1\} \cup X_2)|(|C(a, b)| + 1)$ . By Claim 2, (4) and (6),

$$\begin{aligned} & |N_C(\{x_1\} \cup X_2)|(|C(a, b)| + 1) - |C| \\ & \geq (|Y_1| + 1)(|C(a, b)| + 1) - (2|C(a, b)| + 4|Y_1| - 4) \\ & = (|Y_1| - 1)(|C(a, b)| - 3) + 2 > 0, \end{aligned}$$

a contradiction. Next suppose that  $|Z_1| < \frac{1}{3}(n_2 + 1) - |Y_1|$  or  $N_C(X_2) > N_C(x_2)$  for some  $x_2 \in X_2$ . Then, by Claim 2,  $|N_C(X_2)| \geq |Y_1| + 1$ . By a similar argument as above, we obtain a contradiction. ■

Since  $|Y_1| \geq 2$ , we have  $|X_2| \geq 2$  and by Claim 6 (iii), we can choose  $x_a, x_b$  with  $x_a \neq x_b$ . By Claim 3 and Claims 6 (i) and (ii), we obtain  $|P_0| = |X_2| + |Z_1| - |Y_1| + 1 = \frac{2}{3}(n_2 + 1) - 2|Y_1| + 1$ . On the other hand, by (1) and (5),  $|C(a, b)| = \frac{2}{3}(n_2 + 1) - 2|Y_1| + 1$ . Hence  $C_0$  and  $C$  have the same length. Since  $C(a, b) \cup Y_1$  is contained in a component of  $G - C_0$  and  $|X_2 - P_0| = |Y_1| - 1$ , we have  $\omega(G - C_0) = |Y_1|$ . Note that  $\omega(G - C) = |Y_1| + 1$ . Therefore  $\omega(G - C) > \omega(G - C_0)$ . This contradicts the choice of  $C$ .

*Case 2.2.*  $Z_2 \neq \emptyset$ .

Then  $Y_1 = \emptyset$  by Claim 3. Since  $|U_1| \leq \frac{1}{3}(n_2 + 1) - 1$ ,  $N(u_2) \cap C[b, a] \neq \emptyset$  for any  $u_2 \in U_2$ , that is,  $C[b, a]$  dominates  $U_2$ . Suppose that  $x_a \neq x_b$ . By Claim 3,  $P_0[x_a, x_b]$  dominates  $Z_1$  and  $Z_2$ , and so  $C_0$  is a vertex-dominating cycle. Therefore  $x_a = x_b$ . By the 2-connectivity of  $G$  and the choice of  $x_a$  and  $x_b$ , there exists  $x_d \in X_1 \cup X_2$  such that  $x_d \neq x_a$  and  $N_C(x_d) \cap C(b, a) \neq \emptyset$ , say  $d \in N_C(x_d) \cap C(b, a)$ . Choose  $x_d$  such that  $\min\{|C(b, d)|, |C(d, a)|\}$  as small as possible. Without loss of generality, we may assume that  $|C(b, d)| \geq |C(d, a)|$ . By the choice of  $x_d$ ,  $C[a, d]$  dominates  $X_1$  and  $X_2$ . By Claim 3, there exists a path  $P_3[x_a, x_d]$  in  $G - C$ , which dominates  $Z_1$  and  $Z_2$ . Since  $|C[a, b]| \geq 3$ , we have  $|C(d, a)| \leq \frac{1}{2}(|C| - 2) - 1 \leq 2(\frac{1}{3}(n_2 + 1) - 1) - 1$ . Since  $|C(d, a) \cap V_1|, |C(d, a) \cap V_2| \leq \frac{1}{3}(n_2 + 1) - 1$  and  $Y_1 = Y_2 = \emptyset$ , we can see that  $C[a, d]$  dominates  $C(d, a)$ . Hence  $x_a C[a, d] P_3[x_d, x_a]$  is a vertex-dominating cycle. This completes the proof of Case 2.2 and the proof of Theorem 9. ■

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Received 28 April 2006

Revised 23 February 2007

Accepted 23 February 2007