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DECOMPOSITION OF COMPLETE BIPARTITE DIGRAPHS AND EVEN COMPLETE BIPARTITE MULTIGRAPHS INTO CLOSED TRAILS

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Abstract

It has been shown [3] that any bipartite graph $K_{a,b}$, where a, b are even integers, can be decomposed into closed trails with prescribed even lengths. In this article, we consider the corresponding question for directed bipartite graphs. We show that a complete directed bipartite graph $\overleftrightarrow{K}_{a,b}$ is decomposable into directed closed trails of even lengths greater than 2, whenever these lengths sum up to the size of the digraph. We use this result to prove that complete bipartite multigraphs can be decomposed in a similar manner.

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1. Introduction

Consider a graph G (without loops), whose size we denote by $e(G)$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph or a digraph G. If G is a graph, $\overleftrightarrow{G}$ will denote the digraph obtained from G by replacing each edge $x, y \in E(G)$ by the pair of arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$.

As in [3] we denote by $\mathrm{Lct}(G)$ the set of all integers l such that there is a closed trail of length l in G. A sequence $\tau = (t_1, t_2, \ldots, t_p)$ of integers is called *admissible for a graph* (*digraph*) G if it adds up to $e(G)$ and $t_i \in \mathrm{Lct}(G)$ for all $i \in \{1, 2, \ldots, p\}$. We shall write $((t_1)^{s_1}, \ldots, (t_l)^{s_l})$ for the sequence $(\underbrace{t_1, \ldots, t_1}_{s_1}, \ldots, \underbrace{t_l, \ldots, t_l}_{s_l})$.

Moreover, if $\tau = (t_1, t_2, \ldots, t_p)$ is an admissible sequence for G and G can be edge-disjointly (arc-disjointly) decomposed into (directed) closed trails $T_1, T_2, \ldots, T_p$ ($\overrightarrow{T}_1, \ldots, \overrightarrow{T}_p$) of lengths $t_1, t_2, \ldots, t_p$ respectively, then τ is called *realizable in* G and the sequence $(T_1, T_2, \ldots, T_p)$ is said to be *a* G*-realization of* τ or *a realization of* τ *in* G.

Let $K_{a,b}$ be a complete bipartite graph with two sets of vertices A and B, such that $|A| = a$ and $|B| = b$. In Section 3, we shall prove the first main result. We show that the complete directed bipartite graph $\overleftrightarrow{K}_{a,b}$ can be decomposed as an edge-disjoint union of directed closed trails of lengths $t_1, t_2, \ldots, t_p$ whenever t_i is even and $\sum_{i=1}^{p} t_i = 2ab$.

An undirected graph G is said to be *even* if the degrees of all its vertices are even. Let ${}^rK_{a,b}$ be a complete bipartite graph where each edge xy occurs with multiplicity r. In Section 4, we deal with even multibipartite graphs ${}^rK_{a,b}$. In the first part of this section, we show that if r is even, then for all admissible sequences τ for ${}^rK_{a,b}$, there is a realization of τ in ${}^rK_{a,b}$. In the other part, we focus on graph ${}^rK_{a,b}$, where a, b are even and r is odd.

In [3] M. Horňák, M. Woźniak proved an analogous theorem for a simple bipartite graph.

Theorem 1 ([3]). *If* a, b *are positive even integers, then if* $\sum_{i=1}^{p} t_i = a \cdot b$ *and there is a closed trail of length* t_i *in* $K_{a,b}$ (*for all* $i \in \{1, \ldots, p\}$), *then* $K_{a,b}$ *can be* (*edge-disjointly*) *decomposed into closed trails* $T_1, T_2, \ldots, T_p$ *of lengths* $t_1, t_2, \ldots, t_p$, *respectively.*

Similar problems were first investigated by P.N. Balister.

Theorem 2 ([1]). *Let* $L = \sum_{i=1}^{p} t_i$, $t_i \geqslant 3$, *with* $L = \binom{n}{2}$ *when* n *is odd and* $\binom{n}{2} - \frac{n}{2} - 2 \leqslant L \leqslant \binom{n}{2} - \frac{n}{2}$ *when* n *is even. Then we can write some subgraph of* K_n *as an edge union of circuits of lengths* $t_1, \ldots, t_p$.

Recently, also directed graphs and multigraphs were discussed by the same author, see [2].

Theorem 3 ([2]). *If* $\sum_{i=1}^{p} t_i = 2\binom{n}{2}$ *and* $t_i \geqslant 2$ *for* $i = 1, \ldots, p$, *then* $\overleftrightarrow{K}_n$ *can be decomposed into an edge-disjoint union of directed closed trails of lengths* $t_1, t_2, \ldots, t_p$, *except in the case when* $n = 6$ *and all* $t_i = 3$.

Analogously as before, let ${}^rK_n$ be a complete graph where each edge occurs with multiplicity r.

Theorem 4 ([2]). *Assume* $n \geqslant 3$, $\sum_{i=1}^{p} t_i = r\binom{n}{2}$, *and* $t_i \geqslant 2$ *for* $i = 1, \ldots, p$. *Then* ${}^rK_n$ *can be written as the edge-disjoint union of closed trails of lengths* $t_1, t_2, \ldots, t_p$ *if and only if either*

(a) r *is even, or*

(b) r *and* n *are both odd and* $\sum_{t_i>2} t_i \geqslant \binom{n}{2}$.

2. Terminology

We say a graph (digraph) G is *Eulerian* if and only if it has a (directed) closed trail through every edge (arc) of G. Here and subsequently, a (directed) closed trail T ($\overrightarrow{T}$) of length n is identified with any sequence $(v_1, v_2, \ldots, v_{n+1})$ of vertices of T ($\overrightarrow{T}$) such that $v_i v_{i+1}$ are distinct edges of T ($\overrightarrow{T}$) for $i = 1, 2, \ldots, n$. Notice that we do not require the v_i to be distinct and undoubtedly $v_1 = v_{n+1}$. However, it will be regarded as an Eulerian graph (digraph) of order n.

Moreover, given two edge-disjoint undirected closed trails T_1, T_2 which are *not* disjoint on vertices, we shall write $T_1.T_2$ for their union, see Figure 1.

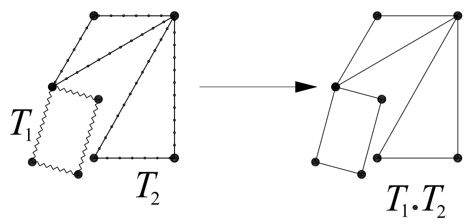

Figure 1. Example of $T_1.T_2$

Observe that $T_1.T_2$ is a closed trail as well.

Notice that if a and b are positive integers, then clearly

$\mathrm{Lct}(K_{2,b}) = \{4i : i = 1, 2, \ldots, \frac{1}{2}b\}$	if b is even,
$\mathrm{Lct}(K_{a,b}) = \{2i : i = 2, 3, \ldots, \frac{1}{2}(ab-4)\} \cup \{ab\}$	if $a, b \geqslant 4$, and a, b are even,
$\mathrm{Lct}({}^rK_{a,b}) = \{2i : i = 1, 2, \ldots, \frac{1}{2}rab\}$	if $a, b \geqslant 1$ and r is even,
$\mathrm{Lct}({}^rK_{a,b}) = \{2i : i = 1, 2, \ldots, \frac{1}{2}rab\}$	if a, b are even and $r \geqslant 2$,
$\mathrm{Lct}(\overleftrightarrow{K}_{a,b}) = \{2i : i = 1, 2, \ldots, ab\}$	if $a, b \geqslant 1$.

3. Decomposition of Complete Bipartite Digraphs into Closed Trails

Let $\overleftrightarrow{K}_{a,b}$ be a complete bipartite digraph with vertices sets A and B. It is easy to see that for all admissible sequences τ for $\overleftrightarrow{K}_{1,b}$, there is a realization of τ in $\overleftrightarrow{K}_{1,b}$.

Theorem 5. *If a sequence $\tau = (t_1, t_2, \ldots, t_p)$ is admissible for $\overleftrightarrow{K}_{a,b}$, then there is a realization of τ in $\overleftrightarrow{K}_{a,b}$.*

Proof. We fix the number of the vertex set B and we will argue by induction on a. For $a = 1$ Theorem 5 is true. The basic idea of the proof is to consider $\overleftrightarrow{K}_{a,b}$ as an arc-disjoint union of $\overleftrightarrow{K}_{a-1,b}$ and $\overleftrightarrow{K}_{1,b}$, each of which has sizes $2(a-1)b$ and $2b$, respectively.

Let $\tau = (t_1, t_2, \ldots, t_p)$ be admissible for $\overleftrightarrow{K}_{a,b}$.
If there exist $i_1, i_2, \ldots, i_l$ such that $\sum_{j=1}^{l} t_{i_j} = 2b$, then we would be done. There exists a realization of $\tau' = (t_{i_1}, \ldots, t_{i_l})$ in $\overleftrightarrow{K}_{1,b}$ and from induction, we obtain the realization of the remaining t_i (namely, all except $t_{i_1}, \ldots, t_{i_l}$) in $\overleftrightarrow{K}_{a-1,b}$. Assume now, that there exists k, such that $\sum_{i=1}^{k-1} t_i < 2b$ and $\sum_{i=1}^{k} t_i > 2b$. Let $t_k = t'_k + t''_k$, such that

$$\sum_{i=1}^{k-1} t_i + t'_k = 2b.$$

Observe that $t'_k, t''_k \geqslant 2$ are even and we may find a $\overleftrightarrow{K}_{1,b}$-realization of the sequence $\tau' = (t_1, \ldots, t'_k)$ and from induction a $\overleftrightarrow{K}_{a-1,b}$-realization of $\tau'' = (t''_k, t_{k+1}, \ldots, t_p)$. Let $(\overrightarrow{T}_1, \ldots, \overrightarrow{T}_{k-1}, \overrightarrow{T}'_k)$ and $(\overrightarrow{T}''_k, \overrightarrow{T}_{k+1} \ldots, \overrightarrow{T}_p)$ are $\overleftrightarrow{K}_{1,b}$ and $\overleftrightarrow{K}_{a-1,b}$-realizations of τ' and τ'', respectively, such that $e(\overrightarrow{T}'_k) = t'_k$ and $e(\overrightarrow{T}''_k) = t''_k$.

Now pick a vertex $v \in B$ of $\overrightarrow{T}'_k$. It is easy to verify that we may write the closed directed trail $\overrightarrow{T}'_k$ as $(v, v_1, \ldots, v_{t'_k-1}, v)$. Now we pick a vertex $w \in B$ of $\overrightarrow{T}''_k$. We may also write $\overrightarrow{T}''_k$ as $(w, w_1, \ldots, w_{t''_k-1}, w)$. Notice that without losing generality, we may have chosen the realization of τ' in such a way that $v = w$. In such a case, if we denote $\overrightarrow{T}_k = (w, w_1, \ldots, w_{t''_k-1}, w, v_1, \ldots, v_{t'_k-1}, w)$, then $\overrightarrow{T}_k$ is a closed directed trail of length t_k and the sequence $(\overrightarrow{T}_1, \ldots, \overrightarrow{T}_p)$ is an $\overleftrightarrow{K}_{a,b}$-realization of τ. ■

4. Decomposition of Complete Bipartite even Multigraphs into Closed Trails

Let ${}^rK_{a,b}$ be the complete bipartite graph with vertices sets A and B such that $|A| = a$ and $|B| = b$. There is no loss of generality in assuming that $a \leqslant b$.

Theorem 6. *If a sequence $\tau = (t_1, t_2, \ldots, t_p)$ is admissible for ${}^rK_{a,b}$, where $a, b \geqslant 1$, r is even, then there is a realization of τ in ${}^rK_{a,b}$.*

Proof. We will argue by induction on r. For $r = 2$, use Theorem 5 and forget the orientations of the edges. The basic idea of the proof is to consider ${}^rK_{a,b}$ as an edge-disjoint union of ${}^2K_{a,b}$ and ${}^{r-2}K_{a,b}$, each of which has sizes $2ab$ and $(r-2)ab$, respectively.

Let $\tau = (t_1, t_2, \ldots, t_p)$ be admissible for ${}^rK_{a,b}$.
If there exist $i_1, i_2, \ldots, i_l$ such that $\sum_{j=1}^{l} t_{i_j} = 2ab$, then we would be done. There exists a realization of $\tau' = (t_{i_1}, \ldots, t_{i_l})$ in ${}^2K_{a,b}$ and we obtain the realization of the remaining t_i (namely, all except $t_{i_1}, \ldots, t_{i_l}$) in ${}^{r-2}K_{a,b}$ by induction.

Assume now that there exists k such that $\sum_{i=1}^{k-1} t_i < 2ab$ and $\sum_{i=1}^{k} t_i > 2ab$. Let us introduce t'_k, t''_k such that $t_k = t'_k + t''_k$ and

$$\sum_{i=1}^{k-1} t_i + t'_k = 2ab.$$

Observe that $t'_k, t''_k \geqslant 2$ are even and the sequence $\tau' = (t_1, \ldots, t'_k)$ is realizable in ${}^2K_{a,b}$, whereas the sequence $\tau'' = (t''_k, t_{k+1}, \ldots, t_p)$, from induction, is realizable in ${}^{r-2}K_{a,b}$. Notice that we may choose these realizations in such a way that the proper T'_k and T''_k have a common vertex. In such a case, if we denote T_k to be $T'_k.T''_k$, then $(T_1, T_2, \ldots, T_p)$ is a ${}^rK_{a,b}$-realization of τ. ■

Notice that if r is odd and a, b are even, then there can be at most $ab\lfloor \frac{r}{2} \rfloor$ trails of length two, so we easily conclude that the sequence $\tau_1 = (2^{\frac{rab}{2}})$ is not realizable in ${}^rK_{a,b}$, then because $6 \notin \mathrm{Lct}(K_{2,b})$ also the sequence $\tau_2 = (2^{\frac{(r-1)ab}{2}}, 6, 2b-6)$ is not realizable in ${}^rK_{2,b}$ $(b \geqslant 6)$. According to the above remark we will consider now for ${}^rK_{a,b}$ an admissible sequence $\tau = (t_1, t_2, \ldots, t_p)$ such that $\sum_{t_i \equiv 0 \pmod 4} t_i + \sum_{t_i \equiv 2 \pmod 4} (t_i - 2) \geqslant ab$ if $a = 2$, or $\sum_{t_i > 2} t_i \geqslant ab$ if $a \geqslant 4$.

Theorem 7. *Let $r \geqslant 1$, r is odd and a, b are even. Then an admissible sequence $\tau = (t_1, t_2, \ldots, t_p)$ for ${}^rK_{a,b}$ is realizable in ${}^rK_{a,b}$ if and only if τ is such that*

$$\sum_{t_i \equiv 0 \pmod 4} t_i + \sum_{t_i \equiv 2 \pmod 4} (t_i - 2) \geqslant ab \quad \text{if } a = 2,$$

$$\sum_{t_i > 2} t_i \geqslant ab \quad \text{if } a \geqslant 4.$$

Before we prove Theorem 7 we shall need two observations.

Observation 8. *Let $r \geqslant 3$, r is odd and let $a = 2$, b be even. If $\tau = (t_1, t_2, \ldots, t_p)$ is admissible for ${}^rK_{2,b}$ and there exist $i_1, i_2, \ldots, i_l$ such that $t_{i_j} \equiv 0 \pmod 4$ and different from them m such that $t_m = t'_m + t''_m$, where $t'_m \geqslant 4$, $t''_m \geqslant 0$ and $\sum_{j=1}^{l} t_{i_j} + t'_m = 2b$, then τ is ${}^rK_{2,b}$-realizable.*

Proof. The basic idea of the proof is to consider ${}^rK_{2,b}$ as an edge-disjoint union of $K_{2,b}$ (G_1) and ${}^{r-1}K_{2,b}$ (G_2), each of which has sizes $2b$ and $2(r-1)b$, respectively. Notice that $t'_m \geqslant 4$ and $t'_m \equiv 0 \pmod 4$. It follows that $\{t_{i_1}, \ldots, t_{i_l}, t'_m\} \subset \mathrm{Lct}(K_{2,b})$ and from Theorem 1 we may decompose graph G_1 into closed trails $T_{i_1}, T_{i_2}, \ldots, T_{i_l}, T'_m$ of lengths $t_{i_1}, t_{i_2}, \ldots, t_{i_l}, t'_m$, respectively.

From Theorem 6, we obtain the realization of the remaining t_i (namely, all except $t_{i_1}, \ldots, t_{i_l}$ and t_m, but including t''_m if $t''_m > 0$) in G_2.

Moreover, if $t''_m > 0$, we may choose these realizations in such a way that the proper T'_m and T''_m have a common vertex. If now T_m is a closed trail being an union of $T'_m.T''_m$ (T'_m if $t''_m = 0$), then $(T_1, T_2, \ldots, T_p)$ make up a ${}^rK_{2,b}$-realization of τ. ■

Observation 9. *Let $r \geqslant 3$, r is odd and let $a, b \geqslant 4$ be even. To show that τ is realizable in ${}^rK_{a,b}$, it is enough to find indicators $i_1, \ldots, i_l, m$ $(l \geqslant 0)$ such that $t_{i_j} \geqslant 4$ $(1 \leqslant j \leqslant l)$, $t_m = t'_m + t''_m$, where $t'_m \geqslant 4$, $t''_m \geqslant 0$ and $\sum_{j=1}^{l} t_{i_j} + t'_m = ab$.*

Proof. Analogously as in the proof of Observation 8 we consider ${}^rK_{a,b}$ as an edge-disjoint union of $K_{a,b}$ (G_1) and ${}^{r-1}K_{a,b}$ (G_2). By our assumptions $t'_m \geqslant 4$ is even and it implies that $\{t_{i_1}, \ldots, t_{i_l}, t'_m\} \subset \mathrm{Lct}(K_{a,b})$. From Theorem 1, we may decompose G_1 into closed trails $T_{i_1}, \ldots, T_{i_l}, T'_m$ of lengths $t_{i_1}, \ldots, t_{i_l}, t'_m$, respectively. Then we may also decompose G_2 into closed trails of lengths equal to t''_m (if $t''_m > 0$) and the remaining t_i (namely, all except $t_{i_1}, \ldots, t_{i_l}$ and t_m), respectively from Theorem 6.

If $t''_m > 0$, then we may carry out these decompositions of G_1 and G_2 in such a way that T'_m and T''_m have a common vertex. If we denote T_m to be $T'_m.T''_m$ (if $t''_m > 0$) or T'_m (if $t''_m = 0$), then $(T_1, T_2, \ldots, T_p)$ is a ${}^rK_{a,b}$-realization of τ. ∎

***Proof of Theorem* 7.**

Necessity. Suppose that τ is an admissible sequence for ${}^rK_{a,b}$. Notice that if we consider, in a closed trail T of ${}^rK_{a,b}$, the set of edges with odd multiplicity, and forget the multiplicity, as the trail is Eulerian, we obtain closed trails in $K_{a,b}$. If r is odd, the set of these trails must cover the edge set of $K_{a,b}$. It follows that:

If $a = 2$ and τ is such that $\sum_{t_i \equiv 0 \pmod 4} t_i + \sum_{t_i \equiv 2 \pmod 4} (t_i - 2) < 2b$, then τ is not realizable in ${}^rK_{2,b}$, because $t_i \in \mathrm{Lct}(K_{2,b})$ if $t_i \equiv 0 \pmod 4$.

If $a \geqslant 4$ and τ is such that $\sum_{t_i > 2} t_i < ab$, then τ is not realizable in ${}^rK_{a,b}$, because $2 \notin \mathrm{Lct}(K_{a,b})$.

Sufficiency. If $r = 1$, then $\sum_{i=1}^{p} t_i = ab$, this implies that Theorem 7 is true by Theorem 1. If $r \geqslant 3$, then the basic idea of the proof is to consider ${}^rK_{a,b}$ as an edge-disjoint union of $K_{a,b}$ (G_1) and ${}^{r-1}K_{a,b}$ (G_2), each of which has sizes ab and $(r-1)ab$, respectively.

Case 1. $a = 2$.

Let ${}^rK_{2,b}$ be a complete bipartite multigraph with vertices sets $\{x, y\}$ and B.

Let M_1, M_2 and M_3 be the sets such that $M_1 = \{i : t_{j_i} \equiv 0 \pmod 4\} = \{1, \ldots, m\}$, $M_2 = \{i : t_{j_i} \equiv 2 \pmod 4, t_{j_i} \geqslant 6\} = \{m+1, \ldots, n\}$ and $M_3 = \{i : t_{j_i} = 2\} = \{n+1, \ldots, p\}$. Notice that if $\sum_{i=1}^{m} t_{j_i} \geqslant 2b$, then τ is realizable in ${}^rK_{2,b}$ from Observation 8.

Let $k \leqslant n$ be the smallest integer such that $\sum_{i=1}^{m} t_{j_i} + \sum_{i=m+1}^{k} (t_{j_i} - 2) \geqslant 2b$. Observe that $t_{j_k} = t'_{j_k} + t''_{j_k}$ such that $\sum_{i=1}^{m} t_{j_i} + \sum_{i=m+1}^{k-1} (t_{j_i} - 2) + t'_{j_k} = 2b$, and $t'_{j_k} \equiv 0 \pmod 4$, $t''_{j_k} \geqslant 2$.

Let $t'_{j_i} = t_{j_i} - 2$, $t''_{j_i} = 2$ for $i = m+1, \ldots, k-1$.

Observe that sequence $\tau' = (t_{j_1}, \ldots, t_{j_m}, t'_{j_{m+1}}, \ldots, t'_{j_{k-1}}, t'_{j_k})$ from Theorem 1 is realizable in $K_{2,b}$ and the sequence $T_{j_1}, \ldots, T_{j_m}, T'_{j_{m+1}}, \ldots, T'_{j_{k-1}}, T'_{j_k}$ is the $K_{2,b}$-realization of τ'.

There is a realization $T''_{j_{m+1}}, \ldots, T''_{j_{k-1}}, T''_{j_k}, T_{j_{k+1}}, \ldots, T_{j_p}$ of sequence $\tau'' = (t''_{j_{m+1}}, \ldots, t''_{j_{k-1}}, t''_{j_k}, t_{j_{k+1}}, t_{j_p})$ in ${}^{r-1}K_{2,b}$ from Theorem 6.

Observe that vertex x or y is in $V(T''_{j_i})$ for $i \in \{m+1,\ldots,k\}$ and the set of vertices $\{x,y\} \subset V(T'_{j_i})$ for $i \in \{m+1,\ldots,k\}$. In such a way T'_{j_i} and T''_{j_i} (for $i \in \{m+1,\ldots,k\}$) have always a common vertex x or y. In such a case, if we denote T_{j_i} to be $T'_{j_i}.T''_{j_i}$ (for $i \in \{m+1,\ldots,k\}$), then $T_{j_1},\ldots,T_{j_m},T_{j_{m+1}},\ldots,T_{j_p}$ is a ${}^{r}K_{2,b}$-realization of τ.

Case 2. $a \geqslant 4$.
We may assume that τ is such that $t_1 \geqslant \cdots \geqslant t_p$. Let k be the smallest integer such that $t_k \geqslant 4$ and $\sum_{i=1}^{k} t_i \geqslant ab$.

Notice that if $\sum_{i=1}^{k-1} t_i \neq ab-2$, then the existence of ${}^{r}K_{a,b}$-realization of τ follows from Observation 9. Assume now that

$$\sum_{i=1}^{k-1} t_i = ab-2.$$

Denote

$$\begin{aligned} t'_{k-1} &= t_{k-1}+2, \\ t'_k &= t_k-2. \end{aligned}$$

Then, since $t'_{k-1} \geqslant 4$, $t'_k \geqslant 2$ we may find a G_1-realization of $\tau_1 = (t_1,\ldots, t_{k-2}, t'_{k-1})$ from Theorem 1 and a G_2-realization of $\tau_2 = (t'_k, t_{k+1},\ldots,t_p)$ from Theorem 6. Let $(T_1,T_2,\ldots,T_{k-2},T'_{k-1})$ and $(T'_k,T_{k+1},\ldots,T_p)$ are G_1 and G_2-realizations of τ_1 and τ_2, respectively, such that $e(T'_{k-1}) = t'_{k-1}$ and $e(T'_k) = t'_k$.

Observe that $e(T'_{k-1}) \geqslant 6$ and $|V(T'_{k-1})| \geqslant 6$, it is easy to verify that we may write T'_{k-1} as $(c_1,\ldots,c_{t'_{k-1}+1})$ with $c_1 \in A$ and $c_4 \in B$. We may also write T'_k as $(d_1,\ldots,d_{t'_k+1})$ with $d_1 \in A$ and $d_2 \in B$. Notice that without losing generality, we may have chosen the realization of τ_2 in such a way that $c_1 = d_1$ and $c_4 = d_2$. In such a case, if we denote $T_{k-1} = (d_1,d_2,c_5,\ldots,c_{t'_{k-1}+1})$ and $T_k = (d_1,c_2,c_3,d_2,d_3,\ldots,d_{t'_k+1})$, then T_{k-1} and T_k are closed trails of lengths t_{k-1} and t_k respectively, and the sequence $(T_1,\ldots,T_p)$ is a ${}^{r}K_{a,b}$-realization of τ. ∎

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