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INFINITE FAMILIES OF TIGHT REGULAR TOURNAMENTS

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Abstract

In this paper, we construct infinite families of tight regular tournaments. In particular, we prove that two classes of regular tournaments, tame molds and ample tournaments are tight. We exhibit an infinite family of 3-dichromatic tight tournaments. With this family we positively answer to one case of a conjecture posed by V. Neumann-Lara. Finally, we show that any tournament with a tight mold is also tight.

Keywords: regular tournament, acyclic disconnection, tight tournament, mold, tame mold, ample tournament, domination digraph.

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1. Introduction

A tournament T is said to be *tight* if for every coloring of its vertex set $V(T)$ with exactly 3 colors, there exists a *heterochromatic directed triangle*, that is, a directed 3-cycle with vertices of different colors. This definition is related to the $\overrightarrow{C}_3$*-free disconnection* of a digraph D introduced by V. Neumann-Lara in [14] and the *heterochromatic number* of a 3*-graph* defined in [1]. The $\overrightarrow{C}_3$-free disconnection $\overrightarrow{\omega}_3(D)$ of a digraph D is the maximum number of colors r such that for every coloring of the vertex set $V(D)$ with exactly r colors, there is no heterochromatic directed triangle. Therefore, a tournament T is tight if and only if $\overrightarrow{\omega}_3(T) = 2$.

We recall that a 3-graph H is a 3-uniform hypergraph and the hyperedges of H are called 3*-edges*. The heterochromatic number $hc(H)$ of a 3-graph H is the minimum number of colors r such that for every coloring of its vertex set $V(H)$ with exactly r colors, there exists a heterochromatic 3-edge. If $hc(H) = 3$, the 3-graph H is defined to be *tight.* From any tournament T, we can construct a 3-graph $H_T = (V, E)$, where $V(H_T) = V(T)$ and $E(H_T)$ is the set of triples corresponding to the vertices of any directed triangle of T. One can easily check that H_T is tight if and only if $\overrightarrow{\omega}_3(T) = 2$, so T is tight if and only if H_T is tight (see [14]).

Infinite families of tight circulant tournaments are given in [10, 11, 14]. In this paper, we are interested in constructing infinite families of tight regular tournaments. In particular, we prove that two classes of regular tournaments (to be defined in the next section), tame molds and ample tournaments are tight. Moreover, any tournament with a tight mold is also tight.

Finally, we exhibit an infinite family of 3-dichromatic tight tournaments. A digraph D is said to be r-dichromatic if its *dichromatic number* $dc(D) = r$. The *dichromatic number* $dc(D)$ of a digraph D is the least number of colors needed to color the vertices of D such a way that each chromatic class is acyclic (for more details, see [13]). This family positively answers to the following conjecture for the special case when $s = 2$ and $r = 3$:

Conjecture 1 ([14], Conjecture 5.8)**.** For every couple of integers (r, s) such that $r \geq s \geq 2$ there is an infinite set of regular tournaments T such that $dc(T) = r$ and $\overrightarrow{\omega}_3(T) = s$.

For the usual terminology on graphs, digraphs and tournaments used in the paper, see [2, 3, 12, 16].

2. Preliminaries

Let $D = (V, A)$ be a digraph. The *order* of D is $|V(D)|$. For an arc of D, we use $uv \in A(D)$. A digraph D is said to be *vertex-transitive* if its automorphism group acts transitively on $V(D)$. If $\varnothing \neq V' \subseteq V(D)$, we denote by $D \setminus V'$ the induced subdigraph $D[V \setminus V']$. Let D' be a nonempty subdigraph of D. We define $D \setminus D' = D \setminus V(D')$. The *in-neighborhood* and the *ex-neighborhood* of a vertex u in D are denoted by $N^-(u; D)$ and $N^+(u; D)$ respectively. The *in-degree* and *ex-degree* of a vertex u in D are $d^-(u, D) = |N^-(u; D)|$ and $d^+(u, D) = |N^+(u; D)|$. Let v and S be a vertex and a nonempty subset of vertices of $V(D)$ respectively. Abusing the notation, we use $N^-(u; S)$ and $N^+(u; S)$ to define the in-neighborhood and ex-neighborhood of the vertex v with respect to the vertex subset S in D, that is,

$$\begin{aligned} N^-(u; S) &= \{v \in S : vu \in A(D)\} \text{ and} \\ N^+(u; S) &= \{v \in S : uv \in A(D)\}. \end{aligned}$$

A *proper r-coloring* of a digraph D is a surjective function $\varphi : V(D) \to \{0, 1, \ldots, r-1\}$. A directed triangle or 3-cycle (x, y, z, x) is a *heterochromatic directed triangle* if the vertices x, y and z receive different colors from φ.

Throughout this paper, we deal with regular tournaments which are well-known to have odd order. We denote by $\mathbb{Z}_{2m+1}$ the group of residues modulo $2m+1$. Let $J \subseteq \mathbb{Z}_{2m+1}$ such that $|J| = m$ and $i + j \not\equiv 0 \pmod{2m+1}$ for every $i, j \in J$. The *circulant* (*or rotational*) *tournament* $T = \overrightarrow{C}_{2m+1}(J)$ (or $RT_{2m+1}(J)$, $m \in \mathbb{N}$, see [16]), is defined by $V(\overrightarrow{C}_{2m+1}(J)) = \mathbb{Z}_{2m+1}$ and

$$A(\overrightarrow{C}_{2m+1}(J)) = \{(i, j) : i, j \in \mathbb{Z}_{2m+1} \text{ and } j - i \in J\}.$$

It is well-known that a circulant tournament is regular and vertex-transitive.

The tournament $\overrightarrow{C}_{2m+1}(I_m)$, where $I_m = \{1, 2, \ldots, m\} \subseteq \mathbb{Z}_{2m+1}$, is called the *cyclic tournament.* It is easy to check that for all m there is only one cyclic tournament up to isomorphism.

If $S \subseteq V(D)$, we say that vertices $u, v \in V(D)$ are *concordant modulo* S (denoted by $u \equiv v \pmod{S}$), if

$$\begin{aligned} N^-(u; S \setminus \{u, v\}) &= N^-(v; S \setminus \{u, v\}) \text{ and} \\ N^+(u; S \setminus \{u, v\}) &= N^+(v; S \setminus \{u, v\}). \end{aligned}$$

The vertices $u, v \in V(D)$ are *discordant modulo* S (denoted by $u \mid v \,(\mathrm{mod}\, S)$), if

$$\begin{aligned} N^+(u; S \setminus \{u,v\}) &= N^-(v; S \setminus \{u,v\}) \text{ and} \\ N^-(u; S \setminus \{u,v\}) &= N^+(v; S \setminus \{u,v\}). \end{aligned}$$

The *domination digraph* $\mathfrak{D}(T)$ of a (not necessarily regular) tournament T is defined as $V(\mathfrak{D}(T)) = V(T)$ and the arcs of $A(\mathfrak{D}(T))$ are of the form uv, where

$$N^+(u; V(T)) \cup N^+(v; V(T)) \cup \{u,v\} = V(T)$$

(see [9]). For a regular tournament T, this definition is equivalent to $V(\mathfrak{D}(T)) = V(T)$ and $A(\mathfrak{D}(T)) = \{uv \in A(T) : u \mid v \,(\mathrm{mod}\, V(T))\}$.

The arcs of the domination digraph are called $\mathfrak{D}$*-arcs*. The underlying graph of a domination digraph $\mathfrak{D}(T)$ of T is the *domination graph* $dom(T)$.

Lemma 1 ([9] Lemma 2.4)**.** *If T' is an induced subtournament of T, then the induced subdigraph of $\mathfrak{D}(T)$ on $V(T')$ is a subdigraph of $\mathfrak{D}(T')$.*

Lemma 2 ([4, 7], [15] Lemma 2)**.** *Let $\mathfrak{D}(T)$ be the domination digraph of a regular tournament T. Then every directed path $P \subset \mathfrak{D}(T)$ is directed in T.*

Lemma 3 ([15] Lemma 3)**.** *Let $P = (u_1, u_2, \ldots, u_k)$, $k \geq 2$, be a directed path in $\mathfrak{D}(T)$, where T is a regular tournament, and let $P_{i,j} = (u_i, u_{i+1}, \ldots, u_j)$ a directed subpath of P, where $1 \leq i < j \leq k$. Then*

(i) *$u_i \equiv u_j (\mathrm{mod}\, S)$ if $j-i$ is even and $u_i \mid u_j (\mathrm{mod}\, S)$ if $j-i$ is odd, where $S = V(T \setminus P_{i,j})$.*

(ii) *$u_j u_i \in A(T)$ if $j-i$ is even and $u_i u_j \in A(T)$ if $j-i$ is odd.*

(iii) *$u_i \mid u_j (\mathrm{mod}\, V(P_{i,j}))$ if $j-i$ is even and $u_i \equiv u_j (\mathrm{mod}\, V(P_{i,j}))$ if $j-i$ is odd.*

Remark 1. Let $P = (u_1, u_2, \ldots, u_k)$ be a directed path in $\mathfrak{D}(T)$, where $k \geq 2$ and T is a regular tournament. Then every ordered triple $(u_i, u_{i+2m+1}, u_{i+2n}, u_i)$, where $0 \leq m < n \leq \frac{k-i}{2}$, is a cyclic triangle in T.

In [5, 6], Cho *et al.* characterized the domination graphs of regular tournaments:

Theorem 1 ([6] Theorems 2.7 and 3.12)**.** *Let G be a graph* (*maybe the edgeless graph*)*. Then G is the domination graph of a regular tournament if and only if G is a cycle of odd order or if G is the disjoint union of m even order paths and n odd order paths* (*maybe the trivial paths*) *such that n is odd and either*

(i) $m = 3$ *or* $m \geq 5$, *or*

(ii) $m = 0, 1, 2, 4$ *and* $m + n \geq 7$.

Remark 2. If $\mathfrak{D}(T)$ is not isomorphic to the directed cycle $\overrightarrow{C}_{2m+1}$ ($m \geq 1$), then it is composed of at least four directed paths.

We will often use the following construction that can be found in [15]:

Remark 3. Let uv be a $\mathfrak{D}$-arc of a regular tournament T of order $2m+3$ ($m \geq 1$) and denote by $T' = T \setminus \{u, v\}$ the residual tournament of uv. By the definition of $\mathfrak{D}(T)$, there exists a natural partition of $V(T')$ into the sets $V^- = N^+(u; T')$ and $V^+ = N^+(v; T')$. Moreover, $V^- = N^-(v, T')$ and $V^+ = N^-(u, T')$. Since T is regular, $|V^-| = m$ and $|V^+| = m + 1$.

Proposition 1 ([15] Lemma 6)**.** *If T is a regular tournament and $u, v \in V(T)$, then the residual tournament $T' = T \setminus \{u, v\}$ is regular if and only if $u \mid v (\operatorname{mod} V(T))$.*

Corollary 1 ([15] Corollary 1)**.** *If T is a regular tournament and $P \subset \mathfrak{D}(T)$ is a directed path of even order, then $T \setminus P$ is regular.*

If T is a regular tournament and every directed path in the domination digraph $\mathfrak{D}(T)$ has order at most 2, then T is said to be a *mold.* By Theorem 1, observe that the domination digraph of a mold is composed of a nonempty set of isolated vertices and a (possibly empty) set of disjoint arcs.

Consider a regular tournament T and let $\{P_i^1\}$ and $\{P_j^2\}$ be the sets of maximal directed paths of odd and even order, respectively, in $\mathfrak{D}(T)$, where $P_i^1 = \{u_{i,1}, u_{i,2}, \ldots, u_{i,2k_i+1}\}$, $P_j^2 = \{v_{j,1}, v_{j,2}, \ldots, v_{j,2m_j}\}$, $i, j, k_i, m_j \in \mathbb{N}$, k_i and m_j are nonnegative integers. Observe that $\{P_j^2\}$ may be empty. Let us define the following sets of vertices:

$$\begin{aligned} V^1 &= \left\{u_{i,1} \in V(T) : u_{i,1} \in P_i^1\right\} \text{ and} \\ V^2 &= \left\{v_{j,1}, v_{j,2} \in V(T) : v_{j,1}v_{j,2} \in A(P_j^2)\right\}. \end{aligned}$$

We define the *mold* M^T of a non-cyclic regular tournament T as follows: $V(M^T) = V^1 \cup V^2$ and $M^T = T[V(M^T)]$, i.e., M^T is the regular tournament induced by $V^1 \cup V^2$ (see Corollary 1).

Consider now a regular tournament T of order $2m+3$ and a $\mathfrak{D}$-arc uv. Then, the residual tournament $T' = T \setminus \{u, v\}$ ($uv \in A(T)$) is regular by Proposition 1 and we say that the arc uv is $\mathfrak{C}$-*residual* if T' is isomorphic to $\overrightarrow{C}_{2m+1}(I_m)$. A mold M is called *tame* if it has a $\mathfrak{C}$-residual arc. A regular tournament T is *tame* if M^T is tame. Finally, we denote by $\mathcal{F}(M)$ the family of regular tournaments whose mold is M.

Theorem 2 ([15], Theorem 4)**.** *Let M be a tame mold and $T \in \mathcal{F}(M)$. Then $dc(M) = 3 = dc(T)$.*

If T is a regular tournament and every directed path in the domination digraph $\mathfrak{D}(T)$ has order at least 3, then T is said to be an *ample tournament.* For more details on tame molds, see [15].

3. Families of Tight Regular Tournaments

Lemma 4. *Let T be a regular tournament and let φ be a proper 3-coloring of $V(T)$ such that there exists a 3-chromatic directed path of order at least 3 in $\mathfrak{D}(T)$. Then T contains a heterochromatic cyclic triangle.*

Proof. Let T be a regular tournament, such that T is not a mold. Consider a proper 3-coloring $\varphi : V(T) \to \{0, 1, 2\}$ of T and a 3-chromatic directed path $P = (v_1, v_2, \ldots, v_l)$ in $\mathfrak{D}(T)$ with $l \geq 3$. For i, j and k such that $1 \leq i < j < k \leq l$, we can suppose that $\varphi(v_i) = 0$, $\varphi(v_j) = 1$ for all $i+1 \leq j \leq k-1$ and $\varphi(v_k) = 2$. Since P is 3-chromatic, such a combination of colors does always occur. We have two cases:

Case 1. $k-i$ is even. By Remark 1, (v_i, v_{i+1}, v_k, v_i) is a heterochromatic cyclic triangle.

Case 2. $k-i$ is odd. By Remark 3, the ex-neighborhoods of vertices v_i and v_{i+1} form a partition of the vertex set of tournament $T \setminus P$. We introduce the following notation: $V^- = N^+(v_i; T \setminus P)$ and $V^+ = N^+(v_{i+1}; T \setminus P)$. Let w be a vertex of V^+.

(i) If $\varphi(w) = 2$, then (w, v_i, v_{i+1}, w) is a heterochromatic cyclic triangle.

(ii) If $\varphi(w) = 1$, then (w, v_i, v_k, w) is a heterochromatic cyclic triangle.

(iii) If $\varphi(w) = 0$, then (w, v_{k-1}, v_k, w) is a heterochromatic cyclic triangle.

By Remark 3, $V^+ \neq \emptyset$ and therefore T contains a heterochromatic cyclic triangle. ■

Let $T = (V, A)$ be a tournament. We define the *converse* (*opposite*) *tournament* $\overleftarrow{T} = (V, \overleftarrow{A})$, where $uv \in A$ if and only if $vu \in \overleftarrow{A}$. Observe that a directed path P belongs to $\mathfrak{D}(T)$ if and only if $\overleftarrow{P}$ belongs to $\mathfrak{D}(\overleftarrow{T})$.

Lemma 5. *Let T be a regular tournament and let φ be a proper 3-coloring of $V(T)$ such that there exists a 2-chromatic directed path of order at least 3 in $\mathfrak{D}(T)$. Then T contains a heterochromatic cyclic triangle.*

Proof. Let T be a regular tournament such that T is not a mold. Consider a proper 3-coloring $\varphi : V(T) \to \{0, 1, 2\}$ of T and a 2-chromatic directed path $P = (v_1, v_2, \ldots, v_l)$ in $\mathfrak{D}(T)$ with $l \geq 3$. Without loss of generality, we can suppose that $\varphi(v_1) = 0$, $\varphi(v_i) = 1$ for some i, $2 \leq i \leq l$ and $\varphi(v_j) = 0$ for every $j < i$. If $i < l$, there are two cases: either $\varphi(v_{i+1}) = 0$ or $\varphi(v_{i+1}) = 1$; and therefore 010 and 011 are possible colorings of three consecutive vertices of P using both colors. There could be three other consecutive vertices with different possible colorings. In fact, since $\varphi(v_{i-1}) = 0$ $(2 \leq i \leq l-2)$, the following three consecutive vertices could have the colorings 100, 101 and 110. But then v_{i-1}, v_i and v_{i+1} in that order, are colored 010 and 011. Finally, if $i = l$, we have the coloring 001 for the last three vertices of P. Therefore, it is enough to prove the following three cases:

Case 1. For $2 \leq i \leq l-1$, consider $v_{i-1}, v_i, v_{i+1} \in V(P)$ such that $\varphi(v_{i-1}) = 0$, $\varphi(v_i) = 1$ and $\varphi(v_{i+1}) = 0$. Since φ is a proper 3-coloring, there exists a vertex $w \in V(T \setminus P)$ such that $\varphi(w) = 2$. Using Remark 3, we have that:

(i) If $v_i w \in A(T)$, then (w, v_{i-1}, v_i, w) is a heterochromatic cyclic triangle.

(ii) If $w v_i \in A(T)$, then (w, v_i, v_{i+1}, w) is a heterochromatic cyclic triangle.

Case 2. For $2 \leq i \leq l-1$, consider $v_{i-1}, v_i, v_{i+1} \in V(P)$ such that $\varphi(v_{i-1}) = 0$, $\varphi(v_i) = 1$ and $\varphi(v_{i+1}) = 1$. Define $V^- = N^-(v_i; T \setminus P)$ and $V^+ = N^+(v_i; T \setminus P)$.

Let $w \in V^+$. Since (w, v_{i-1}, v_i, w) is a cyclic triangle, then for all $w \in V^+$ we suppose that $\varphi(w) \neq 2$. Using that φ is proper, there exists $x \in V^-$, such that $\varphi(x) = 2$. Suppose that there is a vertex in $N^+(x; V^+)$, say u. So, $\varphi(u) \neq 2$. Since (x, u, v_{i+1}, x) and (x, u, v_{i-1}, x) are cyclic triangles, then $\varphi(u) \neq 0$ and $\varphi(u) \neq 1$. Thus, every vertex of V^+ dominates every vertex of color 2 in V^-. Therefore $N^+(x; V^+) = \emptyset$. Moreover, for every $w \in V^+$, we have that $\varphi(w) \neq 0$ since (w, x, v_i, w) is a cyclic triangle. So $\varphi(w) = 1$ for every $w \in V^+$. If there exists a vertex $v_{i-2} \in V(P)$, then $\varphi(v_{i-2}) = 0$ (otherwise, if $\varphi(v_{i-2}) = 1$, then we have a case analogous to case 1). But (x, v_{i-2}, w, x) is a heterochromatic cyclic triangle, therefore v_{i-1} is the first vertex of P and $N^-(x; V^+) = V^+$.

(i) If P is of order $l = 2r$ and $|V(T \setminus P)| = 2m + 1$, then by Remark 3, the first vertex in P dominates exactly the even indexed vertices in P, so v_{i-1} dominates exactly r vertices in P and exactly $|V^-|$ vertices not in P. But every vertex in T has out-degree $r + m$, so $|V^-| = m$. It follows that $|V^+| = m + 1$ and therefore $d^-(x, T \setminus P) \geq m + 1$. This is a contradiction to the fact that every vertex in the subtournament $T \setminus P$ has in-degree m since, by Corollary 1, it is regular.

(ii) If P is of order $l = 2r + 1$ and $|V(T \setminus P)| = 2m$, then the argument used in (i) shows that $|V^-| = m$. It follows that $|V^+| = m$. By repeated use of Remark 3 on the arcs of P, the $r + 1$ odd indexed vertices of P dominate x and all of V^+ dominates x. So $d^-(x, T) \geq (r + 1) + m$, a contradiction to the fact that T is a regular tournament of order $2(r + m) + 1$.

Case 3. $\varphi(v_i) = 0$ for $1 \leq i \leq l - 1$ and $\varphi(v_l) = 1$. Then $\overleftarrow{P} = \{v_l, v_{l-1}, \ldots, v_1\}$ is a directed path of $\mathfrak{D}(\overleftarrow{T})$. Let us define $\varphi' : V(\overleftarrow{T}) \to \{0, 1, 2\}$ such that

$$\varphi'(v) = \begin{cases} 1 \text{ if } \varphi(v) = 0, \\ 0 \text{ if } \varphi(v) = 1, \\ 2 \text{ if } \varphi(v) = 2. \end{cases}$$

Note that $\varphi'(v_l) = 0$ and $\varphi'(v_i) = 1$ for $i = 1, 2, \ldots, l - 1$. Then by Case 2, we have a heterochromatic cyclic triangle in $\overleftarrow{T}$ with the proper 3-coloring φ'. Clearly, there is a heterochromatic cyclic triangle in T with the 3-coloring φ. ■

Theorem 3. *If T is a regular tournament and $\varphi : V(T) \to \{0, 1, 2\}$ is a proper 3-coloring of T without heterochromatic cyclic triangles, then every directed path P of order at least 3 in $\mathfrak{D}(T)$ is monochromatic.*

Proof. A direct consequence of Lemmas 4 and 5. ■

Theorem 4. *Let T be a regular tournament and uv a $\mathfrak{D}$-arc such that $T' = T \setminus \{u, v\}$ is tight. Then T is tight.*

Proof. Let T be a regular tournament and uv a $\mathfrak{D}$-arc such that $T' = T \setminus \{u, v\}$ is tight and $|V(T')| = 2m+1$. Note that T' is regular by Proposition 1. Suppose that $\varphi : V(T) \to \{0, 1, 2\}$ is a proper 3-coloring of T without heterochromatic cyclic triangles. Since T' is tight, we may assume that $\varphi\mid_{T'}: V(T') \to \{0, 1\}$ is a proper coloring of T'. Let $V^+ = N^+(v; T')$, $V^- = N^+(u; T')$, $v^+ \in V^+$ and $v^- \in V^-$ (see Remark 3). We have two cases:

Case 1. $\varphi(u) = \varphi(v) = \{2\}$. If there exist the 2-chromatic arcs $v^-v^+ \in A(T')$ or $v^+v^- \in A(T')$, then the cyclic triangles (u, v^-, v^+, u) or (v, v^+, v^-, v), respectively, are heterochromatic. Consequently, there are no 2-chromatic arcs between vertex sets V^- and V^+ and so T' is monochromatic. It follows that φ is a 2-coloring, a contradiction with the original assumption.

Case 2. Let $\{\varphi(u), \varphi(v)\} = \{1, 2\}$. Since (v^+, u, v, v^+) is a cyclic triangle, we have that $\varphi\mid_{T'}(V^+) = \{1\}$. Then there exists a vertex $j \in V^-$ such that $\varphi(j) = \{0\}$.

(i) If $\varphi(v) = \{2\}$, then $N^-(j; V^+) \neq \emptyset$ since $|V^+| = m + 1$ and $d^+(j, T') = m$. Let $j^- \in N^-(j; V^+)$. The cyclic triangle (v, j^-, j, v) is heterochromatic.

(ii) If $\varphi(u) = \{2\}$, then $N^+(j; V^+) \neq \emptyset$ since $|V^+| = m + 1$ and $d^-(j, T') = m$. Let $j^+ \in N^+(j; V^+)$. The cyclic triangle (u, j, j^+, u) is heterochromatic.

We conclude that if the residual tournament T' of a $\mathfrak{D}$-arc is tight, then the regular tournament T is tight. ■

The following result will be used in the proof of Corollary 2.

Proposition 2 ([14] Proposition 4.4(i))**.** *Let $m \geq 2$ be an integer. Then $\overrightarrow{C}_{2m+1}(I_m)$ is tight.*

Corollary 2. *Tame molds are tight.*

Proof. Let M be a tame mold and uv the $\mathfrak{C}$-residual arc. Then $M \setminus \{u, v\}$ is a cyclic tournament, which is tight by Proposition 2. ■

Corollary 3. *Let T be a regular tournament and P a $\mathfrak{D}$-path of even order such that $T' = T \setminus P$ is tight. Then T is tight.*

Proof. Apply Theorem 4 to the arcs of P. Note that $T \setminus P$ is a regular tournament by Corollary 1. ■

Using this corollary and Theorem 2, we have that tame molds M and all $T \in \mathcal{F}(M)$ tame tournaments are 3-dichromatic and tight. These facts confirm the validity of Conjecture 1 for the special case when $r = 3$ and $s = 2$.

Corollary 4. *There is an infinite set of regular tournaments T such that $dc(T) = 3$ and $\overrightarrow{\omega}_3(T) = 2$.*

Theorem 5. *Ample tournaments are tight.*

Proof. Let T be an ample tournament. By contradiction, suppose that T is not tight, that is, there exists a proper 3-coloring $\varphi : V(T) \to \{0, 1, 2\}$ not inducing heterochromatic cyclic triangles. By Theorem 3, we have that every directed path of order at least 3 in the domination digraph $\mathfrak{D}(T)$ is monochromatic. Since T is ample and by Theorem 1 and Remark 2, there exist at least four directed paths in $\mathfrak{D}(T)$ of order at least 3. Let P_u, P_v and P_w be three directed paths in $\mathfrak{D}(T)$, each one of a different color (since φ is a proper 3-coloring, these directed paths exist). We define

$$\begin{aligned} P_u &= (u_0, u_1, u_2, \ldots, u_k)\,, \\ P_v &= (v_0, v_1, v_2, \ldots, v_l) \text{ and} \\ P_w &= (w_0, w_1, w_2, \ldots, w_m)\,, \end{aligned}$$

where $k, l, m \geq 3$. Without loss of generality, we can assume that u_0v_0, $v_0w_0 \in A(T)$.

(i) If $w_0u_0 \in A(T)$, then (u_0, v_0, w_0, u_0) is a heterochromatic cyclic triangle.

(ii) If $u_0w_0 \in A(T)$, then since w_0v_1 and v_1u_0 are in $A(T)$, by Remark 3, (u_0, w_0, v_1, u_0) is a heterochromatic cyclic triangle.

From (i) and (ii), we obtain a contradiction, φ does not induce heterochromatic cyclic triangles. ■

Theorem 6. *Let M be a tight mold and $T \in \mathcal{F}(M)$. Then T is tight.*

Proof. Let M be a tight mold and $T \in \mathcal{F}(M)$. We have two cases:

Case 1. If T is an ample tournament, then by Theorem 5, T is tight.

Case 2. We suppose that T is not an ample tournament. If $T \cong M$, then T is tight by the supposition of the theorem. If $T \ncong M$, then by the definition of a mold, T has a directed path of order at least 3 in $\mathfrak{D}(T)$. Consider a proper 3-coloring $\varphi : V(T) \to \{0, 1, 2\}$ not inducing heterochromatic cyclic triangles. By Theorem 3, every directed path in $\mathfrak{D}(T)$ of order at least 3 is monochromatic. Since M is tight, we can assume that $\varphi\,|_M \colon V(M) \to \{0, 1\}$ is a proper 2-coloring. By the definition of a mold, M is a proper subtournament of T. Then there exist a vertex $v \in V(T) \setminus V(M)$ such that $\varphi(v) = 2$ and $v \in P$, where P is a directed path of order 3 in $\mathfrak{D}(T)$. Since P is monochromatic, for every vertex $w \in V(P)$ we have that $\varphi(w) = 2$. In particular, the initial vertex of P belonging to $V(M)$ has color 2, a contradiction to the choice of $\varphi\,|_M$. ■

4. Concluding Remarks

Observe that for every mold M, there is an infinite family of ample tournaments T such that $T \in \mathcal{F}(M)$ and all of them are tight by Theorem 5. Moreover, by Theorem 6 we can conclude that there exist an infinite family of non ample tight regular tournaments $T \in \mathcal{F}(M)$, where M is a tame mold.

The non-cyclic circulant tournaments $\overrightarrow{C}_{2n+1}(J)$ are molds since $\mathfrak{D}(T)$ is edgeless by Corollary 3.7 of [9].

The following families are known to be tight:

(i) $\overrightarrow{C}_{2n+1}(J)$, where $J = (I_n \setminus \{s\}) \cup \{-s\}$ and $(2n+1, s) \neq (9, 2)$, $1 \le s \le n$ ([14], Theorem 4.11).

(ii) $\overrightarrow{C}_{2n+1}(J)$, where $J = (I_n \setminus \{s_1, s_2\}) \cup \{-s_1, -s_2\}$, $1 \le s_1 < s_2 \le n$ and $\overrightarrow{C}_{2n+1}(J) \notin \{\overrightarrow{C}_9(-1, 2, 3, -4), \overrightarrow{C}_9(1, -2, -3, 4), \overrightarrow{C}_{15}(1, -2, 3, 4, -5, 6, 7), \overrightarrow{C}_{15}(1, 2, -3, -4, 5, 6, 7)\}$ ([10], Theorem 5).

(iii) $\overrightarrow{C}_p(J)$, where $p \ge 3$ is a prime number ([11], Theorem 3).

We have the following corollary to Theorem 6:

Corollary 5. *Let M be a circulant non-cyclic mold of type* (i), (ii) or (iii). *Then every $T \in \mathcal{F}(M)$ is tight.*

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