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SUBGRAPH DENSITIES IN HYPERGRAPHS

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Abstract

Let $r \geq 2$ be an integer. A real number $\alpha \in [0, 1)$ is a jump for r if for any $\epsilon > 0$ and any integer $m \geq r$, any r-uniform graph with $n > n_0(\epsilon, m)$ vertices and density at least $\alpha + \epsilon$ contains a subgraph with m vertices and density at least $\alpha + c$, where $c = c(\alpha) > 0$ does not depend on ϵ and m. A result of Erdős, Stone and Simonovits implies that every $\alpha \in [0, 1)$ is a jump for $r = 2$. Erdős asked whether the same is true for $r \geq 3$. Frankl and Rödl gave a negative answer by showing an infinite sequence of non-jumps for every $r \geq 3$. However, there are still a lot of open questions on determining whether or not a number is a jump for $r \geq 3$. In this paper, we first find an infinite sequence of non-jumps for $r = 4$, then extend one of them to every $r \geq 4$. Our approach is based on the techniques developed by Frankl and Rödl.

Keywords: Erdős jumping constant conjecture, Lagrangian, optimal vector.

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1. Introduction

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-subsets of V. An *r-uniform graph* G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq \binom{V}{r}$ of edges. In particular, an r-uniform graph is called a *graph* if $r = 2$ and an *r-uniform hypergraph* if $r \geq 3$. We abbreviate r-uniform graph to *r-graph*. The *density* of an r-graph G is defined by $d(G) = \frac{|E(G)|}{|\binom{V(G)}{r}|}$. An r-graph H is a *subgraph* of an r-graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of G if $E(H) = E(G) \cap \binom{V(H)}{r}$.

By a simple argument (c.f. Katona, Nemetz, Simonovits [8]), the average of densities of all induced subgraphs of an r-graph G with $m \geq r$ vertices is $d(G)$. Therefore, there exists a subgraph of G with m vertices and density $\geq d(G)$. A natural question is whether there exists a subgraph of G with m vertices and density $\geq d(G) + c$, where $c > 0$ is a constant? To be more precise, the concept of 'jump' was introduced.

Definition 1.1. Given $r \geq 2$, a real number $\alpha \in [0, 1)$ is a jump for r if there exists a constant $c > 0$ such that for any $\epsilon > 0$ and any integer m, $m \geq r$, there exists an integer $n_0(\epsilon, m)$ such that any r-graph with $n > n_0(\epsilon, m)$ vertices and density $\geq \alpha + \epsilon$ contains a subgraph with m vertices and density $\geq \alpha + c$. A real number $\alpha \in [0, 1)$ is called a non-jump for r if α is not a jump for r.

Erdős and Stone ([4]) proved that every $\alpha \in [0, 1)$ is a jump for $r = 2$. It easily follows from the following classical result.

For an integer $l \geq r$, an r-graph $G = (V, E)$ is called *complete l-partite* if V admits a partition into l classes such that an r-subset of V is an edge if and only if it contains at most one vertex from each class.

Theorem 1.1 (c.f. [4]). *Suppose l is a positive integer. For any $\epsilon > 0$ and any positive integer m, there exists $n_0(m, \epsilon)$ such that any graph G on $n > n_0(m, \epsilon)$ vertices with density $d(G) \geq 1 - \frac{1}{l} + \epsilon$ contains a copy of the complete $(l + 1)$-partite graph with partition classes of size m.*

Note that the density of a complete $(l + 1)$-partite graph with partition classes of size m is greater than $1 - \frac{1}{l+1}$ (approaches $1 - \frac{1}{l+1}$ when $m \to \infty$).

For $r \geq 3$, Erdős proved that every $\alpha \in [0, r!/r^r)$ is a jump. It directly follows from the following:

Theorem 1.2 (c.f. [2]). *For any $\epsilon > 0$ and any positive integer m, there exists $n_0(\epsilon, m)$ such that any r-graph G on $n > n_0(\epsilon, m)$ vertices with density $d(G) \geq \epsilon$ contains a copy of the complete r-partite r-graph with partition classes of size m.*

Note that the density of a complete r-partite r-graph with partition classes of size m is greater than $r!/r^r$ (approaches $r!/r^r$ when $m \to \infty$).

Furthermore, Erdős proposed the following jumping constant conjecture.

Conjecture 1.3. Every $\alpha \in [0, 1)$ is a jump for every integer $r \geq 2$.

In [6], Frankl and Rödl disproved this conjecture by showing the following result.

Theorem 1.4 (c.f. [6])**.** *Suppose $r \geq 3$ and $l > 2r$. Then $1 - \frac{1}{l^{r-1}}$ is not a jump for r.*

Using the techniques developed by Frankl and Rödl in [6], some other non-jumps were given in [7, 10, 11] and [12]. However, there are still a lot of open questions on determining whether or not a number is a jump for $r \geq 3$. A well-known question of Erdős is to determine whether or not $\frac{r!}{r^r}$ is a jump. At this moment, the smallest known non-jump for $r \geq 3$ is $\frac{5r!}{2r^r}$ given in [7]. Another question raised in [7] is whether there is an interval of non-jumps for $r \geq 3$. By the definition of the 'jump', if a number a is a jump, then there exists a constant $c > 0$ such that every number in $[a, a + c)$ is a jump. Consequently, if there is a set of non-jumps whose limits form an interval (number a is a limit of a set A if there is a sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in A$ such that $\lim_{n\to\infty} a_n = a$), then no number in this interval is a jump. We do not know whether such a 'dense enough' set of non-jumps exists or not. In this paper we intend to find more non-jumps in addition to the known non-jumps in [6, 7, 10, 11] and [12]. Our approach is still based on the techniques developed by Frankl and Rödl in [6].

We first work in the case $r = 4$ and find a sequence of non-jumps for $r = 4$. In Sections 3 and 4, we prove the following result.

Theorem 1.5. *Let $l \geq 2$ be an integer. Then $1 - \frac{7}{l^2} + \frac{10}{l^3}$ is not a jump for $r = 4$.*

In Section 5 we extend a special case of Theorem 1.5 ($l = 4$) to all $r \geq 4$. The following result will be proved.

Theorem 1.6. *For $r \geq 4$, $\frac{23r!}{3r^r}$ is not a jump for r.*

Note that when $r = l = 4$, Theorems 1.6 and 1.5 coincide.

In the next section, we introduce the Lagrangian of an r-graph and some other tools to be applied in our proofs.

2. Lagrangians and Other Tools

We first give a definition of the Lagrangian of an r-graph. More studies of Lagrangians were given in [5, 6, 9] and [13].

Definition 2.1. For an r-graph G with vertex set $\{1, 2, \ldots, m\}$, edge set $E(G)$ and a vector $\vec{x} = (x_1, \ldots, x_m) \in R^m$, define

$$\lambda(G, \vec{x}) = \sum_{\{i_1,\ldots,i_r\} \in E(G)} x_{i_1} x_{i_2} \ldots x_{i_r}.$$

x_i is called the *weight* of vertex i.

Definition 2.2. Let $S = \{\vec{x} = (x_1, x_2, \ldots, x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geq 0$ for $i = 1, 2, \ldots, m\}$. The Lagrangian of G, denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in S\}.$$

A vector $\vec{x} \in S$ is called an *optimal vector* for $\lambda(G)$ if $\lambda(G, \vec{x}) = \lambda(G)$.

We note that if H is a subgraph of an r-graph G, then for any vector $\vec{x}$ in S, $\lambda(H, \vec{x}) \leq \lambda(G, \vec{x})$. We formulate this as follows.

Fact 2.1. Let H be a subgraph of an r-graph G. Then

$$\lambda(H) \leq \lambda(G).$$

For an r-graph G and $i \in V(G)$ we define G_i to be the $(r-1)$-uniform graph on $V - \{i\}$ with edge set $E(G_i)$ given by $e \in E(G_i)$ if and only if $e \cup \{i\} \in E(G)$.

We call two vertices i, j of an r-graph G equivalent if for all $f \in \binom{V(G)-\{i,j\}}{r-1}$, $f \in E(G_i)$ if and only if $f \in E(G_j)$.

The following lemma (proved in [6]) will be useful when calculating Lagrangians of certain graphs.

Lemma 2.2 (c.f. [6]). *Suppose G is an r-graph on vertices $\{1, 2, \ldots, m\}$.*

1. *If vertices $i_1, i_2, \ldots, i_t$ are pairwise equivalent, then there exists an optimal vector $\vec{y} = (y_1, y_2, \ldots, y_m)$ of $\lambda(G)$ such that $y_{i_1} = y_{i_2} = \cdots = y_{i_t}$.*
2. *Let $\vec{y} = (y_1, y_2, \ldots, y_m)$ be an optimal vector of $\lambda(G)$ and $y_i > 0$. Let $\hat{y}_i$ be the restriction of $\vec{y}$ on $\{1, 2, \ldots, m\} \setminus \{i\}$. Then $\lambda(G_i, \hat{y}_i) = r\lambda(G)$.*

We also note that for an r-graph G with m vertices, if we take $\vec{u} = (u_1, \ldots, u_m)$, where each $u_i = 1/m$, then

$$\lambda(G) \geq \lambda(G, \vec{u}) = \frac{|E(G)|}{m^r} \geq \frac{d(G)}{r!} - \epsilon$$

for $m \geq m'(\epsilon)$.

On the other hand, we introduce the blow-up of an r-graph G which will allow us to construct r-graphs with large number of vertices and densities close to $r!\lambda(G)$.

Definition 2.3. Let G be an r-graph with $V(G) = \{1, 2, \ldots, m\}$ and $(n_1, \ldots, n_m)$ be a positive integer vector. Define the $(n_1, \ldots, n_m)$ blow-up of G, $(n_1, \ldots, n_m) \otimes G$ as an m-partite r-graph with vertex set $V_1 \cup \cdots \cup V_m$, $|V_i| = n_i, 1 \leq i \leq m$, and edge set $E((n_1, \ldots, n_m) \otimes G) = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} : v_{i_k} \in V_{i_k}$ for $1 \leq k \leq r, \{i_1, i_2, \ldots, i_r\} \in E(G)\}$. We abbreviate $(n, n, \ldots, n) \otimes G$ to $\vec{n} \otimes G$.

We make the following easy Remark used in [10].

Remark 2.3 (c.f. [10])**.** Let G be an r-graph with m vertices and $\vec{y} = (y_1, \ldots, y_m)$ be an optimal vector of $\lambda(G)$. Then for any $\epsilon > 0$, there exists an integer $n_1(\epsilon)$, such that for any integer $n \geq n_1(\epsilon)$,

$$d((\lfloor ny_1 \rfloor, \lfloor ny_2 \rfloor, \ldots, \lfloor ny_m \rfloor) \otimes G) \geq r!\lambda(G) - \epsilon. \tag{1}$$

Let us also state a fact relating the Lagrangian of an r-graph to the Lagrangian of its blow-up used in [6, 7, 10, 11] and [12] as well).

Fact 2.4 (c.f. [6])**.** $\lambda(\vec{n} \otimes G) = \lambda(G)$.

The following lemma proved in [6] gives a necessary and sufficient condition for a number α to be a jump. We need a definition to describe it.

Definition 2.4. For $\alpha \in [0, 1)$ and a family $\mathcal{F}$ of r-graphs, we say that α is a threshold for $\mathcal{F}$ if for any $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon)$ such that any r-graph G with $d(G) \geq \alpha + \epsilon$ and $|V(G)| > n_0$ contains some member of $\mathcal{F}$ as a subgraph. We denote this fact by $\alpha \to \mathcal{F}$.

Lemma 2.5 (c.f. [6])**.** *The following two properties are equivalent.*

1. *α is a jump for r.*
2. *$\alpha \to \mathcal{F}$ for some finite family $\mathcal{F}$ of r-graphs satisfying $\lambda(F) > \frac{\alpha}{r!}$ for all $F \in \mathcal{F}$.*

We also need the following lemma proved in [6].

Lemma 2.6 (c.f. [6]). *For any $\sigma \geq 0$ and any integer $k \geq r$, there exists $t_0(k, \sigma)$ such that for every $t > t_0(k, \sigma)$, there exists an r-graph A satisfying:*

1. *$|V(A)| = t$,*
2. *$|E(A)| \geq \sigma t^{r-1}$,*
3. *For all $V_0 \subset V(A), r \leq |V_0| \leq k$ we have $|E(A) \cap \binom{V_0}{r}| \leq |V_0| - r + 1$.*

The general approach in proving Theorems 1.5 and 1.6 is sketched as follows: Let α be a number to be proved to be a non-jump. Assuming that α is a jump, we will derive a contradiction by the following steps.

Step 1. Construct an r-uniform hypergraph (in Theorem 1.5, $r = 4$) with the Lagrangian close to but slightly smaller than $\frac{\alpha}{r!}$, then use Lemma 2.6 to add an r-graph with enough number of edges but sparse enough (see properties 2 and 3 in this Lemma) and obtain an r-graph with the Lagrangian $\geq \frac{\alpha}{r!} + \epsilon$ for some positive ϵ. Then we 'blow up' this r-graph to an r-graph, say H with large enough number of vertices and density $> \alpha + \frac{\epsilon}{2}$ (see Remark 2.3). If α is a jump, then by Lemma 2.5, α is a threshold for some finite family $\mathcal{F}$ of r-graphs with Lagrangians $> \frac{\alpha}{r!}$. So H must contain some member of $\mathcal{F}$ as a subgraph.

Step 2. We show that any subgraph of H with the number of vertices not greater than $\max\{|V(F)|, F \in \mathcal{F}\}$ has the Lagrangian $\leq \frac{\alpha}{r!}$ and derive a contradiction.

It is easy to construct an r-graph satisfying the property in Step 1, but it is certainly nontrivial to construct an r-graph satisfying the properties in both Steps 1 and 2. In fact, whenever we find such a construction, we can obtain a corresponding non-jump. This method was first developed by Frankl and Rödl in [6], then it was used in [7, 10, 11] and [12] to find more non-jumps by giving this type of construction. The technical part in the proof is to show that the construction satisfies the property in Step 2 (Lemma 3.1).

3. Proof of Theorem 1.5

In this Section, we focus on $r = 4$ and give a proof of Theorem 1.5. Let $\alpha = 1 - \frac{7}{l^2} + \frac{10}{l^3}$. Let t be a large enough integer determined later. We first define a 4-graph $G(l,t)$ on l pairwise disjoint sets $V_1, \dots, V_l$, each of cardinality t. The edge set of $G(l,t)$ consists of all 4-subsets taking exactly one vertex from each of V_i, V_j, V_k, V_s $(1 \le i < j < k < s \le l)$, all 4-subsets taking 2 vertices from V_i, 1 vertex from V_j and 1 vertex from V_k $(1 \le i \le l$, $1 \le j < k \le l$ and i, j, k are pairwise distinct), and all 4-subsets taking 3 vertices from V_i and 1 vertex from V_{i+1} $(1 \le i \le l$ and $V_{l+1} = V_1)$. When $l = 2$ or 3, some of them are vacant.

Note that the density of $G(l,t)$ is close to α if t is large enough. In fact,

$$\begin{aligned} |E(G(l,t))| &= \binom{l}{4}t^4 + l\binom{l-1}{2}\binom{t}{2}t^2 + l\binom{t}{3}t \\ &= \frac{\alpha}{24}l^4t^4 - c_0(l)t^3 + o(t^3), \end{aligned} \tag{2}$$

where $c_0(l)$ is positive (we omit giving the precise calculation here). Let $\vec{u} = (u_1, \dots, u_{lt})$, where $u_i = 1/(lt)$ for each $i, 1 \le i \le lt$, then

$$\lambda(G(l,t)) \ge \lambda(G(l,t), \vec{u}) = \frac{|E(G(l,t))|}{(lt)^4} = \frac{\alpha}{24} - \frac{c_0(l)}{l^4t} + o\left(\frac{1}{t}\right)$$

which is close to $\frac{\alpha}{24}$ when t is large enough.

We will use Lemma 2.6 to add a 4-graph to $G(l,t)$ so that the Lagrangian of the resulting 4-graph is $> \frac{\alpha}{24} + \epsilon(t)$ for some $\epsilon(t) > 0$. The precise argument is given below.

Suppose that α is a jump. In view of Lemma 2.5, there exists a finite collection $\mathcal{F}$ of 4-graphs satisfying the following:

(i) $\lambda(F) > \frac{\alpha}{24}$ for all $F \in \mathcal{F}$, and
(ii) α is a threshold for $\mathcal{F}$.

Set $k_0 = \max_{F\in\mathcal{F}}|V(F)|$ and $\sigma_0 = c_0(l)$. Let $r = 4$ in Lemma 2.6 and $t_0(k_0, \sigma_0)$ be given as in Lemma 2.6. Take an integer $t > \max(t_0, t_1)$, where t_1 is determined in (3) given later. For each $i, 1 \le i \le l$, take a 4-graph $A^i_{k_0,\sigma_0}(t)$ satisfying the conditions in Lemma 2.6 with $V(A^i_{k_0,\sigma_0}(t)) = V_i$. The 4-graph $G^*(l,t)$ is obtained by adding all $A^i_{k_0,\sigma_0}(t)$ to the 4-uniform hypegraph $G(l,t)$. Then

$$\lambda(G^*(l,t)) \geq \lambda(G^*(l,t),\vec{u}) = \frac{|E(G^*(l,t))|}{(lt)^4}.$$

In view of the construction of $G^*(l,t)$ and equation (2), we have

$$\frac{|E(G^*(l,t))|}{(lt)^4} \geq \frac{|E(G(l,t))| + l\sigma_0 t^3}{(lt)^4} \overset{(2)}{\geq} \frac{\alpha}{24} + \frac{c_0(l)}{2l^4 t} \tag{3}$$

for $t \geq t_1$. Consequently,

$$\lambda(G^*(l,t)) \geq \frac{\alpha}{24} + \frac{c_0(l)}{2l^4 t} \tag{4}$$

for $t \geq t_1$.

Now suppose $\vec{y} = (y_1, y_2, \ldots, y_{lt})$ is an optimal vector of $\lambda(G^*(l,t))$. Let $\epsilon = \frac{6c_0(l)}{l^4 t}$ and $n > n_1(\epsilon)$ as in Remark 2.3. Then 4-graph $S_n = (\lfloor ny_1 \rfloor, \ldots, \lfloor ny_{lt} \rfloor) \otimes G^*(l,t)$ has density larger than $\alpha + \epsilon$. Since α is a threshold for $\mathcal{F}$, some member F of $\mathcal{F}$ is a subgraph of S_n for $n \geq \max\{n_0(\epsilon), n_1(\epsilon)\}$. For such $F \in \mathcal{F}$, there exists a subgraph M of $G^*(l,t)$ with $|V(M)| \leq |V(F)| \leq k_0$ so that $F \subset \vec{n} \otimes M$. By Fact 2.1 and Fact 2.4, we have

$$\lambda(F) \overset{\text{Fact 2.1}}{\leq} \lambda(\vec{n} \otimes M) \overset{\text{Fact 2.4}}{=} \lambda(M). \tag{5}$$

Theorem 1.5 will follow from the following lemma to be proved in Section 4.

Lemma 3.1. *Let $G^*(l,t)$ be a 4-graph constructed the same way as above with k_0, σ_0, t replaced by any k, σ, t satisfying $t > t_0(k,\sigma)$ as given in Lemma* 2.6 *respectively. Let M be any subgraph of $G^*(l,t)$ with $|V(M)| \leq k$. Then*

$$\lambda(M) \leq \frac{1}{24}\alpha \tag{6}$$

holds.

Applying Lemma 3.1 to (5), we have

$$\lambda(F) \leq \frac{1}{24}\alpha$$

which contradicts our choice of F, i.e., contradicts the fact that $\lambda(F) > \frac{1}{24}\alpha$ for all $F \in \mathcal{F}$. ■

To complete the proof of Theorem 1.5, what remains is to show Lemma 3.1.

4. Proof of Lemma 3.1

By Fact 2.1, we may assume that M is an induced subgraph of $G^*(l,t)$. For each $s, 1 \leq s \leq l$, let

$$U_s = V(M) \cap V_s = \{v_1^s, v_2^s, \dots, v_{k_s}^s\}.$$

We will apply the following Claim proved in [6].

Claim 4.1 (c.f. [6])**.** *If N is the 4-graph formed from M by removing the edges contained in each U_s and inserting the edges $\{\{v_1^s, v_2^s, v_3^s, v_j^s\} : 1 \leq s \leq l,\ 4 \leq j \leq k_s\}$ then $\lambda(M) \leq \lambda(N)$.*

By Claim 4.1 the proof of Lemma 3.1 will be complete if we show that $\lambda(N) \leq \frac{\alpha}{24}$. Since v_1^s, v_2^s, v_3^s are pairwise equivalent and $v_4^s, \dots v_{k_s}^s$ are pairwise equivalent we can use Lemma 2.2(part 1) to obtain an optimal vector $\vec{z}$ of $\lambda(N)$ such that

$$z_1^s = z_2^s = z_3^s \stackrel{\text{def}}{=} \rho_s, \qquad z_4^s = z_5^s = \dots = z_{k_s}^s \stackrel{\text{def}}{=} \zeta_s.$$

Let w_s be the sum of the total weights in U_s. Let $P = \{s : w_s > 0\}$ and $p = |P|$. Without loss of generality, we may assume that $P = \{1, 2, \dots, p\}$. We may also assume that $p \geq 2$. Otherwise,

$$\lambda(N) = \rho_1^3(1 - 3\rho_1) \leq \frac{1}{256} < \frac{1}{24}\left(1 - \frac{7}{2^2} + \frac{10}{2^3}\right) \leq \frac{1}{24}\left(1 - \frac{7}{l^2} + \frac{10}{l^3}\right) = \frac{\alpha}{24}$$

since $1 - \frac{7}{x^2} + \frac{10}{x^3}$ increases when $x \geq 3$ increases and $1 - \frac{7}{2^2} + \frac{10}{2^3} < 1 - \frac{7}{3^2} + \frac{10}{3^3}$.

So we may assume that $2 \leq p \leq l$. For each $s \in P$ take a vertex $u_s \in U_s$ with positive weight as follows: if $\zeta_s > 0$ then $u_s = v_4^s$ otherwise $u_s = v_1^s$. The vertex u_s receives non-zero weight. Let $\hat{z}^s$ be the restriction of $\vec{z}$ on $V(N_{u_s})$. Then by Lemma 2.2(part 2) we have

$$4\lambda(N) = \lambda(N_{u_s}, \hat{z}^s).$$

Moreover by considering the edges containing vertex u_s we have

$$
\begin{aligned}
\lambda(N_{u_s}, \hat{z}^s) \ &\le \sum_{1\le i<j<k\le p; i,j,k\ne s} w_i w_j w_k + w_s \sum_{1\le i<j\le p; i,j\ne s} w_i w_j \\
&+ \sum_{1\le i<j\le p; i,j\ne s} \left(\frac{w_j^2}{2} w_i + \frac{w_i^2}{2} w_j \right) + \frac{w_s^2}{2} w_{s+1} \\
&+ \left[\frac{1}{6}(w_{s-1} - 3\rho_{s-1})^3 + \frac{3\rho_{s-1}}{2}(w_{s-1} - 3\rho_{s-1})^2 \right. \\
&\quad \left. + 3\rho_{s-1}^2 (w_{s-1} - 3\rho_{s-1}) + \rho_{s-1}^3 \right] + \rho_s^3,
\end{aligned} \tag{7}
$$

where all subscripts are modulo p. Note that

$$
\begin{aligned}
&\frac{1}{6}(w_{s-1} - 3\rho_{s-1})^3 + \frac{3\rho_{s-1}}{2}(w_{s-1} - 3\rho_{s-1})^2 + 3\rho_{s-1}^2(w_{s-1} - 3\rho_{s-1}) + \rho_{s-1}^3 \\
&\le \frac{(w_{s-1} - 3\rho_{s-1})^3 + 9\rho_{s-1}(w_{s-1} - 3\rho_{s-1})^2 + 27\rho_{s-1}^2(w_{s-1} - 3\rho_{s-1}) + 27\rho_{s-1}^3}{6} \\
&\quad -\rho_{s-1}^3 = \frac{w_{s-1}^3}{6} - \rho_{s-1}^3.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
4p\lambda(N) &= \sum_{s=1}^{p} \lambda(N_{u_s}, \hat{z}^s) \\
&\le p \sum_{1\le i<j<k\le p} w_i w_j w_k + \frac{p-2}{2} \sum_{1\le i<j\le p} (w_i^2 w_j + w_j^2 w_i) \\
&\quad + \frac{1}{2} \sum_{s=1}^{p} w_s^2 w_{s+1} + \frac{1}{6} \sum_{s=1}^{p} w_s^3.
\end{aligned} \tag{8}
$$

If $p = 2$, then

$$
8\lambda(N) \le \frac{w_1^3}{6} + \frac{w_2^3}{6} + \frac{w_1^2 w_2}{2} + \frac{w_1 w_2^2}{2} = \frac{(w_1 + w_2)^3}{6} = \frac{1}{6}.
$$

This implies that

$$\lambda(N) \leq \frac{1}{48} = \frac{1}{24}\left(1-\frac{7}{2^2}+\frac{10}{2^3}\right) \leq \frac{1}{24}\left(1-\frac{7}{l^2}+\frac{10}{l^3}\right) = \frac{\alpha}{24}.$$

So we may assume that $p \geq 3$ from now on. We separate the right hand side of (8) into two parts as follows:

$$f(w_1, w_2, \ldots, w_p) = \sum_{1\leq i<j<k\leq p} w_i w_j w_k + \frac{1}{2}\sum_{s=1}^{p} w_s^2 w_{s+1}. \tag{9}$$

$$\begin{aligned} g(w_1, w_2, \ldots, w_p) &= (p-1)\sum_{1\leq i<j<k\leq p} w_i w_j w_k \\ &+ \frac{1}{6}\sum_{s=1}^{p} w_s^3 + \frac{p-2}{2}\sum_{1\leq i<j\leq p}\left(w_i^2 w_j + w_j^2 w_i\right). \end{aligned} \tag{10}$$

Note that

$$f\left(\frac{1}{p},\frac{1}{p},\ldots,\frac{1}{p}\right)+g\left(\frac{1}{p},\frac{1}{p},\ldots,\frac{1}{p}\right)=\frac{p}{6}\left(1-\frac{7}{p^2}+\frac{10}{p^3}\right)\leq\frac{p}{6}\left(1-\frac{7}{l^2}+\frac{10}{l^3}\right)=\frac{p\alpha}{6}.$$

Therefore, Lemma 3.1 follows from the following two Claims.

Claim 4.2. If function $f(a_1, a_2, \ldots, a_p)$ reaches the maximum at $(a_1, a_2, \ldots, a_p)$ under the constraints $\sum_{i=1}^{p} a_i = 1; a_i \geq 0$, then $a_1 = a_2 = \cdots = a_p = \frac{1}{p}$.

The proof of Claim 4.2 will be given later.

Claim 4.3. If function $g(a_1, a_2, \ldots, a_p)$ reaches the maximum at $(a_1, a_2, \ldots, a_p)$ under the constraints $\sum_{i=1}^{p} a_i = 1; a_i \geq 0$, then $a_1 = a_2 = \cdots = a_p = \frac{1}{p}$.

***Proof of Claim* 4.3.** Suppose that $g(a_1, a_2, \ldots, a_p)$ reaches the maximum at $(a_1, a_2, \ldots, a_p)$. We first note that $q = |\{i : a_i > 0\}| \geq 3$. If $q = 1$, then by a direct calculation, $g(1, 0, 0, \ldots, 0) \leq g(\frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p})$ when $p \geq 3$. If $q = 2$, without loss of generality, assume that $a_1 > 0$ and $a_2 > 0$, then it is not difficult to show that

$$g\left(a_1, a_2, 0, \ldots, 0\right) \leq g\left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right) \leq g\left(\frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p}\right).$$

Now we are going to show that $a_1 = a_2 = \cdots = a_p = \frac{1}{p}$. If not, without loss of generality, assume that $a_2 > a_1$, we will show that $g(a_1+\epsilon, a_2-\epsilon, a_3, \ldots, a_p) - g(a_1, a_2, a_3, \ldots, a_p) > 0$ for small enough $\epsilon > 0$ and get a contradiction. In fact

$$\begin{aligned}
&g(a_1+\epsilon, a_2-\epsilon, a_3, \ldots, a_p) - g(a_1, a_2, a_3, \ldots, a_p)\\
&= (p-1)[(a_1+\epsilon)(a_2-\epsilon) - a_1a_2](1-a_1-a_2)\\
&+ \frac{1}{6}\left[(a_1+\epsilon)^3 + (a_2-\epsilon)^3 - a_1^3 - a_2^3\right]\\
&+ \frac{p-2}{2}\left[(a_1+\epsilon)^2(a_2-\epsilon) + (a_1+\epsilon)(a_2-\epsilon)^2 - a_1^2a_2 - a_1a_2^2\right]\\
&= (a_2-a_1)\left[p-1-\left(\frac{p}{2}+\frac{1}{2}\right)(a_1+a_2)\right]\epsilon + o(\epsilon) > 0
\end{aligned}$$

for small enough $\epsilon > 0$ since the coefficient of ϵ, $(a_2-a_1)[p-1-(\frac{p}{2}+\frac{1}{2})(a_1+a_2)]$ is positive under the assumption that $a_2 > a_1$, $p \geq 3$ and $a_1 + a_2 < 1$(since $q \geq 3$). This contradicts to the assumption that g reaches the maximum at $(a_1, a_2, \ldots, a_p)$ and Claim 4.3 follows. ■

Now we will prove Claim 4.2.

***Proof of Claim* 4.2.** We will use induction on p. If $p = 3$, it is enough to show the following Claim.

Claim 4.4.

$$\begin{aligned}
f(a_1, a_2, a_3) &= a_1a_2a_3 + \frac{1}{2}a_1^2a_2 + \frac{1}{2}a_2^2a_3 + \frac{1}{2}a_3^2a_1\\
&\leq f(1/3, 1/3, 1/3) = \frac{5}{54}
\end{aligned} \tag{11}$$

holds under the constraints $\sum_{i=1}^{3} a_i = 1$; $a_i \geq 0$.

***Proof of Claim* 4.4.** By the theory of Lagrange multipliers (see [1]), if $f(a_1, a_2, a_3)$ attains the maximum at (a_1, a_2, a_3), then either $\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = \frac{\partial f}{\partial a_3}$, i.e.,

$$a_2a_3 + \frac{1}{2}a_3^2 + a_1a_2 = a_1a_3 + \frac{1}{2}a_1^2 + a_2a_3 = a_1a_2 + \frac{1}{2}a_2^2 + a_3a_1, \tag{12}$$

or some $a_i = 0$.

If some $a_i = 0$, then it is easy to verify that $f(a_1, a_2, a_3) \leq \frac{2}{27}$.
Now assume that none of a_1, a_2, a_3 is 0, then (12) holds. In this case,

$$\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = \frac{\partial f}{\partial a_3} = a_1 \frac{\partial f}{\partial a_1} + a_2 \frac{\partial f}{\partial a_2} + a_3 \frac{\partial f}{\partial a_3} = 3f(a_1, a_2, a_3).$$

Therefore,

$$\begin{aligned} 9f(a_1, a_2, a_3) &= \frac{\partial f}{\partial a_1} + \frac{\partial f}{\partial a_2} + \frac{\partial f}{\partial a_3} \\ &= 2(a_1a_2 + a_1a_3 + a_2a_3) + \frac{a_1^2 + a_2^2 + a_3^2}{2} \\ &= \frac{1}{2} + (a_1a_2 + a_2a_3 + a_1a_3) \\ &\leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

This implies that $f(a_1, a_2, a_3) \leq \frac{5}{54} = f(1/3, 1/3, 1/3)$ and completes the proof of Claim 4.4. ■

Now let us apply the induction on p and continue the proof of Claim 4.2. Suppose that $f(a_1, \ldots, a_p)$ has the maximum at $(a_1, \ldots, a_p)$. If some $a_i = 0$, say $a_p = 0$, then by induction assumption, $f(a_1, \ldots, a_{p-1}, 0) \leq \frac{1}{6}(1 - \frac{3}{p-1} + \frac{5}{(p-1)^2}) < \frac{1}{6}(1 - \frac{3}{p} + \frac{5}{p^2}) = f(1/p, 1/p, \ldots, 1/p)$. Therefore, each $a_i > 0$ and $\frac{\partial f}{\partial a_1} = \frac{\partial f}{\partial a_2} = \cdots = \frac{\partial f}{\partial a_p}$. By a direct calculation, for each $i, 1 \leq i \leq p$,

$$\frac{\partial f}{\partial a_i} = \sum_{1 \leq j < k \leq p; j,k \neq i} a_j a_k + a_i a_{i+1} + \frac{a_{i-1}^2}{2},$$

where all subscripts here are modulo p. Then for each $i, 1 \leq i \leq p$,

$$\frac{\partial f}{\partial a_i} = \sum_{i=1}^{p} a_i \frac{\partial f}{\partial a_i} = 3f(a_1, \ldots, a_p).$$

Therefore,

$$\begin{aligned} 3pf(a_1, \ldots, a_p) &= \sum_{i=1}^{p} \frac{\partial f}{\partial a_i} \\ &= (p-2) \sum_{1 \leq i < j \leq p} a_i a_j + \sum_{i=1}^{p} \frac{a_i^2}{2} + \sum_{i=1}^{p} a_i a_{i+1}. \end{aligned} \tag{13}$$

If $p \geq 5$, then we apply $a_i a_{i+1} \leq \frac{a_i^2 + a_{i+1}^2}{2}$ to the above inequality and obtain that

$$\begin{aligned} 3pf(a_1, \ldots, a_p) &\leq (p-2) \sum_{1 \leq i < j \leq p} a_i a_j + \sum_{i=1}^{p} \frac{3a_i^2}{2} \\ &= \frac{3}{2} + (p-5) \sum_{1 \leq i < j \leq p} a_i a_j \\ &\leq \frac{3}{2} + (p-5) \frac{\binom{p}{2}}{p^2} = \frac{p^2 - 3p + 5}{2p}. \end{aligned}$$

Therefore,

$$f(a_1, \ldots, a_p) \leq \frac{1}{6} \left(1 - \frac{3}{p} + \frac{5}{p^2}\right) = f(1/p, 1/p, \ldots, 1/p).$$

If $p = 4$, then (13) is equivalent to

$$\begin{aligned} 12f(a_1, a_2, a_3, a_4) &= 2 \sum_{1 \leq i < j \leq 4} a_i a_j + \sum_{i=1}^{4} \frac{a_i^2}{2} + (a_1 a_2 + a_2 a_3 + a_3 a_4 + a_4 a_1) \\ &= \frac{1}{2} + \sum_{1 \leq i < j \leq 4} a_i a_j + (a_1 a_2 + a_2 a_3 + a_3 a_4 + a_4 a_1) \\ &\stackrel{\text{def}}{=} h(a_1, a_2, a_3, a_4). \end{aligned}$$

It is enough to show that

$$h(a_1, a_2, a_3, a_4) \leq h(1/4, 1/4, 1/4, 1/4) = \frac{9}{8}. \tag{14}$$

In fact, $h(a_1, a_2, a_3, a_4)$ has the maximum either at some $a_i = 0$ or satisfy

$$\frac{\partial h}{\partial a_1} = \frac{\partial h}{\partial a_2} = \frac{\partial h}{\partial a_3} = \frac{\partial h}{\partial a_4}.$$

By a direct calculation, the above equation implies that $a_1 = a_2 = a_3 = a_4$.

If $|\{i : a_i = 0, 1 \leq i \leq 4\}| = 3$ or 2, then (14) is clearly true. If one of a_i is 0, without loss of generality, assuming that $a_4 = 0$, then

$$\begin{aligned} h(a_1,a_2,a_3,0) &= \frac{1}{2}+2(a_1a_2+a_2a_3)+a_1a_3 \le \frac{1}{2}+2a_2(1-a_2)+\frac{(1-a_2)^2}{4} \\ &= -\frac{7}{4}\left(a_2-\frac{3}{7}\right)^2+\frac{15}{14}<\frac{9}{8}. \end{aligned}$$

The proof of Claim 4.2 is completed. ■

5. Proof of Theorem 1.6

Theorem 1.6 extends Theorem 1.5 for the case $l=4$ to every integer $r\ge 4$. The proof is based on an extension of the 4-graph $G^*(l,t)$ in Section 3 for the case $l=4$.

Suppose that $\frac{23r!}{3r^r}$ is a jump for $r\ge 4$. In view of Lemma 2.5, there exists a finite collection $\mathcal{F}$ of r-graphs satisfying the following:

(i) $\lambda(F)>\frac{23}{3r^r}$ for all $F\in\mathcal{F}$, and

(ii) $\frac{23r!}{3r^r}$ is a threshold for $\mathcal{F}$.

Set $k_0=\max_{F\in\mathcal{F}}|V(F)|$. Let $\sigma_0=c_0(4)$ be the number defined as in Section 3. Let $r=4$ in Lemma 2.6 and $t_0(k_0,\sigma_0)$ be given as in Lemma 2.6. Take an integer $t>\max(t_0,t_1)$, where t_1 is the number from (3). Now define $G^*(4,t)$ (i.e., $l=4$) the same way as in Section 3. with the new k_0. For simplicity, we simply write $G^*(4,t)$ as $G(t)$.

Since Theorem 1.5 holds, we may assume that $r\ge 5$. Based on the 4-graph $G(t)$, we construct an r-graph $G^{(r)}(t)$ on r pairwise disjoint sets $V_1,V_2,V_3,V_4,V_5,\dots,V_r$, each of cardinality t. The edge set of $G^{(r)}(t)$ consists of all r-subsets in the form of $\{u_1,u_2,u_3,u_4,u_5,\dots,u_r\}$, where $\{u_1,u_2,u_3,u_4\}$ is an edge in $G(t)$ and for each $j, 5\le j\le r, u_j\in V_j$. Notice that

$$|E(G^{(r)}(t))|=t^{r-4}|E(G(t))|. \tag{15}$$

Take $l=4$ in (3), we get

$$|E(G(t))|\ge\frac{23}{3}t^4+\frac{c_0(l)t^3}{2}. \tag{16}$$

Therefore,

$$\begin{aligned} \lambda(G^{(r)}(t)) &\ge \frac{|E(G^{(r)}(t))|}{(rt)^r} \\ &\overset{(15),(16)}{\ge} \frac{23}{3r^r}+\frac{c_0(l)}{2r^rt}. \end{aligned}$$

Similar to the case that Theorem 1.5 follows from Lemma 3.1, Theorem 1.6 follows from the following Lemma.

Lemma 5.1. *Let $M^{(r)}$ be a subgraph of $G^{(r)}(t)$ with $|V(M^{(r)})| \le k_0$. Then*

$$\lambda(M^{(r)}) \le \frac{23}{3r^r} \tag{17}$$

holds.

***Proof of Lemma* 5.1.** By Fact 2.1, we may assume that $M^{(r)}$ is an induced subgraph of $G^{(r)}(t)$. Let $M^{(4)}$ be the 4-graph defined on $\cup_{i=1}^4 V_i$ by taking the edge set to be $\{e \cap (\cup_{i=1}^4 V_i)$, where e is an edge of the r-graph $M^{(r)}\}$. Note that $|V(M^{(4)})| \le |V(M^{(r)})| \le k_0$. Let $\vec{\xi}$ be an optimal vector for $\lambda(M^{(r)})$. Define $U_i = V(M) \cap V_i$ for $1 \le i \le r$. Let a_i be the sum of the weights in $U_i, 1 \le i \le r$ respectively. Let $\vec{\xi^{(4)}}$ be the restriction of $\vec{\xi}$ on $V(M^{(4)})$. In view of the relationship between $M^{(r)}$ and $M^{(4)}$, we have

$$\lambda(M^{(r)}) = \lambda(M^{(4)}, \vec{\xi^{(4)}}) \times \prod_{i=5}^{r} a_i. \tag{18}$$

Applying Lemma 3.1(take $l = 4$ there) with the constraints replaced by $\sum_{i=1}^4 a_i = 1 - \sum_{i=5}^r a_i$, we obtain that

$$\lambda\left(M^{(4)}, \vec{\xi^{(4)}}\right) \le \frac{1}{24}\frac{23}{32}\left(1 - \sum_{i=5}^{r} a_i\right)^4.$$

Therefore,

$$\lambda\left(M^{(r)}\right) \le \frac{1}{24}\frac{23}{32}\left(1 - \sum_{i=5}^{r} a_i\right)^4 \prod_{i=5}^{r} a_i.$$

Since geometric mean is no more than arithmetic mean, we obtain that

$$\lambda(M^{(r)}) \le \frac{1}{24}\frac{23}{32}4^4\frac{1}{r^r} = \frac{23}{3r^r}.$$

This completes the proof of Lemma 5.1. ■

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