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FURTHER RESULTS ON SEQUENTIALLY ADDITIVE GRAPHS

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Abstract

Given a graph G with p vertices, q edges and a positive integer k, a k-sequentially additive labeling of G is an assignment of distinct numbers k, $k+1$, $k+2, \ldots, k+p+q-1$ to the $p+q$ elements of G so that every edge uv of G receives the sum of the numbers assigned to the vertices u and v. A graph which admits such an assignment to its elements is called a k-sequentially additive graph.

In this paper, we give an upper bound for k with respect to which the given graph may possibly be k-sequentially additive using the independence number of the graph. Also, we prove a variety of results on k-sequentially additive graphs, including the number of isolated vertices to be added to a complete graph with four or more vertices to

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be simply sequentially additive and a construction of an infinite family of k-sequentially additive graphs from a given k-sequentially additive graph.

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1. Introduction

For all notation and terminology, we follow Harary [9] and West [13].

Graph labelings, where the *vertices* and *edges* are assigned *real values* or *subsets of a set* subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logico – mathematical). Dozens of graph labeling techniques have been studied in over 600 papers. An enormous body of literature has grown around the subject, especially in the last thirty years or so, and is still getting embellished due to increasing number of application driven concepts (Gallian [7]).

Labeled graphs are becoming an increasingly useful family of *Mathematical Models* for a broad range of applications. While the qualitative labelings of graph elements have inspired research in diverse fields of human enquiry, such as *conflict resolution among individuals,* quantitative labelings of graphs have led to quite intricate fields of application such as *Coding Theory* problems, including the design of good *Radar location codes, Synch-set codes*; *Missile guidance codes* and *convolution codes* with *optimal autocorrelation properties. Labeled graphs* have also been applied in determining ambiguities in *X-Ray Crystallographic analysis, to Design Communication Network addressing Systems*, in determining *Optimal Circuit Layouts and* Radio-Astronomy, etc, (see Bloom [4]).

Given a graph $G = (V, E)$, the set N of nonnegative integers, and a commutative binary operation $* : N \times N \to N$, every vertex function $f : V(G) \to N$ induces an edge function $f* : E(G) \to N$ such that $f^*(uv) = f(u) * f(v) \forall$ uv $\in E(G)$. Often it is of interest to determine the vertex functions f having a specified property P such that the induced edge function f^* has a specified property Q where P and Q may be the same. In literature, one can find several instances of this problem, for example, *graceful labelings* (Golomb) [8], Rosa [12]) and *arithmetic labelings* (Acharya and Hegde [1]).

In this paper, we are interested in the case where the induced edge function is the usual sum. Such a vertex function is said to be additive and henceforth the induced edge function will be written f^+.

An *additive labeling* of a graph G is an injective additive vertex function f such that the induced edge function f^+ is also injective. We adopt the following notation throughout this paper.

$A(G)$ = set of all additive labelings of G.

$f(G) = \{f(u) : u \in V(G)\}$, $\quad f^+(G) = \{f^+(e) : e \in E(G)\}$,

$f_{\min}(G) = \min\limits_{v \in V(G)} f(v)$, $\quad f^+_{\min}(G) = \min\limits_{e \in E(G)} f^+(e)$,

$f_{\max}(G) = \max\limits_{v \in V(G)} f(v)$, $\quad \theta(G) = \min\limits_{f \in A(G)} f_{\max}(G)$.

If $f_{\max}(G) = \theta(G)$, then f is called *optimal.*

Theorem 1.1 (Acharya and Hegde [1])**.** *For any additive vertex function $f : V(G) \to R$, where R is the set of real numbers,*

$$\sum_{e \in E(G)} f^+(e) = \sum_{u \in V(G)} f(u)d(u)$$

where $d(u)$ is the degree of the vertex u.

2. k-SEQUENTIALLY ADDITIVE GRAPHS

An additive labeling $f \in A(G)$ of a graph G with p vertices and q edges is called *k-sequentially additive* (Bange *et al.* [3]) if $f(G) \cap f^+(G) = \phi$ and $f(G) \cup f^+(G) = \{k,\ k+1, \ldots, k+p+q-1\}$. A graph G is called *k-sequentially additive* if it has a k-sequentially additive labeling. When $k = 1$, the k-sequentially additive labeling is called *simply sequential* labeling and the graph admitting simply sequential labeling is called *simply sequential graph.* Let $S_k(G)$ denote the set of all k-sequentially additive labelings of G.

Remark 1. Let k be any positive integer, Clearly, any graph G has an additive labeling f such that $f_{\min}\,(G) = k$ and if $f(G) \cap f^+(G) = \phi$; for example, $f(v_i) = k \times 2^i$ (where $V(G) = \{v_1, v_2, \ldots, v_{p-1}\}$. It follows immediately that every graph can be embedded as an induced subgraph of a k-sequentially additive graph, by appending sufficiently many isolated vertices.

It is easy to create k-sequentially additive graphs, and clearly every graph with no edges is k-sequentially additive for all k. Hence, we assume that in what follows, G always has at least one edge. The following result gives a bound on those k for which such a graph can be k-sequentially additive.

Theorem 2.1. *If f is a k-sequentially additive labeling of G, then $(2k+1) \leq f^{+}_{\min}(G) \leq (2\beta_0(G) + 2k - 1)$, where $\beta_0(G)$ denotes the vertex independence number of G and the bounds are sharp.*

Proof. If G is k-sequential, then the labels on the vertices of any edge are at least k and $k+1$, so label on the edge is at least $2k+1$.

On the other hand, by the definition of $\beta_0(G)$ not all of the vertices labeled k, $k+1, \ldots, k+\beta_0(G)$ can be independent. Therefore two of these values must be on adjacent vertices, and hence

$$f^{+}_{\min}(G) \leq (k + \beta_0(G) - 1) + (k + \beta_0(G)) = 2\beta_0(G) + 2k - 1.$$

To prove that the bounds are sharp it is enough to show that there exists graphs G for which the upper and lower bounds are achieved. For any positive k, the one-edged graph with vertex labels $k, k+1, \ldots, 2k$ (where the edge joins the first two of these) has $f^{+}_{\min}(G) = 2k+1$; and the same is true of the star $K_{1,k}$ with the central vertex labeled k and the others labeled $k+1, \ldots, 2k$.

As regards the *upper* bound, consider the labelings of K_3 given in Theorem 2.2. One can easily see that the upper bounds are achieved in both the cases. ∎

Proposition 2.1. *If G is a k-sequentially additive graph of order p then $k \leq p-1$ and the bound is best possible.*

Proof. Let G be a k-sequentially additive graph with an $f \in S_k(G)$. One can see that f assigns the numbers $k, k+1, \ldots, 2k$ to the $k+1$ vertices of G. Thus G has at least $k+1$ vertices and hence $k \leq p-1$.

To show the bound is best possible consider the star $K_{1,n}$. assign n to its central vertex and $n, n+1, \ldots, 2n$ to the remaining vertices in a one-to-one manner. Then it is not hard to verify that the assignment is an n-sequentially additive labeling of $K_{1,n}$. ∎

Theorem 2.2. *For $n > 1$, the complete graph K_n is k-sequentially additive if and only if the pair (n,k) is $(2,1)$, $(3,1)$ or $(3,2)$.*

Proof. By giving the labelings $\{1,2\}$, $\{1,2,4\}$ and $\{2,3,4\}$, respectively, to the complete graphs with 2, 3 and 3 vertices one can see that the result for the pairs $(n,k) = (2,1), (3,1)$ and $(3,2)$ follows.

Suppose $k = 2$ and K_n is k-sequentially additive. If $n \neq 1$, then $q > 0$ and by Proposition 2.1 we have $n \geq 3$. Then one has to assign the numbers 2, 3 and 4 to the three of the vertices of K_n. Then the numbers 5, 6 and 7 are obtained on the edges. Thus, the number 8 has to be assigned to a vertex. So the numbers 10, 11 and 12 are obtained on the edges. The number 9 is missing and it cannot be obtained on any edge and hence to be assigned to a vertex. This obtains the numbers 11, 12 and 13 on the edges contradicting the injectivity of f^+. Thus when $k = 2$ and $n \geq 4$, K_n is not k-sequentially additive.

Now, suppose $k \geq 3$ and K_n is k-sequentially additive. If $n \neq 1$, then $q > 0$ and by Proposition 2.1, we have $n \geq 4$. Then assigning the numbers k, $k+1$, $k+2$ and $k+3$ to vertices of K_4, we see that $f^+(v_1v_4) = f^+(v_2v_3)$. Thus when $k \geq 3$ and $n \geq 4$, K_n is not k-sequentially additive. ∎

Theorem 2.3. *The star $K_{1,n}$ is k-sequentially additive if and only if $k|n$. Furthermore, for $k \geq 2$ any k-sequentially additive labeling of $K_{1,n}$ assigns k to the central vertex.*

Proof. Let $K_{1,n}$ be a star and assume that $k|n$. Assign k to its central vertex and $k+1$, $k+2, \ldots, 2k$, $3k+1$, $3k+2, \ldots, 4k, \ldots, 2n-k+1$, $2n-k+2, \ldots, 2n$ to the pendant vertices. Then it is easy to verify that $K_{1,n}$ is k-sequentially additive.

We prove the converse by *Strong Induction Principle* (Enderton [5], pp. 87) on n. One can easily verify that the result is true for $n = 1, 2$. Suppose that the result is true for all positive integers less than n, i.e., $K_{1,t}$ is k-sequentially additive for all k dividing $t < n$. Suppose that $K_{1,n}$ is k-sequentially additive. Let t be the label assigned to the central vertex u and m the largest vertex label. Then all the values $m+1$, $m+2, \ldots, 2n+k$ are obtained on the edges and their end vertices have label t less (correspondingly). Consequently, $m = 2n+k-t$. Remove the t vertices with the labels $m+1-t$, $m+2-t, \ldots, m$ (and thus the edges labeled $m+1$, $m+2, \ldots, m+t$ $(= 2n+k)$) (If $t = n$ it is easy to verify that the numbers from $n+1$ to $2n = m$ will appear on the remaining vertices of $K_{1,n}$ and the numbers $2n+1$, $2n+2, \ldots, 3n = f^+_{\max}(G)$, are obtained on the edges of $K_{1,n}$), result is a k-sequentially additive labeling of $K_{1,n-t}$. Thus by strong induction principle, $k|(n-t)$ and k is at the central vertex so $k = t$.

We know that if $k|(n-k)$ then $k|n$. Hence, $K_{1,n}$ is k-sequentially additive for $k = n$ with $f(u) = k$. ■

Corollary 2.1. *For any finite set A of positive integers there is a graph that is k-sequentially additive for all $k \in A$.*

By Theorem 2.3 it follows that $K_{1,4}$ is not 3-sequentially additive but 1-, 2- and 4-sequentially additive. Next, we give a construction of infinite families of k-sequentially additive graphs from a given k-sequentially additive graph.

Conjecture 2.1. For any nonempty finite set B of positive integers there exists a graph G which is k-sequentially additive if and only if $k \in B$.

Next, we consider *caterpillars* that by definition, are trees the deletion of whose end vertices results in a path. We denote $C_{a,b}$ a caterpillar with bipartition $\{A, B\}$ where $A = \{u_1, u_2, \ldots, u_a\}$ and $B = \{v_1, v_2, \ldots, v_b\}$, $a \leq b$. The next result gives a k-sequentially additive labeling of $C_{a,b}$ for $k = a, b$.

Theorem 2.4. *A caterpillar $C_{a,b}$ is k-sequentially additive for $k = a, b$.*

Proof. Let the caterpillar be drawn as a planar graph with the vertices $u_1, u_2, \ldots, u_a$ and $v_1, v_2, \ldots, v_b$ constituting its bipartition. Then define the function $f : V(C_{a,b}) \to N$ by

$$f(u_i) = a + b + i - 1,\ 1 \leq i \leq a,$$

$$f(v_j) = a + j - 1,\ 1 \leq j \leq b.$$

Then we get $f(C_{a,b}) = \{k = a,\ a+1,\ a+2, \ldots, a+b-1,\ a+b,\ a+b+1, \ldots, 2a+b-1\}$ and $f^+(C_{a,b}) = \{2a+b,\ 2a+b+1, \ldots, 2a+b+a+b-2 = a + (a+b) + (a+b-1) - 1 = k+p+q-1\}$. Hence f is an a-sequentially additive labeling of $C_{a,b}$.

Similarly we can give b-sequentially additive labeling of $C_{a,b}$. ■

Remark 2.1. Using Lemma 2.1 below and a star, one can construct an infinite family of simply sequential caterpillars. But *whether a given caterpillar is simply sequential or not* is an interesting problem.

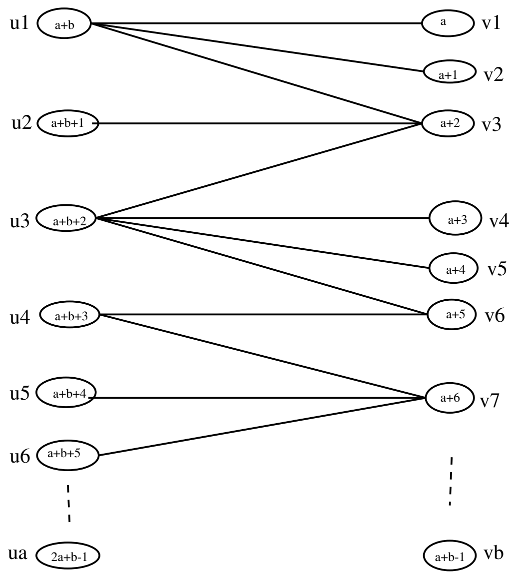

Figure 1. a-sequentially labeling of $C_{a,b}$.

Given a graph G, a vertex u of G and a new set X of Vertices, disjoint from $V(G)$, we define a new graph $G(u, X)$ with $V(G) \cup X$ as its vertex set and $E(G) \cup \{ux : x \in X\}$ as its edge set, respectively.

Lemma 2.1. *Let G be a k-sequentially additive graph with a k-sequentially additive labeling f. Then the graph $H = G(u, X)$ with $|X| = ar$, where $a = f(u)$ and $r \geq 1$, is also a k-sequentially additive graph.*

Proof. Consider an arbitrary vertex u of G and let $f(u) = a$. Then introduce a set X of ar new vertices labeled

$$\begin{array}{c} u_{11},\ u_{12}, \ldots, u_{1a} \\ u_{21},\ u_{22}, \ldots, u_{2a} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ u_{r1},\ u_{r2}, \ldots, u_{ra} \end{array}$$

and construct the graph $H = G(u, X)$ as mentioned above. Then define a function

$F : V(H) \to N$ by

$F(u) = f(u)$ if $u \in V(G)$.

$F(u_{ji}) = k + p + q + i - 1 + 2a(j-1),\ i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, r$.

Then it is not hard to verify that F so defined is an extension of f to H. ■

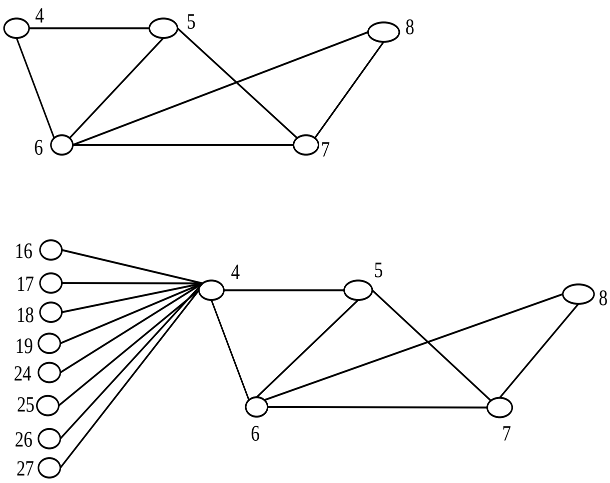

Figure 2. Illustration of Lemma 2.1.

Given a graph G, a set S of vertices of G and a collection $X(S) = \{X_u : u \in S\}$ of disjoint sets, the graph $G(S,\, S(X))$ has $V(G) \cup\ (\cup X_u,\ u \in S)$ as its vertex set and $E(G) \cup\ (\cup E_u,\ u \in S)$ where $E_u = \{uy : y \in X_u, u \in S\}$ as its edge set.

By repeated application of Lemma 2.1, one can prove that the graph $G(S, S(X))$ is also k-sequentially additive.

Theorem 2.5. *Let G be a k-sequentially additive graph with a k-sequentially additive labeling f where X_u is a multiple of $f(u)$ for each $u \in S$. Then the graph $G(S, S(X))$ is also k-sequentially additive.*

A graph G is called (k,d)-*arithmetic* (Acharya and Hegde [1]) if $f^+(G) = \{k, k+d, \ldots, k+(q-1)\,d\}$ for some $f \in A(G)$.

Theorem 2.6. *A graph G is k-sequentially additive if and only if the join $G + v$ has a $(k,1)$-arithmetic labeling F with $F(v) = 0$.*

Proof. Assume that G is a k-sequentially additive graph. Note that $|E(G+v)| = |V(G)| + |E(G)|$. Extend a k-sequentially additive labeling f of G to F by assigning 0 to the vertex v in $G+v$. Then one can verify that the extension of f to $G+v$ is a $(k,1)$-arithmetic labeling of $G+v$.

Conversely, suppose f is a $(k,1)$-arithmetic labeling of $G+v$ with $F(v) = 0$. Then for each $u \in V(G)$ the restriction of F to G yields a k-sequentially additive labeling of G. ■

Since any graph G must have an even number of vertices of odd degree, $G+v$ is an eulerian graph if and only if every vertex of G has odd degree. We call such a graph an odd-degree graph.

Theorem 2.7. *If an odd-degree (p,q)-graph G is k-sequentially additive then*

$$(p+q)(2k+p+q-1) \equiv 0 \pmod 4.$$

Proof. Let G be as in the statement of the theorem. By Theorem 1.1, we get

$$\sum_{u\in V(G)} f(u) + \sum_{e\in E(G)} f^+(e) = \sum_{u\in V(G)} f(u)(1+d(u)) \equiv 0 \pmod{2} \text{ (since } G \text{ is odd).}$$

Also, this sum is $\sum_{i=k}^{k+p+q-1} i = (p+q)k + (p+q-1)(p+q)/2$ and the result follows. ■

As far as the k-sequential additivity of $K_{a,b}$ is concerned, we observe that $K_{a,b}$ is an odd-degree graph if both a and b are odd. Further, if both a and b are odd we get $a+b+ab \equiv 3 \pmod 4$. Then by Theorem 2.7 we get

Corollary 2.2. *If a and b are both odd then $K_{a,b}$ is not k-sequentially additive for any even k.*

For any tree T of order p, we have $p+q = 2p-1 \equiv 1 \pmod 2$ so that $2p-1 \equiv 1$ or $3 \pmod 4$. Hence, if T is an odd tree then $p+q = 2p-1 \equiv 3 \pmod 4$. Then by Theorem 2.7 we get

Corollary 2.3. *If an odd-degree tree is k-sequentially additive then k is odd.*

3. Results on non k-Sequentially Additive Graphs

The study of non-k-sequentially additive graphs (labelings) is as important as that of k-sequentially additive graphs. We define the graph theoretical arithmetic function $R(k,n)$ as the minimum number of edges in a non-k-sequentially additive graph of order $n \geq 4$. We shall give some good bounds for $R(k,n)$.

For an odd positive integer $k \geq 1$, the smallest non-k-sequentially additive graph of order 4 is $2K_2$, so that $R(k,4) = 2$ for odd positive integer k. For any even integer $k \geq 2$, the smallest non k-sequentially additive graph is $K_{1,3}$ so that $R(k,4) = 3$.

When n is an odd positive integer, any graph G of order $n$must contain a point of even degree so that G can never be an odd-degree graph. Hence, in this case, we cannot apply Theorem 2.7 to deduce the non-k-sequential additivity of G.

We must, therefore, discover some other method of deducing the non-k-sequential additivity of a given graph G, or else we must find a structural necessary condition for a graph to be k-sequentially additive. Both of these alternatives seem to pose formidable difficulties. Bange *et al.* [3] have shown that the cycle C_5 is 1-sequentially additive and we have verified all the other graphs of order ≤ 5 and found that C_5 is the smallest graph of order 5 which is not 1-sequentially additive, whence $R(1,5) = 5$.

Theorem 3.1. *Let $n = 2r$, $r \geq 2$.*

(a) *If k is odd, then*

$$R(k,n) \leq \begin{cases} r & \text{if } r \equiv 2 \text{ or } 3 \pmod 4, \\ r+1 & \text{if } r \equiv 0 \pmod 4, \\ r+2 & \text{if } r \equiv 1 \pmod 4. \end{cases}$$

(b) *If k is even, then*

$$R(k,n) \leq \begin{cases} r & \text{if } r \equiv 2 \pmod 4, \\ r+1 & \text{if } r \equiv 3 \pmod 4, \\ r+2 & \text{if } r \equiv 0 \pmod 4, \\ r+3 & \text{if } r \equiv 1 \pmod 4. \end{cases}$$

Proof. For each case, we construct a graph G having the specified number of vertices and edges as given below that is not k-sequentially additive (by Theorem 2.7):

(a) k odd,

$r \equiv 2$ or 3 (mod 4), let $G_1 = rK_2$,

$r \equiv 0$ (mod 4), let $G_2 = K_{1,3} \cup (r-2)\ K_2$,

$r \equiv 1$ (mod 4), let $G_3 = 2K_{1,3} \cup (r-4)\ K_2$.

(b) Let T be the tree obtained from two copies of $K_{1,2}$ with their centers joined by an edge. Then, when k is even and

$r \equiv 0$ (mod 4), let $G_4 = T \cup (r-3)\ K_2$,

$r \equiv 1$ (mod 4), let $G_5 = T \cup K_{1,3} \cup (r-5)\ K_2$,

$r \equiv 2$ (mod 4), let $G_6 = rK_2$,

$r \equiv 3$ (mod 4) then let $G_7 = K_{1,3} \cup (r-2)K_2$. ■

Conjecture 3.1. In Theorem 3.1, equality holds in each case.

Let $C(k,n)$ be the minimum number of edges in a non-k-sequentially additive connected graph of order $n \geq 4$. We shall give some good bounds for $C(k,n)$. One can verify that $C(1,n)$ is not defined for $n \leq 3$ (i.e., all connected graphs with $n \leq 3$ are k-sequentially additive for $k = 1$) and is defined for $n \geq 4$ (as there are graphs with $n \geq 4$ which are not k-sequentially additive for any k).

Theorem 3.2. *For* $6 \leq n \equiv 0 \pmod 2$ *and* $k \equiv 1 \pmod 2$, $C(k,n) \leq n+1$.

Proof. For $n = 6$ let G be the graph shown in Figure 3(a) and for even $n \geq 8$ let G be the graph shown in Figure 3(b). In each case, G is an odd-degree graph and hence by Theorem 2.7, the required inequality follows. ■

Conjecture 3.2. $C(k,n) = n+1$ for all $n \geq 6$ and odd $k \geq 1$.

Next, we give a result on the minimum number of isolated vertices to be added to a graph to make it k-sequentially additive. Let F_n denote the nth *Fibonacci number* given by $F_n = F_{n-1} + F_{n-2}$ for $n > 2$ where $F_1 = 1$ and $F_2 = 2$.

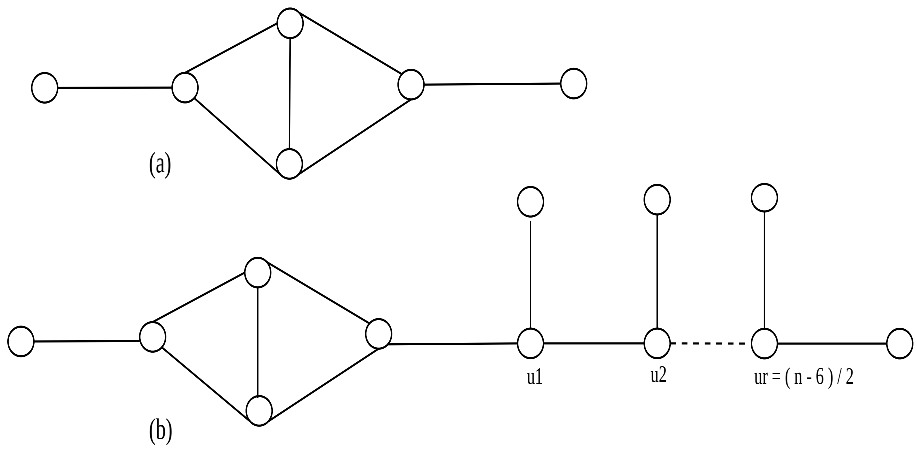

Figure 3. Odd-degree graphs.

Theorem 3.3. *The minimum number of isolated vertices required to be introduced to make the complete graph K_n simply sequentially additive is*

$$F_{n+1} + F_n -^{(n+1)} C_2 - 2$$

where F_n is the nth Fibonacci number.

Proof. Denote the vertices of K_n by $v_1, v_2, \ldots, v_n$. Then define a function $f : V(K_n) \to N$, by

$$f(v_1) = 1, f(v_2) = 2 \quad \text{and} \quad f(v_i) = f(v_{i-1}) + f(v_{i-2}) + 1, \ 3 \leq i \leq n.$$

Then one can see that $f(v_1) = F_2 - 1$, $f(v_2) = F_3 - 1, \ldots, f(v_n) = F_{n+1} - 1$. Thus, f is injective. As $f(v_i) = f(v_{i-1}) + f(v_{i-2}) + 1$, $f^+_{\max}(K_n) = f(v_{n-1}) + f(v_n)$.

Hence, by recursion we have $f^+_{\max}(K_{n-1}) + 1 = f(v_n)$. So that the $n-1$ new edges joining v_n to $v_1, v_2, \ldots, v_{n-1}$ receive the numbers given by

$$f^+(v_n v_1) = f^+_{\max}(K_{n-1}) + 1 + f(v_1),$$

$$f^+(v_n v_2) = f^+_{\max}(K_{n-1}) + 1 + f(v_2), \ldots,$$

$$f^+(v_n v_{n-1}) = f^+_{\max}(K_{n-1}) + 1 + f(v_{n-1}).$$

As f is injective the above numbers are all distinct. Thus, all the edge numbers are distinct with $f(K_n) \cap f^+(K_n) = \phi$.

To show that f as defined above is an optimal additive labeling for K_n, we use induction. One can easily verify that f is an optimal labeling for the complete graph with two and three vertices. Suppose that the result is true for K_{n-1}, i.e., $f_{\max}(K_{n-1}) = \theta(K_{n-1})$. We know that $f^+_{\max}(K_{n-1}) + 1 = f(v_n)$. Any of the missing numbers less than $f^+_{\max}(K_{n-1})$ assigned to v_n will violate the injectivity of either f or f^+.

Hence, $f^+_{\max}(K_{n-1}) + 1 = \theta(K_n)$. Thus, f is optimal.
Now, edge numbers $f^+(K_n)$ and vertex numbers $f(K_n)$ are all distinct and belong to the set $\{1, 2, \ldots, F_n + F_{n+1} - 2\}$, where $F_n + F_{n+1} - 2$ is the maximum number. But K_n has ${}^nC_2$ edges and n vertices. Therefore, number of numbers missing from $\{1, 2, \ldots, F_n + F_{n+1} - 2\}$ is

$$F_n + F_{n+1} - 2 - (n +^n C_2) = F_n + F_{n+1} - 2 -^{n+1} C_2.$$

Hence, the number of isolated vertices to be added to K_n to make it simply sequentially additive is $F_n + F_{n+1} -^{n+1} C_2 - 2$. One can see that the above labeling is optimal for K_n, i.e., there is no other additive labeling in which the largest vertex number is less than the one defined by f. Introduce $F_n + F_{n+1} -^{n+1} C_2 - 2$ isolated vertices and assign the missing numbers to them in a one-to-one manner. Then the resulting graph is simply sequentially additive. ∎

It is interesting to consider the problem of finding the minimum number of isolated vertices required to embed a given graph, or other non k-sequentially additive graphs.

Bange *et al.* [3] have proved that the cycle C_n is simply sequentially additive if and only if $n \equiv 0$ or $1 \pmod 3$. Thus, the cycles C_n where $n \equiv 2 \pmod 3$ are not simply sequentially additive. The following figure shows that the minimum number of isolated vertices to be added to these cycles to make it simply sequentially additive is only one. So, we strongly believe,

Conjecture 3.3. The minimum number of isolated vertices to be added to the cycles C_n where $n \equiv 2 \pmod 3$ to make it simply sequentially additive is exactly one.

4. k-SEGREGATED GRAPHS

A k-sequentially additive labeling is said to be *k-segregated*, if the vertices are labeled k, $k+1, \ldots, k+p-1$, and the edges are labeled $k+p$, $k+p+1, \ldots, k+p+q-1$.

A graph is k-segregatable if it has a k-segregated labeling.

For an example, consider the star $K_{1,4}$ which is k-segregatable for $k = 1, 4$ but not for $k = 2, 3$ (see Figure 4).

Thus, a k-segregated labeling has the property that $f(G) = \{k, k+1, \dots, k+p-1\}$ and $f^+(G) = \{k+p, k+p+1, \dots, k+p+q-1\}$.

Lemma 4.1. *Every connected* 1*-segregatable graph is a tree.*

Proof. Let G be a connected graph. Connectedness of G implies that $q \geq p-1$. As the maximum possible edge label is $2p-1$ it follows that $q = p-1$. Since G is connected and $q = p-1$, it must be a tree. ■

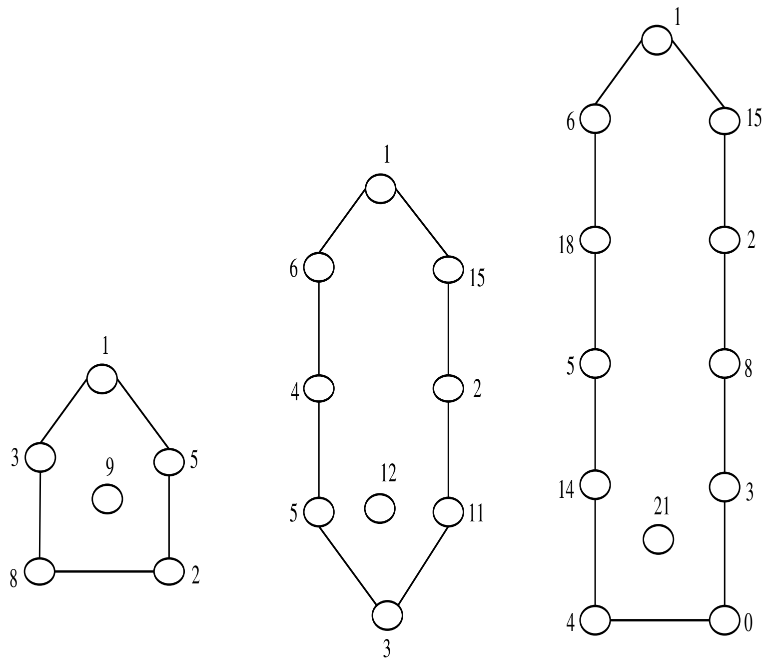

Figure 4. Sequentially additive cycles with one isolated vertex.

Theorem 4.1. *A connected graph G is* 1*-segregatable if and only if it is a star.*

Proof. It is easy to see that if the central vertex of $K_{1,n}$ is assigned $n+1$ and other vertices are numbered $1, 2, \dots, n$, the result is a 1-segregated labeling.

Conversely, assume that G is a connected graph of order p having a 1-segregated labeling f. Let $f(u_i) = i$, $1 \leq i \leq p$. By lemma 4.1, G is a tree. Since $f^+(G) = \{p+1, p+2, \ldots, 2p-1\}$, $f^+(u_iu_j) \geq p+1$. Hence, u_1 must be adjacent to u$_p$, otherwise, if $u_1u_j \in E(G)$ for any j, $2 \leq j \leq p-1$, then $f^+(u_1u_j) = j+1 \leq p < p+1$, a contradiction, so, $j = p$. Assume that u_p is not adjacent to some vertex; let u_i be the one with least i. But then $f^+(u_iu_j) = i+j \leq i+p-1$, which is impossible since f is injective.

Thus, it follows that each of $u_1, u_2, \ldots, u_{p-1}$ must be adjacent to u_p and so that G must be a star. ■

Figure 5 shows a star which is 1-segrated and 4-segregated. From the labeling given in the figure it follows that a star $K_{1,n}$ is 1-, n-segregatable.

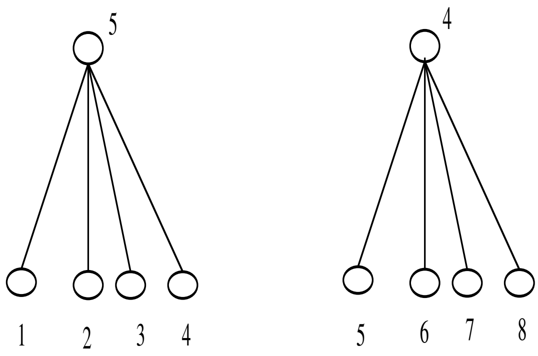

Figure 5. 1-segregatable and 4-segregatable star.

Next, we describe a simple construction of a 1-segregatable graph:

Label the vertices of a totally disconnected graph by $1, 2, \ldots, p$. Successively join a pair of vertices whose labels sum to $p+1$, to $p+2$, and so forth.

The following result shows that at most $p-1$ edges can be added in this way.

Theorem 4.2. *If G is a 1-segregated graph of order p, then it has at most $p-1$ edges and consequently its minimum degree δ is 0 or 1.*

Proof. As the maximum possible edge label is $2p-1$ it follows that $q \leq p-1$. As, $\delta p \leq 2q$ in any graph G with q edges, we get $\delta p/2 \leq p-1$. Hence, $\delta \leq 1$ as claimed. ■

Thus, K_1 and K_2 are the only connected regular graphs which are 1-segregated.

Theorem 4.3. *If G is 1-segregated, then the join $G+\overline{K}_t$, is 1-sequentially additive.*

Proof. Let G be a 1-segregatable graph of order p. Join the vertices of G to every vertex of $\overline{K}_t$. Then assign the numbers

$$f^{+}_{\max}(G)+1+(i-1)(p+1),\ i=1,2,\dots,t$$

to the vertices of $\overline{K}_t$. Then one can see that the above assignment is a 1-sequentially additive labeling of $G+\overline{K}_t$. ■

Next, we give some classes of graphs which are k-sequentially additive for certain values of k.

Corollary 4.1. *The complete tripartite graph $K_{a,b,1}$ is a 1-sequentially additive graph.*

Proof. Since the star $K_{1,a}$ is 1-segregated, the result follows from Theorem 2.5. ■

A (p,q)-graph G is said to be *strongly k-indexable* (Acharya and Hegde [1, 2]) if its vertices can be labeled with $0,1,2,\dots,p-1$ so that the edges receive numbers k, $k+1$, $k+2,\dots,k+q-1$ when each edge is assigned the sum of numbers assigned to its ends.

Enomoto *et al.* [5] have introduced the concept of *super edge magic* labelings: A graph G is said to be super edge magic if it admits a bijection $f : V\cup E \to \{1,2,\dots,p+q\}$ with $f(V)=\{1,2,\dots,p\}$ and $f(E)=\{p+1,p+2,\dots,p+q\}$ such that $f(u)+f(v)+f(uv)=c(f)$, a constant.

Theorem 4.4 (Hegde and Shetty [11])**.** *A graph is super edge magic if and only if it is strongly k-indexable for some positive integer k.*

Theorem 4.5. *A (p,q)-graph G is 1-segregated if and only if it is strongly $(p-q)$-indexable and hence super edge magic.*

Proof. Suppose G has a strong $(p-q)$-indexer f, i.e., $f(u_i)=i$, $0\le i\le p-1$ so that $f^{+}(G)=\{p-q,p-q+1,\dots,p-1\}$. Then, define a function F by $F(u_i)=p-f(u_i)$, $1\le i\le p$. Then for any edge u_iu_j of

G we get $p+1 \leq f(u_i) + f(u_i) \leq p+q$. Since f is injective this implies that the values assigned to the edges of G by F are precisely the numbers $p+1, p+2, \ldots, p+q$. Thus, f is a 1-segregated labeling of G.

The converse can be proved on similar lines. ■

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References

[1] B.D. Acharya and S.M. Hegde, *Arithmetic graphs*, J. Graph Theory **14** (1990) 275–299.

[2] B.D. Acharya and S.M. Hegde, *Strongly indexable graphs*, Discrete Math. **93** (1991) 123–129.

[3] D.W. Bange, A.E. Barkauskas and P.J. Slater, *Sequentially additive graphs*, Discrete Math. **44** (1983) 235–241.

[4] G.S. Bloom, Numbered undirected graphs and their uses: A survey of unifying scientific and engineering concepts and its use in developing a theory of non-redundant homometric sets relating to some ambiguities in x-ray diffraction analysis (Ph. D., dissertation, Univ. of Southern California, Loss Angeles, 1975).

[5] Herbert B. Enderton, Elements of Set Theory (Academic Press, 2006).

[6] H. Enomoto, H. Liadi, A.S.T. Nakamigava and G. Ringel, *Super edge magic graphs*, SUT J. Mathematics **34** (2) (1998) 105–109.

[7] J.A. Gallian, *A dynamic survey of graph labeling*, Electronic J. Combinatorics DS#6 (2003) 1–148.

[8] S.W. Golomb, *How to number a graph*, in: Graph Theory and Computing, (ed. R.C. Read) (Academic Press, 1972), 23–37.

[9] F. Harary, Graph Theory (Addison Wesley, Reading, Massachusetts, 1969).

[10] S.M. Hegde, *On indexable graphs*, J. Combin., Information & System Sciences **17** (1992) 316–331.

[11] S.M. Hegde and Shetty Sudhakar, *Strongly k-indexable labelings and super edge magic labelings are equivalent*, NITK Research Bulletin **12** (2003) 23–28.

[12] A. Rosa, *On certain valuations of the vertices of a graph*, in: Theory of Graphs, Proceedings of the International Symposium, Rome (ed. P. Rosentiehl) (Dunod, Paris, 1981) 349–355.

[13] D.B. West, Introduction to Graph Theory (Prentice Hall of India, New Delhi, 2003).

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