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VERTEX-DOMINATING CYCLES IN 2-CONNECTED BIPARTITE GRAPHS

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Abstract

A cycle C is a vertex-dominating cycle if every vertex is adjacent to some vertex of C. Bondy and Fan [4] showed that if G is a 2-connected graph with $\delta(G) \geq \frac{1}{3}(|V(G)| - 4)$, then G has a vertex-dominating cycle. In this paper, we prove that if G is a 2-connected bipartite graph with partite sets V_1 and V_2 such that $\delta(G) \geq \frac{1}{3}(\max\{|V_1|, |V_2|\} + 1)$, then G has a vertex-dominating cycle.

Keywords: vertex-dominating cycle, dominating cycle, bipartite graph.

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1. Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. We denote the degree of a vertex x in a graph G by $d_G(x)$. Let $\delta(G)$ be the minimum degree of a graph G. We denote the number of components of G by $\omega(G)$. A connected graph G is defined to be *t-tough* if $|S| \geq t \cdot \omega(G - S)$ for every cutset S of $V(G)$. The *toughness* of G, denoted by $t(G)$, is the maximum value of t for which G is t-tough (taking $t(K_n) = \infty$ for all $n \geq 1$). A set S of vertices in a graph G is said to be *d-stable* if the distance of each pair of distinct vertices in S is at least d.

In 1960, Ore introduced a degree sum condition for hamiltonian cycles.

Theorem 1 (Ore [8])**.** *Let G be a graph on $n \geq 3$ vertices. If $d_G(x) + d_G(y) \geq n$ for any nonadjacent vertices x and y, then G is hamiltonian.*

It is observed that weaker conditions guarantee the existence of hamiltonian cycles by putting a further assumption on graphs. For example, Jung (1972) and Moon and Moser (1963) showed that weaker degree sum conditions guarantee hamiltonian cycles in 1-tough graphs and in bipartite graphs, respectively.

Theorem 2 (Jung [6]). *Let G be a 1-tough graph of order $n \geq 11$. If $d_G(x) + d_G(y) \geq n - 4$ for any nonadjacent vertices x and y, then G is hamiltonian.*

Theorem 3 (Moon and Moser [7]). *Let G be a bipartite graph with partite sets V_1 and V_2, where $|V_1| = |V_2| = n$. If $d_G(x) + d_G(y) \geq n + 1$ for each pair of nonadjacent vertices $x \in V_1$ and $y \in V_2$, then G is hamiltonian.*

A cycle C is a *dominating cycle* if every edge is incident with some vertex of C. A cycle C is called a *vertex-dominating cycle* if every vertex is adjacent to some vertex of C. A dominating cycle is can be consider as a generalization of a hamiltonian cycle, and a vertex-dominating cycle as a generalization of a dominating cycle. Therefore there may be weaker sufficient conditions for the existence of dominating cycles or vertex-dominating cycles which correspond to that for hamiltonicity.

Bondy (1980) and Bondy and Fan (1987) gave a degree sum condition for dominating cycles and vertex-dominating cycles, respectively.

Theorem 4 (Bondy [3]). *Let G be a 2-connected graph on n vertices. If $d_G(x) + d_G(y) + d_G(z) \geq n + 2$ for any independent set of three vertices x, y and z, then any longest cycle is a dominating cycle.*

Theorem 5 (Bondy and Fan [4]). *Let $k \geq 2$ and let G be a k-connected graph on n vertices. If $\sum_{x \in S} d_G(x) \geq n - 2k$ for every 3-stable set S of G of order $k + 1$, then G has a vertex-dominating cycle.*

Like hamiltonian cycles, some sufficient conditions for the existence of dominating cycles can be relaxed if we put a further assumption on a graph. In 1989, Bauer, Veldman, Morgana and Schmeichel showed the following result for 1-tough graphs.

Theorem 6 (Bauer *et al.* [2]). *Let G be a 1-tough graph of order n. If $d_G(x) + d_G(y) + d_G(z) \geq n$ for any independent set of three vertices x, y and z, then any longest cycle in G is a dominating cycle.*

In 1984, Ash and Jackson gave a minimum degree condition for a bipartite graph.

Theorem 7 (Ash and Jackson [1]). *Let G be a 2-connected bipartite graph with partite sets V_1 and V_2, where $\max\{|V_1|, |V_2|\} = n$. If $\delta(G) \geq (n+3)/3$, then there exists a longest cycle which is a dominating cycle.*

In 2003, Saito and the author showed that Theorem 5 also admits a similar relaxation under an additional assumption on toughness.

Theorem 8 (Saito and Yamashita [9]). *Let $k \geq 2$ and G be a k-connected graph on n vertices with $t(G) > k/(k+1)$. If $\sum_{x \in S} d_G(x) \geq n - 2k - 2$ for every 4-stable set S of order $k+1$, then G has a vertex-dominating cycle.*

In this paper, we give a minimum degree condition for a bipartite graph to have a vertex-dominating cycle.

Theorem 9. *Let G be a 2-connected bipartite graph with partite sets V_1 and V_2, where $\max\{|V_1|, |V_2|\} = n$. If $\delta(G) \geq (n+1)/3$, then G has a vertex-dominating cycle.*

In Theorem 9, the degree condition is sharp in the following sense. Let m_i, n_i be positive integers, where $1 \leq i \leq 3$. The graph G is obtained from $K_{m_1,n_1} \cup K_{m_2,n_2} \cup K_{m_3,n_3}$, by adding new two vertices x and y, and joining both x and y to every vertex in three partite sets of order n_i. It is easy to see that G is a 2-connected bipartite graph with partite sets V_1 and V_2 and $\delta(G) \leq \max\{|V_1|, |V_2|\}/3$, but has no vertex-dominating cycle.

2. Proof of Theorem 9

Before proving Theorem 9, we prepare some definitions and notations, and refer to Diestel [5] for terminology and notations not defined here. For a subgraph H of G and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. For $X \subset V(G)$, $N_G(X)$ denote the set of vertices in $G - X$ which are adjacent to some vertex in X. Furthermore, for a subgraph H of G and $X \subset V(G) - V(H)$, we sometimes write $N_H(X) := N_G(X) \cap V(H)$. If there is no fear of confusion, we often identify a subgraph H of a graph G with its vertex set $V(H)$. For example, we often write $G - H$ instead of $G - V(H)$.

We write a cycle C with a given orientation by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote by $C[x, y]$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $C[x, y]$ is denoted by $\overleftarrow{C}[y, x]$. We define $C(x, y] = C[x, y] - \{x\}$, $C[x, y) = C[x, y] - \{y\}$ and $C(x, y) = C[x, y] - \{x, y\}$. For convenience, we consider $C[x, x) = \emptyset$. For $x \in V(C)$, we denote the successor and the predecessor of x on $\overrightarrow{C}$ by x^+ and x^-, respectively. A path P connecting x and y is denoted by $P[x, y]$. For a subgraph H of G, a path $P[x, y]$ is called an H-path if $P[x, y] \cap V(H) = \{x, y\}$ and $E(H) \cap E(P) = \emptyset$.

Let S and T be subsets of $V(G)$. Then S is said to *dominate* T if every vertex in T either belongs to S or has a neighbor in S. If S dominates $V(G)$, then S is called a *dominating set*.

We define the following sets $\mathcal{F}_k$ and $\mathcal{H}_k$ of graphs for each odd integer $k \geq 5$. Let $l, b_1, b_2, \ldots, b_l$ be integers with $l \geq 3$ and $b_i \geq (k+1)/2$ $(1 \leq i \leq l)$. Let $\bigcup_{i=1}^{l} K_{(k-3)/2, b_i}$ denote the vertex-disjoint union of $K_{(k-3)/2, b_i}$ for all $i \in \{1, 2, \ldots, l\}$. Then the graph $F_{k, b_1, \ldots, b_l}$ is obtained from $\bigcup_{i=1}^{l} K_{(k-3)/2, b_i}$ by adding two new vertices x and y, and joining both x and y to every vertex of $\bigcup_{i=1}^{l} K_{(k-3)/2, b_i}$ whose degree in $\bigcup_{i=1}^{l} K_{(k-3)/2, b_i}$ is $(k-3)/2$. Let $\mathcal{F}_k$ be the set of all such graphs. To define $\mathcal{H}_k$, let $m, c_1, \ldots, c_m$ be integers at least $(k+1)/2$. The graph $H_{k, c_1, \ldots, c_m}$ is obtained from $\bigcup_{i=1}^{m} K_{1, c_i}$ by adding $(k-1)/2$ new vertices $x_1, \ldots, x_{(k-1)/2}$, and joining each x_i to every vertex of $\bigcup_{i=1}^{m} K_{1, c_i}$ whose degree in $\bigcup_{i=1}^{m} K_{1, c_i}$ is 1. Let $\mathcal{H}_k$ be the set of all such graphs.

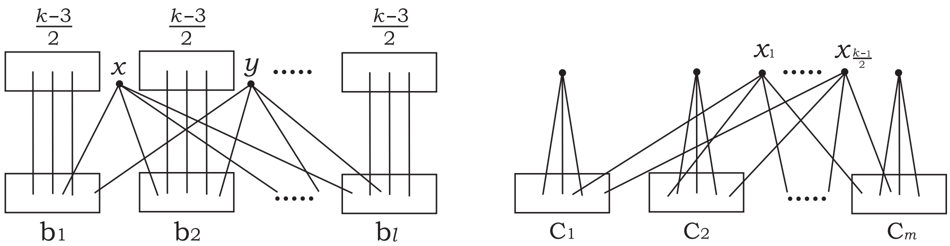

Figure 1. $\mathcal{F}_k$ and $\mathcal{H}_k$

To prove Theorem 9, we use the following result due to Wang.

Theorem 10 (Wang [10]). *Let $k \geq 2$ and let G be a 2-connected bipartite graph with partite sets V_1 and V_2. If $d_G(x) + d_G(y) \geq k + 1$ for every pair of nonadjacent vertices x and y, then G contains a cycle of length at least*

$\min\{2a, 2k\}$ *where* $a = \min\{|V_1|, |V_2|\}$, *unless* $5 \leq k \leq a$, *k is odd and* $G \in \mathcal{F}_k \cup \mathcal{H}_k$.

***Proof of Theorem* 9.** Suppose that G has no vertex-dominating cycle. Let C be a longest cycle in G such that $\omega(G - C)$ is as small as possible, and let $|V_1| = n_1$, $|V_2| = n_2$ and $n_1 \leq n_2$.

Claim 1. $|C| = \frac{2}{3}(2n_2 - 1)$ and $|V_2 - C| = \frac{1}{3}(n_2 + 1)$.

Proof. First suppose that $G \in \mathcal{F}_k$. Since $\delta(G) = \frac{1}{2}(k + 1)$ and $l \geq 3$, we have

$$\frac{1}{3}(n_2 + 1) = \frac{1}{3}\left(\sum_{i=1}^{l} b_i + 1\right) \geq \frac{1}{3}\left(\frac{l(k+1)}{2} + 1\right) = \frac{l}{3}\delta(G) + \frac{1}{3} > \delta(G).$$

This contradicts the degree condition. Hence $G \notin \mathcal{F}_k$. Next suppose that $G \in \mathcal{H}_k$. Since $\delta(G) = \frac{1}{2}(k + 1)$ and $m \geq 3$, we get

$$\frac{1}{3}(n_2 + 1) = \frac{1}{3}\left(\sum_{i=1}^{m} c_i + 1\right) \geq \frac{1}{3}\left(\frac{m(k+1)}{2} + 1\right) = \frac{m}{3}\delta(G) + \frac{1}{3} > \delta(G),$$

a contradiction. Therefore $G \notin \mathcal{H}_k$.

Since $d_G(x) + d_G(y) \geq \frac{2}{3}(n_2 + 1) = \frac{1}{3}(2n_2 - 1) + 1$ for any $x, y \in V(G)$, we obtain $|C| \geq \min\left\{2n_1, \frac{2}{3}(2n_2 - 1)\right\}$ by Theorem 10. Suppose that $|C| \geq 2n_1$. Then $V_1 \subset V(C)$. Since G is 2-connected, $N_C(v_2) \neq \emptyset$ for any $v_2 \in V_2 - C$. Hence C is a vertex-dominating cycle, a contradiction. Suppose that $|C| > \frac{2}{3}(2n_2 - 1)$. Then $|V_1 - C| \leq |V_2 - C| < n_2 - \frac{1}{3}(2n_2 - 1) = \frac{1}{3}(n_2 + 1)$. Since $\delta(G) \geq \frac{1}{3}(n_2 + 1)$, $N_C(v) \neq \emptyset$ for any $v \in V(G - C)$, that is, C is a vertex-dominating cycle, a contradiction. Thus we obtain $|C| = \frac{2}{3}(2n_2 - 1)$ and $|V_2 - C| = \frac{1}{3}(n_2 + 1)$. ■

Note that $\frac{2}{3}(2n_2 - 1)$ and $\frac{1}{3}(n_2 + 1)$ are integers. We shall partition $V_i - C$ $(i = 1, 2)$ into three subsets as follows:

$$X_i := \{x_i \in V_i - C \colon N_C(x_i) \neq \emptyset,\ N_{G-C}(x_i) \neq \emptyset\},$$

$$Y_i := \{y_i \in V_i - C \colon N_{G-C}(y_i) = \emptyset\} \ \text{ and}$$

$$Z_i := \{z_i \in V_i - C \colon N_C(z_i) = \emptyset\}.$$

Claim 2. For any $x_2 \in X_2$, $|N_C(x_2)| \geq \frac{1}{3}(n_2 + 1) - (|X_1| + |Z_1|) \geq |Y_1|$.

Proof. By the degree condition, for any $x_2 \in X_2$, $|N_C(x_2)| \geq \delta(G) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2+1) - (|X_1| + |Z_1|)$. Moreover, it follows from Claim 1 that $|N_C(x_2)| \geq \frac{1}{3}(n_2+1) - (|X_1| + |Z_1|) \geq \frac{1}{3}(n_2+1) - \left(\frac{1}{3}(n_2+1) - |Y_1|\right) \geq |Y_1|$. ■

Claim 3. Let $z_i \in Z_i$. Then $N_G(z_i) = V_{3-i} - C$ and $|V_{3-i} - C| = \frac{1}{3}(n_2+1)$.

Proof. Suppose that $z_i \in Z_i$. By Claim 1 and the definition of Z_i, $\frac{1}{3}(n_2+1) \geq |V_{3-i} - C| \geq d_G(z_i) \geq \frac{1}{3}(n_2+1)$. This implies $|V_{3-i} - C| = d_G(z_i) = \frac{1}{3}(n_2+1)$, and so $N_G(z_i) = V_{3-i} - C$ and $|V_{3-i} - C| = \frac{1}{3}(n_2+1)$. ■

Claim 4. Z_1 or Z_2 is non-empty. If Z_2 is not empty, then $|V_1| = |V_2|$ and Y_1 is empty.

Proof. If $Z_1 = \emptyset$ and $Z_2 = \emptyset$, then C is a vertex-dominating cycle. Hence $Z_1 \neq \emptyset$ or $Z_2 \neq \emptyset$. If $Z_2 \neq \emptyset$ then, by Claims 1 and 3, $|V_1 - C| = |V_2 - C| = \frac{1}{3}(n_2+1)$, that is, $|V_1| = |V_2|$. By Claim 3 and the definition of Y_i, we have $Y_1 = \emptyset$. ■

In view of Claim 4 and the symmetry, we may assume in the rest of the proof that Z_1 is non-empty and consequently Y_2 is empty.

If $X_2 = \emptyset$, let $x_a, x_b \in X_1$; otherwise let $x_a \in X_1 \cup X_2$ and $x_b \in X_2$. By Claims 3 and 4, $X_1 \cup X_2 \cup Z_1 \cup Z_2$ is contained in a component of $G - C$. Hence there exists a path $P_0[x_a, x_b]$ in $G - C$. We can choose x_a, x_b such that (i) $a \in N_C(x_a)$ and $b \in N_C(x_b)$ $(a \neq b)$ are as close as possible on C, and (ii) $|P_0|$ is as large as possible, subject to (i). Let $C_0 = x_b C[b,a] P_0[x_a, x_b]$, $U_i := C(b,a) \cap V_i$ and $U'_i := C(a,b) \cap V_i$. We give an orientation on C such that $|C(a,b)| \leq |C(b,a)|$. By the choice of x_a and x_b, we have

$$(1) \qquad |C(a,b)| \leq \frac{1}{2}|C| - 1 = \frac{1}{3}(2n_2 - 1) - 1 = 2\left(\frac{1}{3}(n_2+1) - 1\right).$$

Claim 5. $C[b,a]$ dominates $X_1 \cup X_2 \cup Y_1 \cup U_1$.

Proof. By the choice of x_a and x_b, $N_G(x) \cap C(a,b) = \emptyset$ for any $x \in X_1 \cup X_2$. Hence $N_G(x) \cap C[b,a] \neq \emptyset$ for any $x \in X_1 \cup X_2$, and so $C[b,a]$ dominates X_1 and X_2. It follows from (1) that $|U_2| \leq \frac{1}{3}(n_2+1) - 1$. Therefore $N_G(y_1) \cap C[b,a] \neq \emptyset$ for any $y_1 \in Y_1$. Moreover, by the choice of x_a and x_b, $N_G(U_1) \cap X_2 = \emptyset$, and so $N_G(u_1) \cap C[b,a] \neq \emptyset$ for any $u_1 \in U_1$. Hence $C[b,a]$ dominates Y_1 and U_1. ■

Case 1. $|C(a,b)|$ is even.
Then $x_a \in X_1$ and $x_b \in X_2$. By Claim 3, $\{x_a, x_b\}$ dominates Z_1 and Z_2. Hence if C_0 dominates U_2 then by Claim 5, C_0 is a vertex-dominating cycle. Thus, we may assume that C_0 does not dominate U_2, that is, there exists $u_2 \in U_2$ such that $N_G(u_2) \subset U_1 \cup Y_1$. By the degree condition, we have

$$(2) \qquad \frac{1}{3}(n_2+1) \le d_G(u_2) \le |U_1| + |Y_1| \le \frac{1}{2}|C(a,b)| + |Y_1|,$$

and by Claim 1,

$$(3) \qquad |C| = \frac{3}{2}(2n_2 - 1) \le 2|C(a,b)| + 4|Y_1| - 2.$$

By combining (1) and (2), we have $|Y_1| \ge 1$. Assume that $|Y_1| \ge 2$. Since $u_2 \ne b^-$, $|C(a,b)| \ge 4$. It follows from Claim 2 and (3) that

$$\begin{aligned}
&(|N_C(X_2)|+1)(|C(a,b)|+1) - |C| \\
&\ge (|Y_1|+1)(|C(a,b)|+1) - (2|C(a,b)| + 4|Y_1| - 2) \\
&= (|Y_1|-1)(|C(a,b)|-3) > 0,
\end{aligned}$$

and so $(|N_C(X_2)|+1)(|C(a,b)|+1) > |C|$. On the other hand, by the choice of x_a and x_b, $C - N_C(\{x_a\} \cup X_2)$ consists of at least $|N_C(X_2)|+1$ paths of order at least $|C(a,b)|$. This implies $|C| \ge (|N_C(X_2)|+1)(|C(a,b)|+1)$. Thus we get a contradiction.

Hence $|Y_1| = 1$, say $y_1 \in Y_1$. By (1) and (2), $|C(a,b)| = |C(b,a)| = 2\left(\frac{1}{3}(n_2+1) - 1\right)$. Therefore $N_C(X_1 \cup X_2) = \{a,b\}$, and so $\{a,b\}$ dominates X_1 and X_2. By using the same argument as the proof of Claim 5, $C[a,b]$ dominates U_1' and Y_1. Hence there exists $u_2' \in U_2'$ such that $N_G(u_2') \subset U_1' \cup Y_1$, otherwise $x_a C[a,b] x_b P_0 x_a$ is a vertex-dominating cycle. Since $|U_1| = |U_1'| = \frac{1}{3}(n_2+1) - 1$, we see that $y_1 \in N_G(u_2)$ and $y_1 \in N_G(u_2')$.

Let $v_2' \in C(a, u_2']$ and $v_2 \in C(b, u_2]$ such that (i) $y_1 \in N_G(v_2)$ and $y_1 \in N_G(v_2')$ and (ii) $C(a,v_2'] \cup C(b,v_2]$ is inclusion-minimal, subject to (i). By the existence of the C-path $v_2 y_1 v_2'$, there exists a C-path $P_1[w_2, w_2']$ joining $C(b,v_2]$ and $C(a,v_2']$. Choose P_1 such that $C(a,w_2'] \cup C(b,w_2]$ is inclusion-minimal. By the choice of v_2' and P_1, $N(w) \cap (Y_1 \cup C(b,w_2)) = \emptyset$ for any $w \in C(a,w_2')$. Thus, since $|C(a,w_2')| \le |C(a,b)| \le 2\left(\frac{1}{3}(n_2+1) - 1\right)$, $N(w) \cap (C[w_2',b] \cup C[w_2,a]) \ne \emptyset$ for any $w \in C(a,w_2')$. Hence $C[w_2',b] \cup C[w_2,a]$ dominates $C(a,w_2')$. Similarly, $C[w_2',b] \cup C[w_2,a]$ dominates $C(b,w_2)$. Moreover, since $u_2 \in C[b,w_2'] \cup C[w_2,a]$, $C[w_2',b] \cup C[w_2,a]$ dominates Y_1. Hence

$x_a \overleftarrow{C}[a, w_2)P_1[w_2, w_2']C(w_2', b]P_0[x_b, x_a]$ is a vertex-dominating cycle. This completes the proof of Case 1.

Case 2. $|C(a,b)|$ is odd.
Note that $x_a \in X_i$ and $x_b \in X_i$ for $i = 1$ or $i = 2$.

Case 2.1. $Z_2 = \emptyset$.
Then $X_2 \neq \emptyset$ and $|X_2| = \frac{1}{3}(n_2 + 1)$, otherwise C is a hamiltonian cycle by Claim 4. By the choice of x_a and x_b, note that $x_a, x_b \in X_2$. By Claim 3, $\{x_a, x_b\}$ dominates Z_1. Hence there exists $u_2 \in U_2$ such that $N_G(u_2) \subset U_1 \cup Y_1$, otherwise C_0 is a vertex-dominating cycle. Since $u_2 \neq a^+, b^-$, we have

$$|C(a,b)| \geq 5. \tag{4}$$

Since $a^+, b^- \in V_2$ and $|C(a,b)|$ is odd,

$$\frac{1}{3}(n_2 + 1) \leq d_G(u_2) \leq |U_1| + |Y_1| \leq \frac{1}{2}(|C(a,b)| - 1) + |Y_1|, \tag{5}$$

and by Claim 1,

$$|C| = \frac{2}{3}(2n_2 - 1) \leq 2|C(a,b)| + 4|Y_1| - 4. \tag{6}$$

By (1) and (5), we have $|Y_1| \geq 2$. Since $C - N_C(X_2)$ has at least $|N_C(X_2)|$ paths of order at least $|C(a,b)|$, we have $|C| \geq |N_C(X_2)|(|C(a,b)| + 1)$. Assume that $|Y_1| \geq 4$. It follows from Claim 2, (4) and (6) that

$$\begin{aligned}
&|N_C(X_2)|(|C(a,b)| + 1) - |C| \\
&\geq |Y_1|(|C(a,b)| + 1) - (2|C(a,b)| + 4|Y_1| - 4) \\
&= (|Y_1| - 2)(|C(a,b)| - 3) - 2 > 0,
\end{aligned}$$

a contradiction. Therefore $|Y_1| = 2$ or $|Y_1| = 3$.

Claim 6. (i) $X_1 = \emptyset$,
(ii) $|Z_1| = \frac{1}{3}(n_2 + 1) - |Y_1|$ and
(iii) $N_C(X_2) = N_C(x_2)$ for any $x_2 \in X_2$.

Proof. First, suppose that $X_1 \neq \emptyset$, say $x_1 \in X_1$. Since $C - N_C(\{x_1\} \cup X_2)$ has at least $|N_C(X_2)| + 1$ paths of order at least $|C(a,b)|$, $|C| \geq |N_C(\{x_1\} \cup X_2)|(|C(a,b)| + 1)$. By Claim 2, (4) and (6),

$$\begin{aligned}&|N_C(\{x_1\} \cup X_2)|(|C(a,b)| + 1) - |C| \\ &\geq (|Y_1| + 1)(|C(a,b)| + 1) - (2|C(a,b)| + 4|Y_1| - 4) \\ &= (|Y_1| - 1)(|C(a,b)| - 3) + 2 > 0,\end{aligned}$$

a contradiction. Next suppose that $|Z_1| < \frac{1}{3}(n_2 + 1) - |Y_1|$ or $N_C(X_2) > N_C(x_2)$ for some $x_2 \in X_2$. Then, by Claim 2, $|N_C(X_2)| \geq |Y_1| + 1$. By a similar argument as above, we obtain a contradiction. ■

Since $|Y_1| \geq 2$, we have $|X_2| \geq 2$ and by Claim 6 (iii), we can choose x_a, x_b with $x_a \neq x_b$. By Claim 3 and Claims 6 (i) and (ii), we obtain $|P_0| = |X_2| + |Z_1| - |Y_1| + 1 = \frac{2}{3}(n_2 + 1) - 2|Y_1| + 1$. On the other hand, by (1) and (5), $|C(a,b)| = \frac{2}{3}(n_2 + 1) - 2|Y_1| + 1$. Hence C_0 and C have the same length. Since $C(a,b) \cup Y_1$ is contained in a component of $G - C_0$ and $|X_2 - P_0| = |Y_1| - 1$, we have $\omega(G - C_0) = |Y_1|$. Note that $\omega(G - C) = |Y_1| + 1$. Therefore $\omega(G - C) > \omega(G - C_0)$. This contradicts the choice of C.

Case 2.2. $Z_2 \neq \emptyset$.
Then $Y_1 = \emptyset$ by Claim 3. Since $|U_1| \leq \frac{1}{3}(n_2 + 1) - 1$, $N(u_2) \cap C[b,a] \neq \emptyset$ for any $u_2 \in U_2$, that is, $C[b,a]$ dominates U_2. Suppose that $x_a \neq x_b$. By Claim 3, $P_0[x_a, x_b]$ dominates Z_1 and Z_2, and so C_0 is a vertex-dominating cycle. Therefore $x_a = x_b$. By the 2-connectivity of G and the choice of x_a and x_b, there exists $x_d \in X_1 \cup X_2$ such that $x_d \neq x_a$ and $N_C(x_d) \cap C(b,a) \neq \emptyset$, say $d \in N_C(x_d) \cap C(b,a)$. Choose x_d such that $\min\{|C(b,d)|, |C(d,a)|\}$ as small as possible. Without loss of generality, we may assume that $|C(b,d)| \geq |C(d,a)|$. By the choice of x_d, $C[a,d]$ dominates X_1 and X_2. By Claim 3, there exists a path $P_3[x_a, x_d]$ in $G - C$, which dominates Z_1 and Z_2. Since $|C[a,b]| \geq 3$, we have $|C(d,a)| \leq \frac{1}{2}(|C| - 2) - 1 \leq 2\left(\frac{1}{3}(n_2 + 1) - 1\right) - 1$. Since $|C(d,a) \cap V_1|, |C(d,a) \cap V_2| \leq \frac{1}{3}(n_2 + 1) - 1$ and $Y_1 = Y_2 = \emptyset$, we can see that $C[a,d]$ dominates $C(d,a)$. Hence $x_a C[a,d] P_3[x_d, x_a]$ is a vertex-dominating cycle. This completes the proof of Case 2.2 and the proof of Theorem 9. ■

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