# ERDŐS REGULAR GRAPHS OF EVEN DEGREE* 

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#### Abstract

In 1960, Dirac put forward the conjecture that $r$-connected 4 critical graphs exist for every $r \geq 3$. In 1989, Erdős conjectured that for every $r \geq 3$ there exist $r$-regular 4-critical graphs. A method for finding $r$-regular 4 -critical graphs and the numbers of such graphs for $r \leq 10$ have been reported in $[6,7]$. Results of a computer search for graphs of degree $r=12,14,16$ are presented. All the graphs found are both $r$-regular and $r$-connected.


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## 1. Introduction

A simple graph is 4-critical if it is 4-chromatic and removing any of its edges leads to a 3-chromatic graph. Erdős conjectured that for every $r \geq 3$ there exist $r$-regular 4-critical graphs [8]. Dirac posed the conjecture that vertex $r$-connected 4-critical graphs exist for every $r \geq 3[3,4]$. Regular graphs

[^0]satisfying the conjectures of Erdős and Dirac will be called Erdős and Dirac graphs, respectively.

It follows from the theorem of Brooks [1] that $K_{4}$ is the only 3 -regular 4 -critical graph. Various constructions of 4-regular 4-critical graphs were presented in works $[2,9,10,12,14,16,17,18,24]$. An infinite family of 5 regular 4 -critical graphs was constructed in [13]. More detailed information on critical graphs and related topics can be found in the book [15]. Examples of regular 4 -critical graphs of degree $4,6,8$ and 10 have been recently reported in $[5,6,7,21]$. In this paper, we describe results of a computer search for graphs of degree $r=12,14,16$ that are both Erdős and Dirac graphs.

## 2. A Theoretical Basis

For positive integers $1 \leq a_{0}<a_{1}<a_{2}<\cdots<a_{k} \leq n / 2$, denote by $C\left(n ; a_{0}, a_{1}, \ldots, a_{k}\right)$ the graph having the vertex set $V=\{1,2, \ldots, n\}$ and the edge set $E=\left\{i j:|i-j| \equiv a_{0}, a_{1}, \ldots, a_{k-1}\right.$, or $\left.a_{k}(\bmod n)\right\}$. Such graphs are known as circulants. Their edges defined by $a_{i}$ are called $a_{i}$-edges. It is clear that a circulant is a regular vertex-transitive graph of degree $2 k+2$ if $a_{k} \neq n / 2$, and of degree $2 k+1$, otherwise. As an illustration, the structure of circulant $C(97 ; 1,23,38)$ is shown in Figure 1.

A circulant is called proper if $a_{0}=1$ and $\left(n, a_{i}\right)=1$ for every $i=$ $1,2, \ldots, k$ where $(a, b)$ is the greatest common divisor of $a$ and $b$. Each proper circulant can be represented as the union of $k+1$ Hamiltonian cycles spanned by its $a_{i}$-edges for $i=0,1, \ldots, k$ (we call them $a_{i}$-cycles). The Hamiltonian 1-cycle $12 \ldots n$ is the main cycle of a circulant. Denote by $A^{o}$ and $A^{e}$ the subsets of all odd and even elements of the set $A=\left\{a_{1}, a_{2}, \ldots, a_{k}\right\}$, respectively. Since a proper circulant of even order is always bipartite, only circulants of odd order will be considered. Let $a$ be any element of $\{1,2, \ldots, n-1\}$ for which $(n, a)=1$ holds. Define the function $r_{n, a}(b)=\min \{r \geq 0 \mid r a \equiv \pm b(\bmod n)\}$ for $b \in\{0,1, \ldots, n-1\}$. It is clear that $\left\{r_{n, a}(b) \mid 0 \leq b \leq n-1\right\}=\left\{0,1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$.

A proper circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$ is called normal if
(a) $n \equiv 1(\bmod 6)$ and $a_{i} \equiv 2(\bmod 3)$ for every $i \in\{1,2, \ldots, k\}$, and
(b) $r_{n, a}(b) \equiv 2(\bmod 3)$ for every $a \in A$ and $b \in(A \cup\{1\}) \backslash\{a\}$.

It is easy to verify that if we consider the $a_{1}$-cycle in a proper circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$ as the main cycle, then we obtain the circulant
$C\left(n ; r_{n, a_{1}}(1), 1, r_{n, a_{1}}\left(a_{2}\right), \ldots, r_{n, a_{1}}\left(a_{k}\right)\right)$ which is just another representation of the initial circulant. Such a representation is called an inversion of the circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$. Using $a_{i}$-cycle for $i=1,2,3, \ldots, k$ as the main cycle, one can obtain $k$ inversions of the initial circulant. For instance, the circulant $C(97 ; 1,23,38)$ has inversions $C(97 ; 38,1,11)$ and $C(97 ; 23,44,1)$. There exist circulants for which all their inversions coincide, as happens for $C(13 ; 1,5)$ and $C(289 ; 1,38,110,134)$.

It follows from the next lemma that every normal 4-chromatic circulant is 4 -critical. Lemmas $1-3$ have been proved in $[6,7,21]$.


Figure 1. Circulant $C(97 ; 1,23,38)$.

Lemma 1. If $G$ is a normal circulant, then for every edge e the graph $G \backslash e$ is 3-chromatic.

Suppose that $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$ is a 3 -chromatic circulant. Denote by $f_{i} \in$ $\{1,2,3\}$ the color of the vertex $i$ in some proper 3 -coloring, $i=1,2, \ldots, n$. Extend the color sequence $f$ in both directions using the rule $f_{i+m n}=f_{i}$ for every integer $m$ and $i=1,2, \ldots, n$. Then we obtain an $n$-periodic infinite word over the alphabet $\{1,2,3\}$ having the property $f_{i} \neq f_{j}$ if $|i-j| \in$ $A \cup\{1\}$. The vertex $i$ is called outer if $f_{i-1} \neq f_{i+1}$ and inner, otherwise. Denote by $c=\left(c_{1}, c_{2}, \ldots, c_{s}\right)$ the subsequence of indices of all outer vertices. A coloring $f$ (possibly not proper) is called periodic if $f_{i} \neq f_{i+1}$ and $c_{j+1}-c_{j}$ is odd for every $i, j$. This means that the number of inner vertices between any two consecutive outer vertices $c_{i}, c_{i+1}$ in a periodic coloring is even and equals, say, $2 l_{i}$. In other words, every maximal subword induced by any two colors has an even length. A 3 -chromatic circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$ is periodic if all of its proper 3-colorings are periodic.

There are several known sufficient conditions for a 3-chromatic circulant to be periodic. They are collected in the following lemma proved in [7].

Lemma 2. $A 3$-chromatic circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$ is periodic if there are some $p, q$ and $r$ (possibly some of them are equal) such that
(1) $a_{p}=a_{q}+3$, or
(2) $a_{p}+a_{q}-2=a_{r}$, or
(3) $a_{p}+a_{q}=n-3$, or
(4) $a_{p}+a_{q}+a_{r}=n+2$.

It should be noted that a periodic circulant can have a non-periodic inversion.

The next lemma provides the necessary and sufficient conditions for 3 -colorability of periodic circulants (for proof of this lemma see [7]).

Lemma 3. $A$ circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$ has a proper periodic 3 -coloring if and only if there exists a nonnegative integer $t$ such that
(1) for every $a \in A^{o}$ there exists a nonnegative integer $m_{a} \leq\left\lceil\frac{a-5}{6}\right\rceil$ such that

$$
n \geq 6 a t+3 a-6 m_{a} n \geq-n, \quad \text { and }
$$

(2) for every $a \in A^{e}$ there exists a nonnegative integer $m_{a} \leq\left\lceil\frac{a-8}{6}\right\rceil$ such that

$$
4 n \geq 6 a t+3 a-6 m_{a} n \geq 2 n
$$

It follows from Lemmas $1-3$ that a normal circulant, which satisfies the conditions of Lemma 2 but does not satisfy the conditions of Lemma 3, must be 4-critical, i.e., an Erdős graph. Such a circulant is also a Dirac graph due to the result of Mader and Watkins that the vertex connectivity of every connected vertex-transitive graph without $K_{4}$ is equal to its maximum degree [19, 20, 22, 23].

Given a circulant $C\left(n ; 1, a_{1}, \ldots, a_{k}\right)$, the conditions of Lemmas 1-3 are not difficult to verify. But how to find normal circulants? A natural idea is to write a system of Diophantine equations of the variables $n, a_{1}, a_{2}, \ldots, a_{k}$ corresponding to the conditions of the normality of a circulant. This technique has been applied for obtaining 6-, 8-, and 10-regular circulants reported in $[6,7]$. However, this approach is not practically suitable for large values of $n$ and $k$ because the system becomes very difficult to solve. In the next section we present a modified technique for searching for normal circulants with many vertices.

## 3. Search Method

The main idea of the method is to reduce the search for normal circulants to finding cliques in an auxiliary graph $H_{n}$.

Let $n \equiv 1(\bmod 6)$ be given, and we want to find all normal circulants on $n$ vertices. A number $v \in\{2,3, \ldots,(n-1) / 2\}$ is a vertex of $H_{n}$ if and only if $(n, v)=1 ; v \equiv 2(\bmod 3)$; and $r_{n, v}(1) \equiv 2(\bmod 3)$. Two vertices $u$ and $v$ of $H_{n}$ are connected by an edge if and only if $r_{n, u}(v) \equiv 2(\bmod 3)$ and $r_{n, v}(u) \equiv 2(\bmod 3)$. The method is based on the following lemma.

Lemma 4. A circulant $C\left(n ; 1, a_{1}, a_{2}, \ldots, a_{k}\right)$ is normal if and only if $a_{1}, a_{2}, \ldots, a_{k}$ are vertices of $H_{n}$ and they induce a clique in $H_{n}$.

Lemma 4 follows immediately from the definitions.
Let $K$ be a clique of $H_{n}$ induced by the vertices $a_{1}, a_{2}, \ldots, a_{k}$. Then the circulant $C\left(n ; 1, a_{1}, a_{2}, \ldots, a_{k}\right)$ is called the corresponding circulant to $K$. In order to search for all 4-critical normal circulants on $n$ vertices, it is sufficient to find all maximal cliques in $H_{n}$. It is known that this type
of problem is generally not polynomially solvable [11]. Fortunately, the order and the degree of $H_{n}$ are quite small even for large values of $n$. The procedure FindNormalCirculants presented below finds all normal circulants on $n$ vertices. Denote by $N_{H}(v)$ the neighborhood of a vertex $v$ in a graph $H$.
procedure FindNormalCirculants ( $n$ : the order of a graph, $n \equiv 1(\bmod 6)$ ); begin
$V:=\emptyset ;\left\{\right.$ vertices of $\left.H_{n}\right\}$
for all $v \in\{2,3, \ldots,(n-1) / 2\}$ do
if $(n, v)=1 \& v \equiv 2(\bmod 3) \& r_{n, v}(1) \equiv 2(\bmod 3)$ then $V:=V \cup\{v\} ;$
List $:=\emptyset ;\left\{\right.$ adjacency list of $\left.H_{n}\right\}$
for all $\{u, v\} \subset V \times V$ do

$$
\text { if } r_{n, u}(v) \equiv 2(\bmod 3) \& r_{n, v}(u) \equiv 2(\bmod 3) \text { then }
$$

$$
\text { begin } \text { List }_{u}:=\text { List }_{u} \cup\{v\} ; \text { List }_{v}:=\text { List }_{v} \cup\{u\} ; \text { end; }
$$

for all $v \in V$ do AddVertexToClique $\left(H_{n}, \emptyset, v\right)$;
end;
The procedure AddVertexToClique finds all maximal by inclusion cliques in a given graph. A simple recursion version of this procedure is shown below. It tries to increase the current clique $K$ of the current graph $H$ by adding a new vertex $v$.
procedure AddVertexToClique ( $H$ : graph; $K$ : clique; $v$ : vertex);
begin
$K:=K \cup\{v\} ;$
if $N_{H}(v)=\emptyset$ then CheckLemmas $(K)\{K$ is a clique $\}$ else for all $u \in N_{H}(v)$ do AddVertexToClique $\left(\left\langle N_{H}(v)\right\rangle, K, u\right)$;
end;
The procedure CheckLemmas verifies the conditions of Lemma 2 and Lemma 3 for all inversions of the circulant corresponding to the clique $K$. Of course, the symmetry of circulants and other similar properties should be used for reducing calculations.

The presented algorithm finds all normal circulants of order $n$ irrespect of their degree $r$, i.e., none of normal circulants has been skipped. Therefore, there are no 4 -critical normal circulants with up to 53000 vertices except those reported in [7] and in the Appendix.

Since the number of cliques in $H_{n}$ becomes very large when $n$ increases, our approach is limited by available computing tools. Therefore, in our opinion, application of other maximal clique enumeration algorithms can not essentially help.

## 4. Results of a Computer Search

As a result of the described approach, new Erdős and Dirac graphs have been obtained.

Theorem. The circulants listed in the Appendix are $r$-regular $r$-connected 4 -critical graphs for $r=12,14,16$ (45, 36 and 6 graphs, respectively), i.e., they are Erdös and Dirac graphs.

Some of the obtained circulants have the same order. One can check that they are non-isomorphic since they have different numbers of small cycles.

By canonical representation of a circulant we mean the lexicographic minimum among all its inversions. For every circulant, the inversion meeting Lemma 2 is presented. The corresponding equation is written after a circulant and its parameters are marked by bold font.

There are no other normal circulants on at most 53000 vertices which satisfy the conditions of Lemma 2 but do not satisfy the conditions of Lemma 3. Nevertheless, we obtain many normal circulants (approx. 300 graphs) for which both Lemmas 2 and 3 do not hold. This means that such circulants have no proper periodic 3 -coloring but may possibly have a nonperiodic one. Their chromatic numbers should be found by other methods. A complete list of these "suspicious" normal circulants is available from the authors. It is unknown whether the list of sufficient conditions of Lemma 2 are complete. Therefore, some suspicious normal circulants might be periodic. It is possible that graphs of the list may provide a new lower bound for the order of normal circulants.

Appendix. $r$-regular 4-critical graphs for $r=12,14,16$.
$r=12$

1. $(\mathbf{4 1 5 3} ; 1,53,386,431,737,2075)$
2. $(4153 ; 1808, \mathbf{1 6 4 6}, \mathbf{1 6 4 9}, 1439,1046,1)$
3. $(4453 ; 791,938,1,1910,80,1832)$
4. $(\mathbf{4 5 6 7} ; 1286, \mathbf{2 2 8 2}, 1196,755,1,665)$
5. $(4837$; 104, 206, 1370, 2207, 2189, 1)
6. $(5557 ; \mathbf{2 4 8 6}, 1,2489,1175,1289,836)$
7. $(5629 ; \mathbf{2 1 7 4}, 2642,1,626,404, \mathbf{2 1 7 7})$
8. $(5629 ; 2504,401,1,1124,944,404)$
9. (5725; 1, 107, 131, 476, 593, 2567)
10. (5725; 2036, 302, 1, 602, 107, 1439)
11. (5893; 791, 1, 587, 1892, 1889, 2237)
12. (5953; 1, 20, 719, 857, 1016, 1574)
13. (6019; 2312, 2663, 1, 233, 230, 464)
14. (6451; 2228, 524, 1, 2609, 695, 2612)
15. (6913; 2591, 1544, 1049, 2309, 2013, 1)
16. (8011; 914, 1, 917, 3017, 149, 1754)
17. (8731; 2834, 1, 341, 3686, 680, 2300)
18. (8917; 1043, 2897, 1, 3572, 980, 4304)
19. (9217; 1, 266, 530, 3521, 3956, 4217)
20. (9805; 4724, 2789, 1, 671, 2837, 2363)
21. (10105; 3551, 4319, 1, 3743, 2753, 2621)
$2075+2075=4153-3$
$1649=1646+3$
$80+1832=1910+2$
$2282+2282=4567-3$
$104+104=206+2$
$2489=2486+3$
$2177=2174+3$
$404=401+3$
$593+2567+2567=5725+2$
$302+302=602+2$
$1892=1889+3$
$719+857=1574+2$
$233=230+3$
$2612=2609+3$
$1544+1049=2591+2$
$917=914+3$
$341+341=680+2$
$1043+3572+4304=8917+2$
$266+266=530+2$
$2363+2363=4724+2$
$3743+3743+2621=10105+2$
$221+2186=2405+2$
$4040+2048+5291=11377+2$
$833+833=1664+2$
$566=563+3$
$2774+2774=5546+2$
$542+4202=4742+2$
$419+4094=4511+2$
$3233+3233=6464+2$
$962=959+3$
$6842+7433+1052=15325+2$
$2096+2096=4190+2$
$3503+3968=7469+2$
$215+215=428+2$
$1655+6032=7685+2$
$2669+2669=5336+2$
$2510+2510=5018+2$
$278+278=554+2$
22. (26719; 5018, 3617, 3224, 1, 12350, 2510)
23. (27349; 1, 278, 554, 5912, 6632, 10082)
24. (29779; 13721, 7241, 6482, 12590, 1, 13808)
25. (32161; 635, 7433, 6800, 12692, 3464, 1)
26. (34213; 11810, 6974, 4358, 2618, 1, 15434)
27. $(35347 ; \mathbf{9 8 1 8}, \mathbf{7 6 1 9}, 12401,1,386,2201)$
28. (36661; 1, 221, 224, 2198, 11192, 14057)
$7241+6482=13721+2$
$635+6800=7433+2$
$4358+2618=6974+2$
$7619+2201=9818+2$
$224=221+3$
29. (41071; 9800, 182, 17618, 1, 179, 7640)
$182=179+3$
30. (43177; 16622, 6320, 1328, 4463, 15296, 1)
```
r=14
1. (\mathbf{14275};1862, 1, 3683, 3221, 6953, 3239, 5297)
    2. (17785; 5813, 3461, 1, 1802, 3464, 7253, 1817)
    3. (17971; 2852, 5633, 5432, 2855, 2435, 3374, 1)
    4. (17971; 5960, 1, 3374, 2855, 2852, 5432, 2435)
    5. (22075; 4568, 2141, 5147, 9134, 7052, 4853, 1)
    6. (22207; 3305, 8645, 1, 4967, 4343, 8684, 137)
    7. (22327; 1, 140, 1001, 1004, 4853, 6005, 6281)
    8. (25411; 5504, 1, 6746, 608, 6749, 3509, 1052)
    9. (26599; 1, 146, 1070, 2138, 4262, 4748, 7652)
10. (27619; 11957, 6062, 1, 5897, 12092, 878, 5480)
11. (30487; 14519, 9911, 1, 7298, 2861, 3374, 7985)
12. (31183; 692, 5192, 5486, 13241, 1, 3998, 1382)
13. (32059; 1, 686, 3140, 6836, 7790, 11498, 12182)
14. (32107; 9896, 1, 4409, 11036, 1268, 3143, 6524)
15. (32737; 5615, 695, 1, 599, 11504, 1145, 12197)
16. (32737; 13916, 10919, 1, 1844, 13847, 3653, 14570)
17. (32821; 9017, 4607, 1, 10379, 2762, 9212, 1046)
18. (33493; 4331, 1, 10337, 2537, 15392, 5882, 12872)
19. (33937; 2282, 1403, 980, 13607, 1, 8924, 2381)
20. (34213; 14777, 14774, 3335, 13910, 3593, 16007, 1)
21. (34483; 11594, 665, 7145, 1, 1079, 6602, 1328)
22. (35287; 3572, 1, 11036, 8990, 14606, 7376, 3845)
23. (36259; 1, 137, 2054, 4106, 6416, 15989, 16289)
24. (36697; 14987, 7826, 1, 11645, 3344, 16943, 17759)
25. (37687; 1, 224, 410, 5243, 13838, 16103, 16325)
26. (38629; 7967, 2267, 14222, 14645, 3488, 17708, 1)
27. (38953; 16283, 11912, 1, 15281, 2906, 12785, 3500)
28. (39271; 1, 1118, 8909, 9389, 13232, 19031, 19124)
29. (39493; 2579, 19481, 10250, 8753, 2582, 19409, 1)
30. (40099; 10526, 1, 14642, 19382, 10193, 19322, 10286)
31. (40345; 572, 4538, 1, 3968, 13244, 7877, 6386)
32. (40687; 2186, 5138, 1, 11300, 8162, 389, 2183)
33. (40711; 1, 458, 746, 3956, 4700, 6998, 10073)
34. (40771; 12710, 3305, 7814, 14786, 11117, 1, 17357)
35. (42397; 12983, 19823, 2288, 15269, 3551, 8756, 1)
36. (43621; 10769, 14867, 1, 11048, 21566, 4100, 830)
3683+5297+5297=14275+2
3464=3461+3
2855=2852+3
2855=2852+3
4568+4568=9134+2
4343+4343 = 8684+2
1004=1001+3
6749=6746+3
1070+1070 = 2138+2
6062+5897 = 11957+2
14519+7985+7985=30487+2
692+692=1382+2
686+11498=12182+2
1268+3143 = 4409+2
695+11504 = 12197+2
10919+3653 = 14570 +2
4607+4607 = 9212+2
10337+2537 = 12872+2
1403+980=2381+2
14777 = 14774+3
665+665=1328+2
3572+11036 = 14606 + 2
2054+2054=4106+2
11645+3344 = 14987+2
224+16103 = 16325+2
14222+3488=17708+2
12785+3500=16283+2
1118+19031+19124=39271+2
2582=2579+3
10526 + 19382+10193 = 40099 +2
572+3968=4538+2
2186 = 2183+3
746+3956 = 4700+2
3305+7814 = 11117+2
12983+2288=15269+2
10769+4100 = 14867+2
r=16
1. (19897; 4382, 6887, 4052, 1, 3824, 4151, 326, 230) }\quad3824+230=4052+
2. (28279; 13604, 10049, 8408, 1, 10868, 6545, 5660,6962) 10868+10868+6545 = 28279+2
3. (31339; 6095, 13025, 7328, 5018, 1355, 3665, 1, 6929) }3665+3665=7328+
4. (34987; 11339, 1, 11639, 14141, 8015, 1793, 8012,6104) 8015=8012+3
5. (41779; 16415, 10130, 3539, 7694, 4022, 10133, 1, 12485) 10133=10130+3
6. (48055; 6542, 13772, 13697, 1, 6887, 6701, 3704, 3113) 6887+6887 = 13772+2
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