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ERDŐS REGULAR GRAPHS OF EVEN DEGREE*

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Abstract

In 1960, Dirac put forward the conjecture that r-connected 4-critical graphs exist for every $r \geq 3$. In 1989, Erdős conjectured that for every $r \geq 3$ there exist r-regular 4-critical graphs. A method for finding r-regular 4-critical graphs and the numbers of such graphs for $r \leq 10$ have been reported in [6, 7]. Results of a computer search for graphs of degree $r = 12, 14, 16$ are presented. All the graphs found are both r-regular and r-connected.

Keywords: vertex coloring, 4-critical graph, circulant, regular graph, vertex connectivity.

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1. Introduction

A simple graph is 4-*critical* if it is 4-chromatic and removing any of its edges leads to a 3-chromatic graph. Erdős conjectured that for every $r \geq 3$ there exist r-regular 4-critical graphs [8]. Dirac posed the conjecture that vertex r-connected 4-critical graphs exist for every $r \geq 3$ [3, 4]. Regular graphs

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satisfying the conjectures of Erdős and Dirac will be called *Erdős* and *Dirac graphs*, respectively.

It follows from the theorem of Brooks [1] that K_4 is the only 3-regular 4-critical graph. Various constructions of 4-regular 4-critical graphs were presented in works [2, 9, 10, 12, 14, 16, 17, 18, 24]. An infinite family of 5-regular 4-critical graphs was constructed in [13]. More detailed information on critical graphs and related topics can be found in the book [15]. Examples of regular 4-critical graphs of degree 4, 6, 8 and 10 have been recently reported in [5, 6, 7, 21]. In this paper, we describe results of a computer search for graphs of degree $r = 12, 14, 16$ that are both Erdős and Dirac graphs.

2. A THEORETICAL BASIS

For positive integers $1 \leq a_0 < a_1 < a_2 < \cdots < a_k \leq n/2$, denote by $C(n;\ a_0, a_1, \ldots, a_k)$ the graph having the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{ij\ :\ |\, i - j\,| \equiv a_0,\ a_1, \ldots, a_{k-1},\ \text{or}\ a_k \pmod{n}\}$. Such graphs are known as *circulants*. Their edges defined by a_i are called *a_i-edges*. It is clear that a circulant is a regular vertex-transitive graph of degree $2k+2$ if $a_k \neq n/2$, and of degree $2k+1$, otherwise. As an illustration, the structure of circulant $C(97;\ 1, 23, 38)$ is shown in Figure 1.

A circulant is called *proper* if $a_0 = 1$ and $(n, a_i) = 1$ for every $i = 1, 2, \ldots, k$ where (a, b) is the greatest common divisor of a and b. Each proper circulant can be represented as the union of $k + 1$ Hamiltonian cycles spanned by its a_i-edges for $i = 0, 1, \ldots, k$ (we call them *a_i-cycles*). The Hamiltonian 1-cycle $12 \ldots n$ is the *main cycle* of a circulant. Denote by A^o and A^e the subsets of all odd and even elements of the set $A = \{a_1, a_2, \ldots, a_k\}$, respectively. Since a proper circulant of even order is always bipartite, only circulants of odd order will be considered. Let a be any element of $\{1, 2, \ldots, n-1\}$ for which $(n, a) = 1$ holds. Define the function $r_{n,a}(b) = \min\{r \geq 0 \mid ra \equiv \pm b \pmod{n}\}$ for $b \in \{0, 1, \ldots, n-1\}$. It is clear that $\{r_{n,a}(b) \mid 0 \leq b \leq n-1\} = \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

A proper circulant $C(n;\ 1, a_1, \ldots, a_k)$ is called *normal* if

(a) $n \equiv 1 \pmod{6}$ and $a_i \equiv 2 \pmod{3}$ for every $i \in \{1, 2, \ldots, k\}$, and

(b) $r_{n,a}(b) \equiv 2 \pmod{3}$ for every $a \in A$ and $b \in (A \cup \{1\}) \setminus \{a\}$.

It is easy to verify that if we consider the a_1-cycle in a proper circulant $C(n;\ 1, a_1, \ldots, a_k)$ as the main cycle, then we obtain the circulant

$C(n;\ r_{n,a_1}(1), 1, r_{n,a_1}(a_2), \ldots, r_{n,a_1}(a_k))$ which is just another representation of the initial circulant. Such a representation is called an *inversion* of the circulant $C(n;\ 1, a_1, \ldots, a_k)$. Using a_i-cycle for $i = 1, 2, 3, \ldots, k$ as the main cycle, one can obtain k inversions of the initial circulant. For instance, the circulant $C(97;\ 1, 23, 38)$ has inversions $C(97;\ 38, 1, 11)$ and $C(97;\ 23, 44, 1)$. There exist circulants for which all their inversions coincide, as happens for $C(13;\ 1, 5)$ and $C(289;\ 1, 38, 110, 134)$.

It follows from the next lemma that every normal 4-chromatic circulant is 4-critical. Lemmas 1–3 have been proved in [6, 7, 21].

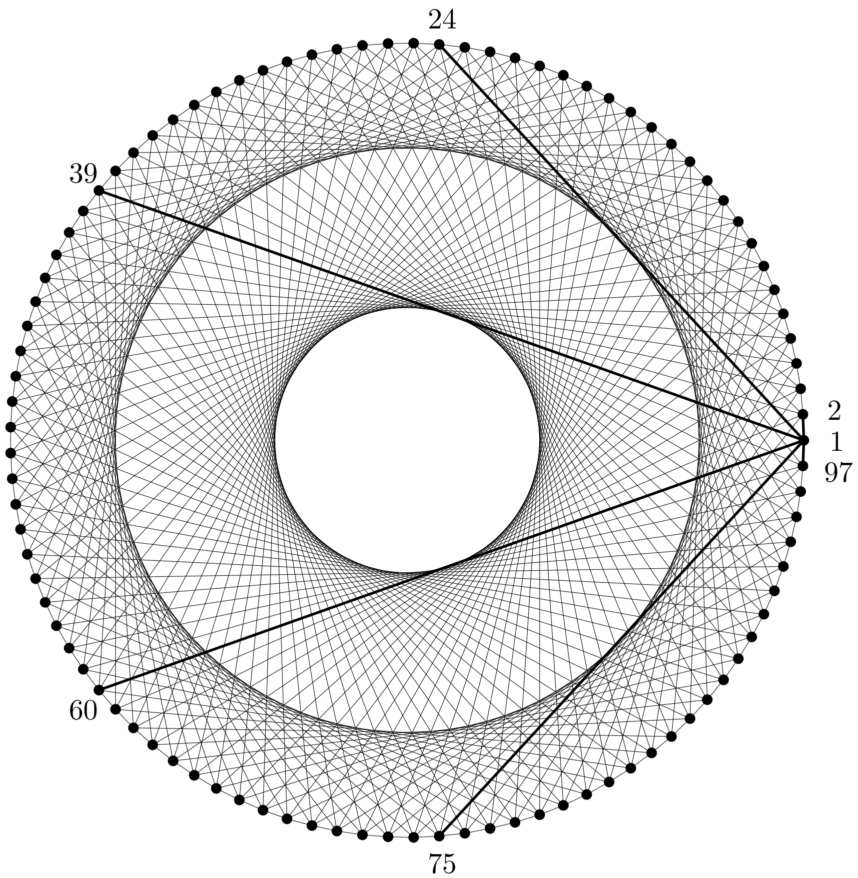

Figure 1. Circulant $C(97;\ 1, 23, 38)$.

Lemma 1. *If G is a normal circulant, then for every edge e the graph $G \setminus e$ is 3-chromatic.*

Suppose that $C(n;\ 1, a_1, \dots, a_k)$ is a 3-chromatic circulant. Denote by $f_i \in \{1,2,3\}$ the color of the vertex i in some proper 3-coloring, $i = 1, 2, \dots, n$. Extend the color sequence f in both directions using the rule $f_{i+mn} = f_i$ for every integer m and $i = 1, 2, \dots, n$. Then we obtain an n-periodic infinite word over the alphabet $\{1,2,3\}$ having the property $f_i \neq f_j$ if $|i-j| \in A \cup \{1\}$. The vertex i is called *outer* if $f_{i-1} \neq f_{i+1}$ and *inner*, otherwise. Denote by $c = (c_1, c_2, \dots, c_s)$ the subsequence of indices of all outer vertices. A coloring f (possibly not proper) is called *periodic* if $f_i \neq f_{i+1}$ and $c_{j+1} - c_j$ is odd for every i, j. This means that the number of inner vertices between any two consecutive outer vertices c_i, c_{i+1} in a periodic coloring is even and equals, say, $2l_i$. In other words, every maximal subword induced by any two colors has an even length. A 3-chromatic circulant $C(n;\ 1, a_1, \dots, a_k)$ is *periodic* if all of its proper 3-colorings are periodic.

There are several known sufficient conditions for a 3-chromatic circulant to be periodic. They are collected in the following lemma proved in [7].

Lemma 2. *A 3-chromatic circulant $C(n;\ 1, a_1, \dots, a_k)$ is periodic if there are some p, q and r (possibly some of them are equal) such that*

(1) $a_p = a_q + 3$, *or*
(2) $a_p + a_q - 2 = a_r$, *or*
(3) $a_p + a_q = n - 3$, *or*
(4) $a_p + a_q + a_r = n + 2$.

It should be noted that a periodic circulant can have a non-periodic inversion.

The next lemma provides the necessary and sufficient conditions for 3-colorability of periodic circulants (for proof of this lemma see [7]).

Lemma 3. *A circulant $C(n;\ 1, a_1, \dots, a_k)$ has a proper periodic 3-coloring if and only if there exists a nonnegative integer t such that*

(1) *for every $a \in A^o$ there exists a nonnegative integer $m_a \leq \lceil \frac{a-5}{6} \rceil$ such that*

$$n \geq 6at + 3a - 6m_a n \geq -n, \quad \text{and}$$

(2) *for every* $a \in A^e$ *there exists a nonnegative integer* $m_a \le \lceil \frac{a-8}{6} \rceil$ *such that*

$$4n \ge 6at + 3a - 6m_a n \ge 2n.$$

It follows from Lemmas 1–3 that a normal circulant, which satisfies the conditions of Lemma 2 but does not satisfy the conditions of Lemma 3, must be 4-critical, i.e., an Erdős graph. Such a circulant is also a Dirac graph due to the result of Mader and Watkins that the vertex connectivity of every connected vertex-transitive graph without K_4 is equal to its maximum degree [19, 20, 22, 23].

Given a circulant $C(n;\ 1, a_1, \dots, a_k)$, the conditions of Lemmas 1-3 are not difficult to verify. But how to find normal circulants? A natural idea is to write a system of Diophantine equations of the variables $n, a_1, a_2, \dots, a_k$ corresponding to the conditions of the normality of a circulant. This technique has been applied for obtaining 6-, 8-, and 10-regular circulants reported in [6, 7]. However, this approach is not practically suitable for large values of n and k because the system becomes very difficult to solve. In the next section we present a modified technique for searching for normal circulants with many vertices.

3. Search Method

The main idea of the method is to reduce the search for normal circulants to finding cliques in an auxiliary graph H_n.

Let $n \equiv 1 \pmod 6$ be given, and we want to find all normal circulants on n vertices. A number $v \in \{2, 3, \dots, (n-1)/2\}$ is a vertex of H_n if and only if $(n, v) = 1$; $v \equiv 2 \pmod 3$; and $r_{n,v}(1) \equiv 2 \pmod 3$. Two vertices u and v of H_n are connected by an edge if and only if $r_{n,u}(v) \equiv 2 \pmod 3$ and $r_{n,v}(u) \equiv 2 \pmod 3$. The method is based on the following lemma.

Lemma 4. *A circulant* $C(n;\ 1, a_1, a_2, \dots, a_k)$ *is normal if and only if* $a_1, a_2, \dots, a_k$ *are vertices of* H_n *and they induce a clique in* H_n.

Lemma 4 follows immediately from the definitions.

Let K be a clique of H_n induced by the vertices $a_1, a_2, \dots, a_k$. Then the circulant $C(n;\ 1, a_1, a_2, \dots, a_k)$ is called the *corresponding* circulant to K. In order to search for all 4-critical normal circulants on n vertices, it is sufficient to find all maximal cliques in H_n. It is known that this type of problem is generally not polynomially solvable [11]. Fortunately, the order and the degree of H_n are quite small even for large values of n. The procedure *FindNormalCirculants* presented below finds all normal circulants on n vertices. Denote by $N_H(v)$ the neighborhood of a vertex v in a graph H.

procedure FindNormalCirculants (n: the order of a graph, $n \equiv 1 \pmod 6$);
begin
  $V := \emptyset$; {vertices of H_n}
  for all $v \in \{2, 3, \ldots, (n-1)/2\}$ **do**
    if $(n, v) = 1$ & $v \equiv 2 \pmod 3$ & $r_{n,v}(1) \equiv 2 \pmod 3$ **then**
      $V := V \cup \{v\}$;
  List $:= \emptyset$; {adjacency list of H_n}
  for all $\{u, v\} \subset V \times V$ **do**
    if $r_{n,u}(v) \equiv 2 \pmod 3$ & $r_{n,v}(u) \equiv 2 \pmod 3$ **then**
      begin $List_u := List_u \cup \{v\}$; $List_v := List_v \cup \{u\}$; **end;**

  for all $v \in V$ **do** AddVertexToClique (H_n, $\emptyset$, v);
end;

The procedure *AddVertexToClique* finds all maximal by inclusion cliques in a given graph. A simple recursion version of this procedure is shown below. It tries to increase the current clique K of the current graph H by adding a new vertex v.

procedure AddVertexToClique (H: graph; K: clique; v: vertex);
begin
  $K := K \cup \{v\}$;
  if $N_H(v) = \emptyset$ **then** CheckLemmas (K) {K is a clique} **else**
    for all $u \in N_H(v)$ **do** AddVertexToClique ($\langle N_H(v) \rangle$, K, u);
end;

The procedure *CheckLemmas* verifies the conditions of Lemma 2 and Lemma 3 for all inversions of the circulant corresponding to the clique K. Of course, the symmetry of circulants and other similar properties should be used for reducing calculations.

The presented algorithm finds all normal circulants of order n irrespect of their degree r, i.e., none of normal circulants has been skipped. Therefore, there are no 4-critical normal circulants with up to 53000 vertices except those reported in [7] and in the Appendix.

Since the number of cliques in H_n becomes very large when n increases, our approach is limited by available computing tools. Therefore, in our opinion, application of other maximal clique enumeration algorithms can not essentially help.

4. Results of a Computer Search

As a result of the described approach, new Erdős and Dirac graphs have been obtained.

Theorem. *The circulants listed in the Appendix are r-regular r-connected 4-critical graphs for $r = 12, 14, 16$ (45, 36 and 6 graphs, respectively), i.e., they are Erdős and Dirac graphs.*

Some of the obtained circulants have the same order. One can check that they are non-isomorphic since they have different numbers of small cycles.

By canonical representation of a circulant we mean the lexicographic minimum among all its inversions. For every circulant, the inversion meeting Lemma 2 is presented. The corresponding equation is written after a circulant and its parameters are marked by bold font.

There are no other normal circulants on at most 53000 vertices which satisfy the conditions of Lemma 2 but do not satisfy the conditions of Lemma 3. Nevertheless, we obtain many normal circulants (approx. 300 graphs) for which both Lemmas 2 and 3 do not hold. This means that such circulants have no proper periodic 3-coloring but may possibly have a non-periodic one. Their chromatic numbers should be found by other methods. A complete list of these "suspicious" normal circulants is available from the authors. It is unknown whether the list of sufficient conditions of Lemma 2 are complete. Therefore, some suspicious normal circulants might be periodic. It is possible that graphs of the list may provide a new lower bound for the order of normal circulants.

Appendix. r-regular 4-critical graphs for $r = 12, 14, 16$.

$r = 12$

1. (**4153**; 1, 53, 386, 431, 737, **2075**)	$2075 + 2075 = 4153 - 3$
2. (4153; 1808, **1646**, **1649**, 1439, 1046, 1)	$1649 = 1646 + 3$
3. (4453; 791, 938, 1, **1910**, **80**, **1832**)	$80 + 1832 = 1910 + 2$
4. (**4567**; 1286, **2282**, 1196, 755, 1, 665)	$2282 + 2282 = 4567 - 3$
5. (4837; **104**, **206**, 1370, 2207, 2189, 1)	$104 + 104 = 206 + 2$
6. (5557; **2486**, 1, **2489**, 1175, 1289, 836)	$2489 = 2486 + 3$
7. (5629; **2174**, 2642, 1, 626, 404, **2177**)	$2177 = 2174 + 3$
8. (5629; 2504, **401**, 1, 1124, 944, **404**)	$404 = 401 + 3$
9. (**5725**; 1, 107, 131, 476, **593**, **2567**)	$593 + 2567 + 2567 = 5725 + 2$
10. (5725; 2036, **302**, 1, **602**, 107, 1439)	$302 + 302 = 602 + 2$
11. (5893; 791, 1, 587, **1892**, **1889**, 2237)	$1892 = 1889 + 3$
12. (5953; 1, 20, **719**, **857**, 1016, **1574**)	$719 + 857 = 1574 + 2$
13. (6019; 2312, 2663, 1, **233**, **230**, 464)	$233 = 230 + 3$
14. (6451; 2228, 524, 1, **2609**, 695, **2612**)	$2612 = 2609 + 3$
15. (6913; **2591**, **1544**, **1049**, 2309, 2013, 1)	$1544 + 1049 = 2591 + 2$
16. (8011; **914**, 1, **917**, 3017, 149, 1754)	$917 = 914 + 3$
17. (8731; 2834, 1, **341**, 3686, **680**, 2300)	$341 + 341 = 680 + 2$
18. (**8917**; **1043**, 2897, 1, **3572**, 980, **4304**)	$1043 + 3572 + 4304 = 8917 + 2$
19. (9217; 1, **266**, **530**, 3521, 3956, 4217)	$266 + 266 = 530 + 2$
20. (9805; **4724**, 2789, 1, 671, 2837, **2363**)	$2363 + 2363 = 4724 + 2$
21. (**10105**; 3551, 4319, 1, **3743**, 2753, **2621**)	$3743 + 3743 + 2621 = 10105 + 2$
22. (11131; **221**, 2711, 3314, 1, **2186**, **2405**)	$221 + 2186 = 2405 + 2$
23. (**11377**; 3437, 1214, **4040**, **2048**, 1, **5291**)	$4040 + 2048 + 5291 = 11377 + 2$
24. (11581; 1, **833**, 1037, **1664**, 3608, 4754)	$833 + 833 = 1664 + 2$
25. (12025; 1727, 1, **566**, **563**, 3632, 4586)	$566 = 563 + 3$
26. (12961; 1397, 1, 3008, **2774**, 2759, **5546**)	$2774 + 2774 = 5546 + 2$
27. (13093; **4742**, **542**, 5699, 1109, 1, **4202**)	$542 + 4202 = 4742 + 2$
28. (13687; **419**, 1, 1670, **4511**, **4094**, 3710)	$419 + 4094 = 4511 + 2$
29. (14185; 1, 11, **3233**, 3281, **6464**, 6632)	$3233 + 3233 = 6464 + 2$
30. (14761; 1799, 1, **962**, **959**, 4280, 5522)	$962 = 959 + 3$
31. (**15325**; **6842**, **7433**, **1052**, 3902, 1, 6476)	$6842 + 7433 + 1052 = 15325 + 2$
32. (16051; 2939, 5627, **2096**, 1, 3293, **4190**)	$2096 + 2096 = 4190 + 2$
33. (16189; 2420, 1, **3503**, 2483, **3968**, **7469**)	$3503 + 3968 = 7469 + 2$
34. (16519; 1, **215**, **428**, 1385, 1904, 7094)	$215 + 215 = 428 + 2$
35. (19999; 830, 1, **1655**, 7346, **6032**, **7685**)	$1655 + 6032 = 7685 + 2$
36. (21997; 1856, 1, 8393, **2669**, 7403, **5336**)	$2669 + 2669 = 5336 + 2$
37. (26719; **5018**, 3617, 3224, 1, 12350, **2510**)	$2510 + 2510 = 5018 + 2$
38. (27349; 1, **278**, **554**, 5912, 6632, 10082)	$278 + 278 = 554 + 2$
39. (29779; **13721**, **7241**, **6482**, 12590, 1, 13808)	$7241 + 6482 = 13721 + 2$
40. (32161; **635**, **7433**, **6800**, 12692, 3464, 1)	$635 + 6800 = 7433 + 2$
41. (34213; 11810, **6974**, **4358**, **2618**, 1, 15434)	$4358 + 2618 = 6974 + 2$
42. (35347; **9818**, **7619**, 12401, 1, 386, **2201**)	$7619 + 2201 = 9818 + 2$
43. (36661; 1, **221**, **224**, 2198, 11192, 14057)	$224 = 221 + 3$
44. (41071; 9800, **182**, 17618, 1, **179**, 7640)	$182 = 179 + 3$
45. (43177; **16622**, 6320, **1328**, 4463, **15296**, 1)	$1328 + 15296 = 16622 + 2$

$r = 14$

1. (**14275**; 1862, 1, **3683**, 3221, 6953, 3239, **5297**) — $3683 + 5297 + 5297 = 14275 + 2$
2. (17785; 5813, **3461**, 1, 1802, **3464**, 7253, 1817) — $3464 = 3461 + 3$
3. (17971; **2852**, 5633, 5432, **2855**, 2435, 3374, 1) — $2855 = 2852 + 3$
4. (17971; 5960, 1, 3374, **2855**, **2852**, 5432, 2435) — $2855 = 2852 + 3$
5. (22075; **4568**, 2141, 5147, **9134**, 7052, 4853, 1) — $4568 + 4568 = 9134 + 2$
6. (22207; 3305, 8645, 1, 4967, **4343**, **8684**, 137) — $4343 + 4343 = 8684 + 2$
7. (22327; 1, 140, **1001**, **1004**, 4853, 6005, 6281) — $1004 = 1001 + 3$
8. (25411; 5504, 1, **6746**, 608, **6749**, 3509, 1052) — $6749 = 6746 + 3$
9. (26599; 1, 146, **1070**, **2138**, 4262, 4748, 7652) — $1070 + 1070 = 2138 + 2$
10. (27619; **11957**, **6062**, 1, **5897**, 12092, 878, 5480) — $6062 + 5897 = 11957 + 2$
11. (**30487**; **14519**, 9911, 1, 7298, 2861, 3374, **7985**) — $14519 + 7985 + 7985 = 30487 + 2$
12. (31183; **692**, 5192, 5486, 13241, 1, 3998, **1382**) — $692 + 692 = 1382 + 2$
13. (32059; 1, **686**, 3140, 6836, 7790, **11498**, **12182**) — $686 + 11498 = 12182 + 2$
14. (32107; 9896, 1, **4409**, 11036, **1268**, **3143**, 6524) — $1268 + 3143 = 4409 + 2$
15. (32737; 5615, **695**, 1, 599, **11504**, 1145, **12197**) — $695 + 11504 = 12197 + 2$
16. (32737; 13916, **10919**, 1, 1844, 13847, **3653**, **14570**) — $10919 + 3653 = 14570 + 2$
17. (32821; 9017, **4607**, 1, 10379, 2762, **9212**, 1046) — $4607 + 4607 = 9212 + 2$
18. (33493; 4331, 1, **10337**, **2537**, 15392, 5882, **12872**) — $10337 + 2537 = 12872 + 2$
19. (33937; 2282, **1403**, **980**, 13607, 1, 8924, **2381**) — $1403 + 980 = 2381 + 2$
20. (34213; **14777**, **14774**, 3335, 13910, 3593, 16007, 1) — $14777 = 14774 + 3$
21. (34483; 11594, **665**, 7145, 1, 1079, 6602, **1328**) — $665 + 665 = 1328 + 2$
22. (35287; **3572**, 1, **11036**, 8990, **14606**, 7376, 3845) — $3572 + 11036 = 14606 + 2$
23. (36259; 1, 137, **2054**, **4106**, 6416, 15989, 16289) — $2054 + 2054 = 4106 + 2$
24. (36697; **14987**, 7826, 1, **11645**, **3344**, 16943, 17759) — $11645 + 3344 = 14987 + 2$
25. (37687; 1, **224**, 410, 5243, 13838, **16103**, **16325**) — $224 + 16103 = 16325 + 2$
26. (38629; 7967, 2267, **14222**, 14645, **3488**, **17708**, 1) — $14222 + 3488 = 17708 + 2$
27. (38953; **16283**, 11912, 1, 15281, 2906, **12785**, **3500**) — $12785 + 3500 = 16283 + 2$
28. (**39271**; 1, **1118**, 8909, 9389, 13232, **19031**, **19124**) — $1118 + 19031 + 19124 = 39271 + 2$
29. (39493; **2579**, 19481, 10250, 8753, **2582**, 19409, 1) — $2582 = 2579 + 3$
30. (**40099**; **10526**, 1, 14642, **19382**, **10193**, 19322, 10286) — $10526 + 19382 + 10193 = 40099 + 2$
31. (40345; **572**, **4538**, 1, **3968**, 13244, 7877, 6386) — $572 + 3968 = 4538 + 2$
32. (40687; **2186**, 5138, 1, 11300, 8162, 389, **2183**) — $2186 = 2183 + 3$
33. (40711; 1, 458, **746**, **3956**, **4700**, 6998, 10073) — $746 + 3956 = 4700 + 2$
34. (40771; 12710, **3305**, **7814**, 14786, **11117**, 1, 17357) — $3305 + 7814 = 11117 + 2$
35. (42397; **12983**, 19823, **2288**, **15269**, 3551, 8756, 1) — $12983 + 2288 = 15269 + 2$
36. (43621; **10769**, **14867**, 1, 11048, 21566, **4100**, 830) — $10769 + 4100 = 14867 + 2$

$r = 16$

1. (19897; 4382, 6887, **4052**, 1, **3824**, 4151, 326, **230**) — $3824 + 230 = 4052 + 2$
2. (**28279**; 13604, 10049, 8408, 1, **10868**, **6545**, 5660, 6962) — $10868 + 10868 + 6545 = 28279 + 2$
3. (31339; 6095, 13025, **7328**, 5018, 1355, **3665**, 1, 6929) — $3665 + 3665 = 7328 + 2$
4. (34987; 11339, 1, 11639, 14141, **8015**, 1793, **8012**, 6104) — $8015 = 8012 + 3$
5. (41779; 16415, **10130**, 3539, 7694, 4022, **10133**, 1, 12485) — $10133 = 10130 + 3$
6. (48055; 6542, **13772**, 13697, 1, **6887**, 6701, 3704, 3113) — $6887 + 6887 = 13772 + 2$

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