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VARIATIONS ON A SUFFICIENT CONDITION FOR HAMILTONIAN GRAPHS

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Abstract

Given a 2-connected graph G on n vertices, let G^* be its *partially square graph*, obtained by adding edges uv whenever the vertices u, v have a common neighbor x satisfying the condition $N_G(x) \subseteq N_G[u] \cup N_G[v]$, where $N_G[x] = N_G(x) \cup \{x\}$. In particular, this condition is satisfied if x does not center a claw (an induced $K_{1,3}$). Clearly $G \subseteq G^* \subseteq G^2$, where G^2 is the square of G. For any independent triple $X = \{x, y, z\}$ we define

$$\overline{\sigma}_3(X) = d(x) + d(y) + d(z) - |N(x) \cap N(y) \cap N(z)|\,.$$

Flandrin *et al.* proved that a 2-connected graph G is hamiltonian if $\overline{\sigma}_3(X) \geq n$ holds for any independent triple X in G. Replacing X in G by X in the larger graph G^*, Wu *et al.* improved recently this result. In this paper we characterize the nonhamiltonian 2-connected graphs G satisfying the condition $\overline{\sigma}_3(X) \geq n - 1$ where X is independent in G^*. Using the concept of dual closure we (i) give a short proof of the above results and (ii) we show that each graph G satisfying this condition is hamiltonian if and only if its dual closure does not belong to two well defined exceptional classes of graphs. This implies that it takes a polynomial time to check the nonhamiltonicity or the hamiltonicity of such G.

Keywords: cycles, partially square graph, degree sum, independent sets, neighborhood unions and intersections, dual closure.

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1. Introduction

We use the book of Bondy and Murty [7] for terminology and notation not defined here and consider simple graphs only $G = (V, E)$. If A, B are disjoint sets of V, we denote by $E(A, B)$ the set of edges with an end in A and the other in B. Also $G[A]$ is the subgraph induced by A. A vertex x is dominating if $d(x) = |V| - 1$ and we note $\Omega := \{d | d \text{ is dominating}\}$.

For any vertex u of G, $N(u)$ denotes its neighborhood set and $N[u] = \{u\} \cup N(u)$. If $X \subset V$, we denote by $N_X(u)$ the set of vertices of X adjacent to u. For $1 \leq k \leq \alpha$, we put $I_k(G) = \{Y \mid Y \text{ is a } k\text{-independent set in } G\}$, where α stands for the independence number of G. With each pair (a, b) of vertices such that $d(a, b) = 2$ (vertices at distance 2), we associate the set $J(a, b) := \{x \in N(a) \cap N(b) \mid N_G[x] \subseteq N_G[u] \cup N_G[v]\}$.

The partially square graph G^* (see [4]) of a given graph $G = (V, E)$ is the graph $(V, E \cup \{uv \mid d(u, v) = 2, J(u, v) \neq \emptyset\})$. Clearly $G \subseteq G^* \subseteq G^2$, where G^2 is the square of G and every partially square graph is claw-free. For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ on disjoint vertex sets we let $G_1 \cup G_2$ denote the union of G_1 and G_2 with $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and we let $G_1 \vee G_2$ denote the join of G_1 and G_2 with $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup (V_1 \times V_2))$. Moreover $\overline{K}_p$ denotes the empty graph on p vertices.

For each set $S \in I_k(G)$, $k \geq 1$ we adopt a partition of V by defining $S_i := \{u \in V \mid |N_S(u)| = i\}$ and $s_i := |S_i|$, $i = 0, ..., k$. We also put $\sigma_S := \sum_{x \in S} d(x)$. Obviously, we have $|N(S)| = \sum_{i=1}^{k} s_i$ and $\sigma_S = \sum_{i=1}^{k} i s_i$. We point out that any 2-connected graph G for which $\alpha(G^*) \leq 2$ is hamiltonian (see [4]). For any set $S := \{x, y, z\} \in I_3(G^*)$ in a graph G, such that $\alpha(G^*) \geq 3$ we define

$$\overline{\sigma}_3(S) = d_G(x) + d_G(y) + d_G(z) - |N_G(x) \cap N_G(y) \cap N_G(z)| .$$

Alternatively we may write $\overline{\sigma}_3(S) = s_1 + 2s_2 + 2s_3$ if S is fixed. As in [1], for each pair (a, b) of nonadjacent vertices we associate:

$$T_{ab}(G) := V \backslash (N_G[a] \cup N_G[b]),\ \overline{\alpha}_{ab}(G) := 2 + |T_{ab}| = |V \backslash N(a) \cup N(b)| ,$$

$$\delta_{ab}(G) = \min\{d(x) | x \in T_{ab}\} \text{ if } T_{ab} \neq \varnothing \text{ and } \delta_{ab}(G) = \delta(G) \text{ otherwise.}$$

If there is no confusion, we may omit G and/or the subscript ab. In [8], Bondy and Chvátal introduced the concept of the k-*closure* for graph. Ainouche and Christofides [1] proposed the 0-*dual closure* $c_0^*(G)$ as an extension of the n-*closure*. To define the 0-*dual closure*, we use the following weaker condition than that obtained in ([1]).

Theorem 1.1 ([1]). *Let G be a 2-connected graph and let a, b be two nonadjacent vertices. If*

$$(1) \qquad |N(a) \cup N(b)| + \delta_{ab} \geq n \ (\textit{or equivalently} \ \ \overline{\alpha}_{ab}(G) \leq \delta_{ab}),$$

then G is hamiltonian if and only if $(G + ab)$ is hamiltonian.

The 0-dual closure $c_0^*(G)$ is the graph obtained from G by successively joining nonadjacent vertices satisfying (1). Clearly $c_0^*(G)$ is polynomially obtained from G. As a consequence of Theorem 1.1, G is hamiltonian if and only if $c_0^*(G)$ is hamiltonian. Flandrin *et al.* [9] proved the following result:

Theorem 1.2. *A 2-connected graph G of order n is hamiltonian if*

$$(2) \qquad \overline{\sigma}_3(S) \geq n \ \textit{holds for all} \ S \in I_3(G).$$

This result is strong enough to dominate a large spectra of sufficient conditions involving degrees and/or neighborhood of pairs or triple of vertices (see for instance [5]).

Recently Wu *et al.* [10], improved Theorem 1.2 by using a weaker condition.

Theorem 1.3. *A 2-connected graph G of order n is hamiltonian if*

$$(3) \qquad \overline{\sigma}_3(S) \geq n \ \textit{holds for all} \ S \in I_3(G^*).$$

In this paper we go further by allowing exceptional classes of nonhamiltonian graphs. More precisely, we prove:

Theorem 1.4. *Let G be a 2-connected graph of order n. If $\overline{\sigma}_3(S) \geq n - 1$ holds for all $S \in I_3(G^*)$, then G is nonhamiltonian if and only if either (1) $c_0^*(G) = (K_r \cup K_s \cup K_t) \vee K_2$ where r, s, t are positive integers or (2) $c_0^*(G) = K_{\frac{n-1}{2}} \vee \overline{K}_{\frac{n+1}{2}}$.*

Note that the two classes of graphs are not 1-tough since $\omega(G - \Omega) > |\Omega|$, where $\omega(H)$ stands for the number of components of the graph H. They are of course nonhamiltonian. Theorem 1.4 is sharp even for the class of 1-tough graphs. For instance for the Petersen graph we have $\overline{\sigma}_3(S) = 8 = n - 2$ for any independent triple $\{x, y, z\}$ such that $|N(x) \cap N(y) \cap N(z)| = 1$. The graph $(K_r \cup K_s \cup K_t \cup T) \vee \overline{K}_1$, where $2 \leq r, s, t$ and T is a triangle having a vertex from each complete graph of $(K_r \cup K_s \cup K_t)$ is 1-tough, nonhamiltonian and $\overline{\sigma}_3(S) = n - 2$. In both cases, $S \in I_3(G^*)$. Moreover it is possible to answer in a polynomial time if a graph satisfying the condition of Theorem 1.4 is hamiltonian or not. Indeed (i) the closure is obtained in a polynomial time, (ii) the set Ω of dominating vertices is easily identified, in which case (iii) it suffices to check whether $\omega(c_0^*(G) - \Omega) > |\Omega|$ or not.

2. Preliminaries

Let C be a longest cycle for which an orientation is given. For $x \in V(C)$, x^+ (resp. x^-) denotes its successor (resp. predecessor) on C. More generally, if $A \subseteq V$ then $A^+ := \{x \in C \mid x^- \in A\}$ and $A^- := \{x \in C \mid x^+ \in A\}$. Given the vertices a, b of C we let $C[a, b]$ denote the subgraph of C from a to b (and including both a and b) in the chosen direction. We shall write $C(a, b]$, $C[a, b)$ or $C(a, b)$ if a and b or both a and b are respectively excluded. The same vertices, in the reverse order are denoted $\overleftarrow{C}(a, b]$, $\overleftarrow{C}[a, b)$ or $\overleftarrow{C}(a, b)$ respectively. Let H be a component of $G - C$ and let $d_1, \ldots, d_m$ be the vertices of the set $D = N_C(H)$, occurring on C in a consecutive order. For $i \geq 1$, we set $P_i := C(d_i, d_{i+1})$, where the subscripts are taken modulo m and $n_i = |P_i|$. We define a relation $\sim$ on C by the condition $u \sim v$ if there exists a path with endpoints u, v in C and no internal vertex in C. Such a path is called a *connecting path* between u and v. We say that two connecting paths are *crossing* at $x, y \in C$ if there exist two consecutive vertices a, b of C such that $a \sim x$, $b \sim y$ and either $a, b \in C(x, y)$, $a = b^+$ or $a, b \in C(y, x)$, $a = b^-$. We note that the two connecting paths from a to x and from b to y must be internally disjoint since C is a longest cycle. In this paper, most of the time the connecting paths are edges.

For all $i \in \{1, 2, \ldots, m\}$, a vertex u of P_i is *insertible* if there exist $w, w^+ \in C - P_i$ such that $u \sim w$ and $u \sim w^+$. The edge ww^+ is referred as an insertion edge of u. A vertex $x \notin C$ is *C-insertible* if there exist $w, w^+ \in C$ such that $w \sim w^+$ and the path connecting w and w^+ passes through x. Paths and cycles in $G = (V, E)$ are considered as subgraphs, vertex sets or edge sets.

Throughout, H is a component of $G - C$, x_0 is any vertex of H and for all $i \in \{1, \ldots, m\}$, x_i is the first noninsertible vertex (if it exists) on P_i.

Clearly $m \geq 2$ if G is 2-connected. For all $i \in \{1, \ldots, m\}$ for which x_i exists, we define $W_i = V(C(d_i, x_i])$. Set $X := \{x_0, x_1, \ldots, x_m\}$ and $X_0 := \{x_1, \ldots, x_m\}$. Similarly we define the sets $Y := \{x_0, y_1, \ldots, y_m\}$, $Y_0 := \{y_1, \ldots, y_m\}$, where y_i is the last noninsertible vertex (if it exists) on P_i.

The following key-lemma is mainly an adaptation of Lemmas proved in [3] and [4].

Lemma 2.1. *Let C be a longest cycle of a connected nonhamiltonian graph. Let i, j be two distinct integers in $\{1, \ldots, m\}$ and let $u_i \in W_i$, $u_j \in W_j$. Then*

1. *x_i and y_i exist.*
2. *$u_i \nsim u_j$ and there are no crossing paths at u_i, u_j.*
3. *Any set $W = \{x_0\} \cup \{w_i \in W_i \mid 1 \leq i \leq m\}$ and in particular X is independent in G.*
4. *$N(u_i) \cap N(u_j) \subset V(C) \setminus \cup_{i=1}^{m} W_i$.*
5. *X, Y are independent sets in G^*.*
6. *For each i, we may assume that $N(x_i) \cap C[d_i, x_i) = \{x_i^-\}$.*

Proof. The proof of statements 1 to 4 is given in [2], while the proof of 5 is given in [4]. To prove (6), let $u_i \in C[d_i, x_i)$ be the first vertex along C, adjacent to x_i and assume that $C(u_i, x_i)$ is not empty. The vertices of $C(u_i, x_i)$ are insertible by definition. For $i = 1, \ldots, m$, let F_i be the set of insertion edges of vertices of $C(d_i, x_i)$. We proved in [2] that $F_i \cap F_j = \varnothing$ whenever $j \neq i$. Moreover $E(W_i, W_j) = \varnothing$ by (2). Therefore the vertices of $C(u_i, x_i)$ can be easily inserted into $C - P_i$. ■

The next general Lemma is an extension of Lemma 2.1. Set $S := \{x_i, x_j, x_k\}$, where i, j, k are pairwise distinct integers in $\{0, \ldots, m\}$.

Lemma 2.2. $|S_0 \cap C| \geq s_2 + s_3$.

Proof. To prove the Lemma, it suffices to show that an injection $\theta : S_2 \cup S_3 - > S_0 \cap C$ exists. By Lemma 2.1(4), $S_2 \cup S_3 \subset V(C) \setminus \cup_{i=1}^{m} W_i$ and by definition, the sets S_0, S_1, S_2, S_3 are disjoint. Choose $S := \{x_i, x_j, x_k\}$ and let $a \in S_2 \cup S_3$. As a first case, we suppose that $a \notin D$ and without loss of generality assume $a \in (N(x_j) \cap N(x_k) \cap C(x_k, d_j)) \backslash D$. If $a^+ \in S_0 \cap C$ then we are done with $\theta(a) = a^+$, otherwise we must have $a^+ \in S_1$. Clearly $a^+ \notin N(x_j)$ since x_j is noninsertible and $a^+ \notin N(x_k)$ by Lemma 2.1(2).

Thus $a^+ \in N(x_i)$. If $i = 0$ then $a^+ = d_h \in D \cap C(d_k, d_j]$. But then $d_h^+ = a^{++} \in S_0 \cap C$ and we set $\theta(a) = a^{++}$. If $i > 0$, then by Lemma 2.1(2), $x_i \in C(d_{(h+1) \bmod m}, d_j)$ in which case $a^{++} \in S_0 \cap C$ since $a^{++} \notin N(x_j) \cup N(x_k)$ by Lemma 2.1(2) and x_i is noinsertible. We set again $\theta(a) = a^{++}$.

As a second case, we suppose that $a = d_h$. If $h = j$ then $x_j \in S_0 \cap C$ and we are done. So, we assume $d_h \in C(d_k, d_j)$. If $x_i = x_0$ then clearly $a^+ = d_h^+ \in S_0 \cap C$. If $i > 0$ the arguments are the same as in the previous case. The proof is now complete. ∎

Lemma 2.3. *Let G be a nonhamiltonian graph satisfying the conditions of Theorem* 1.4. *Then*

1. $S_0 \cap (G - C) = \{x_0\}$, $|S_0 \cap C| = |S_2 \cup S_3|$ *and* $\overline{\sigma}_3(S) = n - 1$.
2. *For each* $v \in S_0 \cap C$, *either* $v^- \in S_2 \cup S_3$ *or* $v^{--} \in S_2 \cup S_3$, *in which case* $v^- \in S_1$.
3. $X_0 = D^+$ *and* $Y_0 = D^-$.

Proof. Among all possible components of $G - C$ we assume that H is chosen so that $|N_C(H)| = m$ is maximum.

(1) Set $\overline{\sigma}_3(S) = s_1 + 2s_2 + 2s_3 = n - 1 + \delta$ with $\delta \geq 0$. By definition, $n = s_0 + s_1 + s_2 + s_3$. Thus $\overline{\sigma}_3(S) = s_1 + 2s_2 + 2s_3 = n - 1 + \delta = s_0 + s_1 + s_2 + s_3 - 1 + \delta$. It follows that $s_2 + s_3 = s_0 - 1 + \delta$. As $s_0 = |S_0 \cap C| + |S_0 \cap (G - C)|$, $x_0 \in S_0 \cap (G - C)$ and $|S_0 \cap C| \geq s_2 + s_3$ by Lemma 2.2 we must have equality throughout. Thus (1) is proved, that is $S_0 \cap (G - C) = \{x_0\}$, $|S_0 \cap C| = |S_2 \cup S_3|$ and $\overline{\sigma}_3(S) = n - 1$.

(2) Follows from the proof of Lemma 2.2 and the fact that $|S_0 \cap C| = |S_2 \cup S_3|$ by (1).

(3) Suppose first $m \geq 3$ and assume without loss of generality that $x_1 \neq d_1^+$. If we set $S := \{x_0, x_2, x_3\}$ then $W_1 \subset S_0 \cap C$. This contradicts (2) since $d_1^{++} \in S_0 \cap C$, $d_1 \in S_2 \cup S_3$ but $d_1^+ \notin S_1$. Suppose next $m = 2$ and $x_1 \neq d_1^+$. If $d_1^+ \notin N(x_1)$ then $d_1^+ \in S_0 \cap C$ and we are done. Otherwise, by Lemma 2.1 (6), $x_1 = d_1^{++}$ and $x_1 d_1 \notin E$. Set $S := \{x_0, x_1, x_2\}$. Since $x_1 \in S_0 \cap C$ and $d_1^+ \in N(x_1)$ we have $d_1 \in N(x_0) \cap N(x_2)$. Let ww^+ be an insertion edge of d_1^+. It follows that $w^+ \neq d_1^-$ by Lemma 2.1 (2). Since x_1 is not insertible then $N(x_1) \cap \{w, w^+, w^{++}\} = \varnothing$ (see [3]). Moreover $N(x_2) \cap \{w^+, w^{++}\} = \varnothing$ by Lemma 2.1(2). Thus $\{w^+, w^{++}\} \subset S_0 \cap C$. This is a contradiction to (2). We have proved that $X_0 = D^+$. By changing the orientation of C, we get by symmetry $Y_0 = D^-$. ∎

3. PROOFS

3.1. A new proof of Theorems 1.2 and 1.3

Proof. This is a direct consequence of Lemma 2.3 (1). If G is nonhamiltonian then $\overline{\sigma}_3(S) = n - 1$, $(S \subset X) \in I_3(G^*)$. By hypothesis, $\overline{\sigma}_3(S) \geq n$, a contradiction implying that G must be hamiltonian. ■

3.2. Proof of Theorem 1.4

By contradiction, we suppose that G satisfies the hypothesis of Theorem 1.4 but $c_0^*(G) \neq K_n$.

Proof. By Lemma 2.3, $X_0 = D^+$ and $Y_0 = D^-$ and we assume that H is chosen so that $m := |N_C(H)|$ is maximum. Two distinct cases are needed. Each one leads to an exceptional class of nonhamiltonian graphs, whose dual-closure is well characterized.

Case 1. $m = 2$.

(1) $N[x_i] = P_i \cup N_D(x_i)$, $i = 1, 2$.

Without loss of generality and by contradiction suppose that there exists $v \in P_2 \backslash N(x_2)$. Choose v as close to d_2 as possible. If $v \in N(x_1)$ then $v \neq y_2$ since x_1 is noninsertible. Moreover, by setting $S := \{x_0, x_1, x_2\}$, we see that $v^+ \in S_0 \cap C$ by Lemma 2.1(2) and the fact that x_1 is noninsertible. In that case $v \in N(x_1) \cap N(x_2)$ since clearly $v^- \notin N(x_0) \cap N(x_2)$. This is a contradiction to our assumption. Therefore $v \in S_0 \cap C$ and by the above arguments, $v^- \in N(x_1) \cap N(x_2)$. At this point we need two subcases. Suppose first $v^+ \in N(x_2)$. Clearly $G - v$ contains a cycle $C' = C \cup H$. Since C is a longest cycle, we must have $H = \{x_0\}$ and $d(x_0) = 2$. Moreover we may assume $d(v) = 2$ for otherwise, we choose C' instead of C. In particular $N_{G-C}(v) = \varnothing$. As it is easy to check that $\{x_0, x_1, v\}$ is independent in G^*, we have $d(x_1) + 4 \geq n - 1 + |N(v) \cap N(x_0) \cap N(x_1)|$. If $v = y_2$ then $|N(v) \cap N(x_0) \cap N(x_1)| = 1$ and hence $d(x_1) \geq n - 4$, that is $N(x_1) = V \backslash \{x_0, x_1, x_2, v\}$. If $v \neq y_2$ then $d(x_1) \geq n - 5$ and more precisely $N(x_1) = V \backslash \{x_0, x_1, x_2, y_2, v\}$. So, in either case, $x_1 x_2^+ \in E$, implying the existence of a cycle $C'' = C \cup H$ in $G - x_2$. As previously for the cycle C', we obtain $d(x_2) = 2$. This is a contradiction since $N(x_2) \supseteq \{d_2, x_2^+, v^+\}$.

Next, suppose $v^+ \notin N(x_2)$. If $v^+ \in N(x_1) \backslash D$, we use the above arguments to get $v^+ \in N(x_1) \cap N(x_2)$, a contradiction to the choice of v.

If $v^+ \notin N(x_1) \cup N(x_2)$ then $v, v^+ \in S_0 \cap C$, a contradiction to Lemma 2.3 (1). So, it remains to consider the case where $v^+ = d_1 \notin N(x_2)$. Now $vx_2 = y_2x_2 \notin E$ by assumption and $y_1x_2 \notin E$ as y_1 is noninsertible. Therefore, setting $S := \{x_0, y_1, y_2\}$ we obtain $x_2 \in S_0 \cap C$ and hence $x_2^+ \in N(y_1) \cap N(y_2)$. It follows that $G - x_2$ contains the cycle

$$H[d_1, d_2]\overleftarrow{C}[d_2, x_1]x_1v^-\overleftarrow{C}[v^-, x_2^+]x_2^+vd_1$$

in $C \cup H$ and consequently $d(x_2) = 2$. Similarly (recall that $x_2^+ \in N(y_1) \cap N(y_2)$) $G - y_2$ contains a cycle in $C \cup H$ and hence $d(y_2) = d(v) = 2$. This, in turn implies that $P_2 = x_2x_2^+v$. Obviously $\{x_0, x_2, v\}$ is independent in G^*. Then $d(x_0) + d(x_2) + d(v) = 6 \geq n - 1 + |N(x_0) \cap N(x_2) \cap N(v)| = n - 1$. It follows that $n \leq 7$ and $P_1 = x_1$. This is a contradiction since now $G - x_1$ contains a cycle $C \cup H$, implying $d(x_1) = 2$. This is a contradiction since $N(x_1) = \{d_1, v, x_1^+\}$. The proof of (1) is now complete.

For $i = 1, 2$ we let u_i be any vertex of P_i.

(2) $E(P_1, P_2) = \varnothing$ and $\{x_0, u_1, u_2\}$ is independent in G^*.

By contradiction suppose $u_1u_2 \in E$. Clearly $(u_1, u_2) \neq (x_1, x_2), (y_1, y_2)$. So, we may assume $u_i \in P_i \backslash \{x_i, y_i\}$, $i = 1, 2$. But then the cycle

$$H[d_1, d_2]\overleftarrow{C}[d_2, u_1)u_1^+x_1C[x_1, u_1]u_1u_2\overleftarrow{C}[u_2, x_2]x_2u_2^+C(u_2, d_1]$$

is hamiltonian. Now we show that the set $\{x_0, u_1, u_2\}$ is independent in G^*. Since $E(P_1, P_2) = \varnothing$, $N(u_1) \cap N(u_2) \subseteq D$. If there exists $v \in J(u_1, u_2) = \emptyset$, then $v \in D$ and a contradiction arises since there is a vertex of $H \cap N(D)$ which cannot be adjacent to neither u_1 nor to u_2. Similarly $J(x_0, u_1) = \emptyset$ since $N(x_0) \cap N(u_1) \subseteq D$ and $y_2 = d_1^- \notin N(x_0) \cup N(u_1)$. The same arguments apply to $J(x_0, u_2)$.

(3) $c_0^*(G) = (K_r \cup K_s \cup K_s)$ with r, s, t are positive integers.

First of all, we point out that we may have $G - C \neq H$. By Lemma 2.3 (1), $S_0 \cap (G - C) = \{x_0\}$, implying that $(G - C \cup H) = N_{G-C}(x_1) \cup N_{G-C}(x_2)$. For simplicity, set $H_i := N_{G-C}(x_i)$ for $i = 1, 2$. We observe that $H_1 \cap H_2 = \varnothing$ for otherwise $x_1 \sim x_2$, a contradiction to Lemma 2.1 (2) and $H \cap H_i = \varnothing$ for $i = 1, 2$ by maximality of C.

Since G is nonhamiltonian by assumption, its 0-dual closure $c_0^*(G)$ is not complete. Choose $S := \{x_0, u_1, u_2\}$ and set $d(u_i) = n_i + |H_i| + |N_D(u_i)| - 1 - \varepsilon_i$ where $\varepsilon_i \geq 0$ for $i = 1, 2$, $d(x_0) = |H| + |N_D(x_0)| - 1 - \varepsilon_0$ where

$\varepsilon_0 \geq 0$. By (**2**), $S \in I_3(G^*)$ and hence $\sigma_S = d(x_0)+d(u_1)+d(u_2) \geq n-1+s_3$. Since $n = 2 + n_1 + n_2 + |H| + |H_1| + |H_2|$ and $\sum_{i=0}^{2} |N_D(u_i)| \leq 6$ we get $4 + s_3 + \sum_{i=0}^{2} \varepsilon_i \leq \sum_{i=0}^{2} |N_D(u_i)| \leq 6$. We remark that $\sum_{i=0}^{2} |N_D(u_i)| = 5 \Rightarrow s_3 = 1$ and $\sum_{i=0}^{2} |N_D(u_i)| = 6 \Rightarrow s_3 = 2$. Therefore

$$\sum_{i=0}^{2} \varepsilon_i = 0 \text{ and } 4 + s_3 \leq \sum_{i=0}^{2} |N_D(u_i)| \leq 6. \tag{4}$$

As an immediate consequence of (4) we have (i) $G[H]$ is complete, (ii) $N[u_i] = N_D(u_i) \cup P_i \cup H_i$ for $i = 1, 2$ since $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$. In particular $N(x_1) \cap N(x_1^+) \supseteq H_1$ if $x_1^+ \neq d_2$, in which case $H_1 = \varnothing$ by maximality of C. If $x_1^+ = d_2$ then clearly $C \cup H - x_1$ contains a cycle C' for which $|N_{C''}(x_1)| \geq 3 > m$, a contradiction to the choice of H. Similarly we have $H_2 = \varnothing$, that is $G - C = H$.

It remains now to show that each vertex of D is dominating in $c_0^*(G)$, that is $D = \Omega$. Without loss of generality, suppose $d_1u_2 \notin E(c_0^*(G))$ and $|N(d_1) \cap S| < 3$ is minimum. If $|N(d_1) \cap S| = 0$ then $\sum_{i=0}^{2} |N_D(u_i)| \leq 3$, a contradiction to (4). If $|N(d_1) \cap S| = 1$ then $|N(d_2) \cap S| \geq 3$ and $s_3 \geq 1$, leading to again a contradiction. So we may assume $|N(d_1) \cap S| = 2$ and hence $N(d_1) \supset H \cup P_1$ since x_0, u_1 are arbitrarily chosen. But then $\overline{\alpha}_{d_1u_2} \leq |\{d_1, d_2, u_2\}| = 3$ (recall that $N[u_2] \supseteq P_2$). Because $d(d_2) \geq 3 \geq \overline{\alpha}_{d_1u_2}$, we contradict the assumption $d_1u_2 \notin E(c_0^*(G))$ by Theorem 1.1. Therefore $N(d_i) \supseteq V \backslash D$ is true for $i = 1, 2$. It is also easy to see that $d_2d_1 \in E(c_0^*(G))$ since $\overline{\alpha}_{d_1d_2}(c_0^*(G)) \leq 2$. As claimed $d_{c_0^*(G)}(d_i) = n-1$, $i = 1, 2$. Since H, P_1, P_2 are distinct complete components of $G - C$ we obviously have, as claimed, $c_0^*(G) = (K_r \cup K_s \cup K_s) \vee K_2$ where, $r = |H|$, $s = n_1$, $t = n_2$ and K_2 is induced by D.

Case 2. $m > 2$.

We have already proved in Case 1 (3) that $(G - C) = H$ if $m > 2$. We next prove

(1) $G - x_i$ ($G - y_i$ resp.) is hamiltonian for all $i = 0, \ldots, m$ and hence $d(x_i) \leq m$ ($d(y_i) \leq m$ resp.).

By setting $S := \{x_1, x_2, x_3\}$, we get $H \subset S_0 \cap (G-C)$ and hence $G-C = \{x_0\}$ by Lemma 2.3. Thus (1) is true for $i = 0$. Obviously (1) is true whenever $n_i = 1$. Otherwise, suppose for instance $n_1 > 1$ and set $S := \{x_0, x_2, x_3\}$. Clearly $x_1^+ \notin S_0 \cap C$ by Lemma 2.2 since $x_1 \notin S_1 \cup S_2 \cup S_3$. Therefore

$x_1^+ \in N(x_2) \cup N(x_3)$. Whether $x_2x_1^+ \in E$ or $x_3x_1^+ \in E$, $G - x_1$ is obviously hamiltonian and (1) is true. From now on and by the choice of C, we may assume $d(x_i) \leq m$ ($d(y_i) \leq m$ by symmetry. As a next step we prove.

(2) $|N_{X_0}(d_i)| \geq m-1$ and $|N_{Y_0}(d_i)| \geq m-1$ holds for any $d_i \in D$.

Otherwise choose x_i, x_j with $1 < i < j \leq m$ such that $N(d_1) \cap \{x_i, x_j\} = \emptyset$. Set $S := \{x_1, x_i, x_j\}$. Clearly $x_1 \in S_0 \cap C$ and hence, $d_1 = x_1^- \in S_1$ and $d_1^- = y_m \in N(x_i) \cap N(x_j)$. Suppose first $m \geq 4$, set $S := \{x_h, x_i, x_j\} \subset X_0$ and assume $i < j$. Choose, if possible, i minimum. If $h > i$ then $x_hd_1 \notin E$ by Lemma 2.1 (2). By the choice of i, we must have $j = m$, $i = m-1$ and $1 < h < i$. Moreover $x_hd_1 \in E$ for otherwise x_1, d_1 are consecutive elements of $S_0 \cap C$. Consider now $S := \{x_0, x_1, x_h\}$. Clearly $y_m \in S_0 \cap C$ since x_1, x_h are noninsertible. But then $y_m^- \in N(x_1) \cup N(x_h)$, a contradiction to Lemma 2.1 (2). It remains to consider the case $m = 3$, in which case $d_1^- = y_3 \in N(x_1) \cap N(x_2)$. This implies in turn that $n_2 \geq 2$ and $n_3 \geq 2$. Since $d(y_3) \leq m = 3$, we get $N(y_3) = \{d_1, x_2, x_3, y_3^-\}$, implying $x_3 = y_3^-$ and hence $n_3 = 2$. In G^*, we clearly have $x_0x_1 \notin E(G^*)$ and $x_0y_3 \notin E(G^*)$. It is now easy to check that $x_1y_3 \notin E(G^*)$ since $N(x_1) \cap N(y_3) \subset \{d_1\}$ and $x_0 \in N(d_1) \setminus \{x_1, y_3\}$. Therefore $\{x_0, x_1, y_3\} \in I_3(G^*)$. Thus $d(y_3) + d(x_0) + d(x_1) \geq n - 1 + s_3 = n$. As $n_2 \geq 2$, $n_3 \geq 2$ we must have $n_1 = 1$, $n_2 = 2$, $n_3 = 2$ and $d(x_1) = d(x_2) = d(x_3) = 3$. We note that $x_3d_1 \notin E$ for otherwise we have edges crossing at x_2 and x_3, $x_3d_2 \notin E$ for otherwise replacing d_2x_2 by $d_2x_3y_3x_2$ and $d_3x_3y_3d_1$ by $d_3x_0d_1$ in C we get a hamiltonian cycle. Moreover $x_3y_2 \notin E$ since x_3 is noninsertible and $x_3x_2 \notin E$. Thus $N(x_3) = \{d_2, y_3\}$, a contradiction to the fact that $d(x_3) = 3$. The proof of (2) is now complete.

(3) $X = Y$.

By contradiction, suppose $X \neq Y$. As a first step, we show that (3) is true if there exists $x_i \in X_0$ such that $N_D(x_i) = D$. Without loss of generality, assume $N_D(x_1) = D$. Since $d(x_1) \leq m$, we deduce that $N(x_1) = D$ and hence $x_1 = y_1$, that is $n_1 = 1$. Suppose next $n_i > 1$ for some i, $2 \leq i < m$ and set $S := \{x_0, x_1, x_{i+1}\}$. Clearly $y_i \in S_0 \cap C$ and hence $y_i^- \in N(S)$. Obviously $y_i^- \notin N(x_0) \cup N(x_1)$ and consequently $y_i^- \in N(x_{i+1})$, $y_i^{--} \in N(x_0) \cap N(x_1)$. This means that $y_i^{--} = d_i$, a contradiction since then $y_i^- = x_i$. Therefore $n_i = 1$ for any $i, 1 \leq i < m$. To prove that $n_m = 1$ it suffices to consider $S := \{x_0, x_1 = y_1, y_{m-1}\}$ and to use the same arguments.

For the remainder we assume that $|N_D(x_i)| < m$ is true for all $x_i \in X_0$. Consider the graph $G[D \cup X_0]$. By (2) we have $|E(D, X_0)| \geq m(m-1)$. On the other hand we have $|E(X_0, D)| \leq m(m-1)$ since $|N_D(x_i)| < m$ for all

$x_i \in X_0$. Therefore the equality holds and $|N_D(x_i)| = m-1$ for all $x_i \in X_0$ and $|N_{X_0}(d_i)| = m-1$ for all $d_i \in D$. By symmetry $|N_D(y_i)| = m-1$ for all $y_i \in Y_0$ and $|N_{Y_0}(d_i)| = m-1$ for all $d_i \in D$. Suppose now that $d_1x_i \notin E$ in $c_0^*(G)$ for some $i > 1$. By (3), $N_X(d_1) = X \backslash \{x_i\}$ and $N_Y(d_1) = X \backslash \{y_j\}$ for some $j > 0$. Therefore $T_{d_1x_i} \subseteq \{y_j\}$ and $\overline{\alpha}_{d_1x_i} \leq 3$. As $d(y_j) \geq 3$ we have $d_1x_i \in E(c_0^*(G))$ by Theorem 1.1. With this contradiction, (3) is proved.

(4) $c_0^*(G) = K_{\frac{n-1}{2}} \vee \overline{K}_{\frac{n+1}{2}}$.

Consider again the dual closure $c_0^*(G)$ and suppose $x_1d_h \notin E$ for some $h > 0$. By (3) and the fact that $|N_D(x_h)| = m-1$, $N(x_1) \cup N(d_h) \cup \{x_1, d_h\} = V$, implying $x_1d_h \in E(c_0^*(G))$. Therefore $N_D(x_i) = D$ holds for any $x_i \in X_0$. It remains to show that D is a clique in $c_0^*(G)$. Indeed, if $d_1d_j \notin E$ then $\overline{\alpha}_{d_1dj} \leq |D| = m$ and $\delta_{d_1dj} \geq m$ since $T_{d_1dj} \subset D$ and $d(d_i) \geq m$ for any $d_i \in D$. By Theorem 1.1, $d_1d_j \in E(c_0^*(G))$. It remains to note that $|D| = m = \frac{n-1}{2}$ by (3) and hence $c_0^*(G) = K_{\frac{n-1}{2}} \vee \overline{K}_{\frac{n+1}{2}}$. ■

4. Concluding Remarks

For any independent triple $S = \{a, b, c\}$, we set $\lambda_{\min}(S) := \min\{\lambda_{ab}, \lambda_{bc}, \lambda_{ca}\}$, where λ_{xy}, $xy \notin E$ stands for the number of vertices adjacent to both x and y. In [6], we obtained the following result, related to Theorem 1.4.

Theorem 4.1. *Let G be a 2-connected graph. If*

$$S \in I_3(G) \Rightarrow \sigma_S \geq n - 1 + \lambda_{\min}(S) \tag{5}$$

then $c_0^*(G) \in \{C_7, K_n, (K_r \cup K_s \cup K_t) \vee K_2, K_{(\frac{n-1}{2})} \vee \overline{K}_{(\frac{n+1}{2})}\}$.

The graph C_7 is the cycle on 7 vertices. In fact this result is still valid if we change the condition $S \in I_3(G)$ by $S \in I_3(G^*)$. From this result one can derive nearly twenty corollaries which are improvements of known sufficient conditions (see [6]).

Since $\lambda_{\min}(S) \geq s_3$, a natural open question is the following:

Problem 4.2. A 2-connected graph G satisfying the condition $S \in I_3(G^*) \Rightarrow \overline{\sigma}_S \geq n-1$ is hamiltonian if and only if $c_0^*(G) \in \{C_7, K_n\}$.

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