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MORE ON EVEN $[a, b]$-FACTORS IN GRAPHS

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Abstract

In this note we give a characterization of the complete bipartite graphs which have an even (odd) $[a, b]$-factor. For general graphs we prove that an a-edge connected graph G with n vertices and with $\delta(G) \geq \max\{a+1, \frac{an}{a+b} + a - 2\}$ has an even $[a, b]$-factor, where a and b are even and $2 \leq a \leq b$. With regard to the edge-connectivity this result is slightly better than one of the similar results obtained by Kouider and Vestergaard in 2004 and unlike their results, this result has no restriction on the order of graphs.

Keywords: $[a, b]$-factor; spanning graph; edge-connectivity.

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1. Introduction

We follow the notations and terminology of [6] except otherwise stated. All graphs in this paper are simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $\deg_G(v)$ the degree of a vertex v in G and $\delta(G)$ the minimum degree of G. Let (X, Y) be an ordered pair of distinct subsets of $V(G)$. In this note, $e(X, Y)$ denotes the number of edges with one endvertex in X and the other in Y. Let $a,\ b$ be two integers with $1 \leq a \leq b$. A spanning subgraph F of G is called an $[a, b]$-*factor* of G if $a \leq \deg_F(v) \leq b$ for all $v \in V(G)$. An $[a, b]$-factor F is *even* (*odd*) if both a and b are even (odd) and $\deg_F(v)$ is even (odd), respectively, for each $v \in V(F)$. When $a = b = k$, an $[a, b]$-factor is called a k-*factor*.

The definition of $[a, b]$-factor can be generalized as follows ([5]). Let g, f be two functions from $V(G)$ into non-negative integers and let $g(v) \leq f(v)$

for all $v \in V(G)$. Then F is called a $[g, f]$-factor of G if F is a factor of G with $g(v) \leq \deg_F(v) \leq f(v)$ for all $v \in V(G)$. Lovász' parity $[g, f]$-factor theorem (Theorem 5, or see [1, 5]) gives a characterization of the graphs which have a $[g, f]$-factor. Applying this theorem, Kouider and Vestergaard obtained various sufficient conditions for a graph to have an even $[a, b]$-factor.

Theorem 1 ([2, 3]). *Let a and b be even integers such that $2 \leq a \leq b$ and let G be a connected graph of order n.*

1. *If $a = 2$, $n \geq 3$ and G is 2-edge connected with $\delta(G) \geq \max\{3, \frac{2n}{b+2}\}$, then G has an even $[2, b]$-factor.*
2. *If $a \geq 4$, $n \geq \frac{(a+b)^2}{b}$ and G is 2-edge connected with $\delta(G) \geq \frac{an}{a+b} + \frac{a}{2}$, then G has an even $[a, b]$-factor.*
3. *If $a \geq 4$, $n \geq \frac{(a+b)^2}{b}$, G is k-edge connected with $k \geq a + \min\{\sqrt{a}, \frac{b}{a}\}$ and $\delta(G) \geq \frac{an}{b+2}$, then G has an even $[a, b]$-factor.*

In addition, Kouider and Vestergaard [2] obtained a characterization of the complete bipartite graphs which have an even $[2, b]$-factor. The reader may consult [4] for a recent survey on connected factors in graphs. In this paper, we first generalize this result to even $[a, b]$-factor. Then we give a characterization of the complete bipartite graphs which have an odd $[a, b]$-factor. For general graphs, we modify the proof of Theorem 2 of [2] to prove that an a-edge connected graph G with n vertices and with $\delta(G) \geq max\{a + 1, \frac{an}{a+b} + a - 2\}$ has an even $[a, b]$-factor, where a and b are even and $2 \leq a \leq b$. The edge-connectivity condition required in our result is slightly weaker than the one given in Part (3) of Theorem 1. Moreover, unlike Part (2) and Part (3) of Theorem 1, in our result there is no condition on the order of the graph. But we require a slightly larger minimum degree for G.

2. $[a, b]$-Factors for Complete Bipartite Graphs

In this section we first prove that the complete bipartite graph $K_{p,q}$ has an even $[a, b]$-factor if and only if $aq \leq bp$, where a, b are even, $2 \leq a \leq b$ and $a + 1 \leq p \leq q$. This generalizes Theorem 1 of [2]. Then we prove that $K_{p,q}$ has an odd $[a, b]$-factor if and only if $aq \leq bp$ and $p \equiv q \pmod 2$, where a, b are odd, $1 \leq a \leq b$ and $a + 1 \leq p \leq q$. We make use of the following well-known result in this section [6].

Lemma 2. *The complete bipartite graph $K_{n,n}$ has an m-factor for $1 \leq m \leq n$.*

Theorem 3. *Let a, b be even, $2 \leq a \leq b$ and $a+1 \leq p \leq q$. Then $K_{p,q}$ has an even $[a,b]$-factor if and only if $aq \leq bp$.*

Proof. Let $\mathcal{F}$ be an even $[a,b]$-factor of $K_{p,q}$. Then obviously $aq \leq |E(\mathcal{F})| \leq bp$.

Conversely, assume $aq \leq bp$. If $b > q$, it suffices to prove that $K_{p,q}$ has an even $[a,q]$ or $[a,q-1]$-factor, depending on the parity of q. Because

$$aq \leq (p-1)q = pq - q \leq pq - p = (q-1)p \leq qp,$$

this case can be reduced to the case $b = q$ or $b = q-1$, depending on the parity of q. So we may assume $b \leq q$. Define $X = \{x_i \mid 0 \leq i \leq p-1\}$ and $Y = \{y_j \mid 0 \leq j \leq q-1\}$.

Case 1. q is even.
Let M_i be the complete bipartite graph with partite sets $\{x_{ai+j} \mid 0 \leq j \leq a-1\}$ and $\{y_{2i}, y_{2i+1}\}$ for $0 \leq i \leq (q-2)/2$. Notice that the addition in the subscript of x is modulo p. Since $a(q/2) \leq (b/2)p$ it follows that each $x \in X$ is a vertex of at most $b/2$ complete bipartite graphs M_i. Define

$$\mathcal{F} = \bigcup_{i=0}^{(q-2)/2} M_i.$$

Obviously, $\deg_{\mathcal{F}}(y) = a$ for all $y \in Y$. By the construction of $\mathcal{F}$ we have $\mid \deg_{\mathcal{F}}(x_i) - \deg_{\mathcal{F}}(x_j) \mid \leq 2$ for $i, j \in \{0, 1, 2, \ldots, p-1\}$. On the other hand, $\sum_{x \in X} \deg_{\mathcal{F}}(x) = \sum_{y \in Y} \deg_{\mathcal{F}}(y) = aq$. Hence, $a \leq \deg_{\mathcal{F}}(x) \leq b$ for all $x \in X$.

Case 2. q is odd.
If $p = q$ we apply Lemma 2. For $p \leq q-1$ we proceed as follows. Let $\mathcal{F}$ be an even $[a,b]$-factor of $K_{p,q-1}$ as in Case 1.

Claim 1. $\deg_{\mathcal{F}}(x_i) \leq b-2$ for $p-(a/2) \leq i \leq p-1$.
If $\deg_{\mathcal{F}}(x) \leq b-2$ for every $x \in X$, then the result follows since $p \geq a+1 \geq a/2$. Now assume that there exists an $x \in X$ with $\deg_{\mathcal{F}}(x) = b$. The construction of $\mathcal{F}$ enforces that $\deg_{\mathcal{F}}(x) \in \{b, b-2\}$, for every $x \in X$. Let

α and β be the number of degree b and of degree $b-2$ vertices of X in $\mathcal{F}$, respectively. Then we have

$$\alpha + \beta = p \text{ and } \alpha b + \beta(b-2) = (q-1)a.$$

Solving for β, we obtain $\beta = \frac{bp-aq+a}{2} \geq \frac{a}{2}$. Now the result follows by the construction of $\mathcal{F}$.

For convenience, we may relabel the vertices of X such that $\deg_{\mathcal{F}}(x_1) \leq \deg_{\mathcal{F}}(x_2) \leq \cdots \leq \deg_{\mathcal{F}}(x_p)$. Now, let $X_1 = \{x_1, x_2, \ldots x_{a/2}\}$ and $X_2 = \{x_{(a/2)+1}, x_{(a/2)+2}, \ldots, x_p\}$. By Claim 1, every vertex of X_1 has degree $\leq b-2$ in $\mathcal{F}$.

Claim 2. There are $\{y_{i_1}, y_{i_2}, \ldots y_{i_{a/2}}\} \subset Y' = Y \setminus \{y_{q-1}\}$ and $\{x_{i_1}, x_{i_2}, \ldots x_{i_{a/2}}\} \subset X_2$ such that $|\{x_{i_1}, x_{i_2}, \ldots x_{i_{a/2}}\}| = a/2$, $x_j y_{i_j} \notin E(\mathcal{F})$ and $x_{i_j} y_{i_j} \in E(\mathcal{F})$ for $1 \leq j \leq a/2$. Notice that the elements of $\{y_{i_1}, y_{i_2}, \ldots y_{i_{a/2}}\}$ are not necessarily distinct.

Since $\deg_{\mathcal{F}}(x_1) \leq b-2$ there exists $y_{i_1} \in Y'$ such that $x_1 y_{i_1} \notin E(\mathcal{F})$. Now the fact that $\deg_{\mathcal{F}}(y_{i_1}) = a$ implies that there exists $x_{i_1} \in X_2$ with $x_{i_1} y_{i_1} \in E(\mathcal{F})$. Assume there exists y_{i_t} and x_{i_t} with the required properties for $1 \leq t \leq (a/2) - 1$. We have $\deg_{\mathcal{F}}(x_{t+1}) \leq b-2$ and, hence, there exists $y_{i_{t+1}} \in Y'$ such that $x_{t+1} y_{i_{t+1}} \notin E(\mathcal{F})$. Again, the fact that $\deg_{\mathcal{F}}(y_{i_{t+1}}) = a$ together with $|\{x_{i_j} \mid 1 \leq j \leq t\} \cup \{x_k \mid 1 \leq k \leq (a/2)\}| < a$ show that there exists $x_{i_{t+1}} \in (X_2 \setminus \{x_{i_j} \mid 1 \leq j \leq t\})$ with $x_{i_{t+1}} y_{i_{t+1}} \in E(\mathcal{F})$. Now the result follows.

Define

$$\mathcal{F}' = (\mathcal{F} \setminus \{x_{i_j} y_{i_j} \mid 1 \leq j \leq a/2\}) \cup \{x_j y_{i_j}, x_j y_{q-1}, x_{i_j} y_{q-1} \mid 1 \leq j \leq a/2\}.$$

Obviously, $\mathcal{F}'$ is an even $[a,b]$-factor of $K_{p,q}$. ■

Theorem 4. *Let a, b be odd, $1 \leq a \leq b$ and $a+1 \leq p \leq q$. Then $K_{p,q}$ has an odd $[a,b]$-factor if and only if $aq \leq bp$ and $p \equiv q \pmod 2$.*

Proof. Suppose that $\mathcal{F}$ is an odd $[a,b]$-factor of $K_{p,q}$, then $aq \leq |E(\mathcal{F})| \leq bp$. Since the number of odd vertices is even for any graph and $\mathcal{F}$ has odd degree vertices only, then $|V(\mathcal{F})| = p+q$ is even. Therefore $p \equiv q \pmod 2$.

Conversely, assume $aq \leq bp$ and $p \equiv q \pmod 2$. For fixed a, b and p, by induction on q, we will prove that $K_{p,q}$, for $q \in \{k \mid p \leq k,\ p \equiv k \pmod 2 \text{ and } ak \leq bp\}$, has an odd $[a,b]$-factor $\mathcal{F}$ such that $\deg_{\mathcal{F}}(y_i) = a$

for $0 \le i \le q-1$ and the degree difference of any two vertices of X in $\mathcal{F}$ is at most 2.

Let $X = \{x_i \mid 0 \le i \le p-1\}$ and $Y = \{y_j \mid 0 \le j \le q-1\}$ be the partite sets of $K_{p,q}$. By Lemma 2 the statement is true for $q = p$. Suppose that the statement is true for $q \le k-2$, where $k \ge p+2$ and $k \equiv p \pmod 2$. We will prove that it is true for $q = k$ if $ak \le bp$. Since $ak \le bp$ and $k \equiv p \pmod 2$ it follows that $a(k-2) \le bp$ and $k-2 \equiv p \pmod 2$. Now by the induction hypotheses, $K_{p,k-2}$ has an odd $[a,b]$-factor $\mathcal{F}'$ such that $\deg_{\mathcal{F}'}(y_i) = a$ for $0 \le i \le k-3$ and the degree difference of any two vertices of X in $\mathcal{F}'$ is at most 2. For convenience, let us relabel the vertices of X in $\mathcal{F}'$ such that $\deg_{\mathcal{F}'}(x_0) \le \deg_{\mathcal{F}'}(x_1) \le \cdots \le \deg_{\mathcal{F}'}(x_{p-1})$. If $\deg_{\mathcal{F}'}(x_{a-1}) = b$ then $\deg_{\mathcal{F}'}(x_\ell) = b$ for $a-1 \le \ell \le p-1$. Now we have

$$a(k-2) = |E(\mathcal{F}')| > (b-2)a + b(p-a)$$

which implies that $ak > bp$, a contradiction. Therefore $\deg_{\mathcal{F}'}(x_{a-1}) \le b-2$. Now define

$$\mathcal{F} = \mathcal{F}' \cup \bigcup_{i=0}^{a-1} \{y_{k-2}x_i, y_{k-1}x_i\}.$$

Obviously, $\mathcal{F}$ is the required odd $[a,b]$-factor of $K_{p,q}$. ■

3. Even $[a,b]$-Factors for General Graphs

Kouider and Vestergaard [3] apply Lovász' parity $[g,f]$-factor theorem (Theorem 5 below) to find simple sufficient conditions for graphs to contain even $[a,b]$-factors. In this section we also apply Theorem 5 to obtain different sufficient conditions for graphs to have even $[a,b]$-factors. With regard to the edge-connectivity our result is slightly better than Part (3) of Theorem 1 and with regard to the minimum degree our result is slightly worse. But our result has no restriction on the order of the graph.

Let g, f be be two functions from $V(G)$ into the non-negative integers such that $g(v) \le f(v)$ for each $v \in V(G)$, and let (X,Y) be an ordered pair of disjoint subsets of $V(G)$. We use $h(X,Y)$ for the number of components C of $G \setminus (X \cup Y)$ with $e(V(C),Y) + \sum_{v \in V(C)} f(v)$ odd. Now let us state Lovász' parity $[g,f]$-factor theorem.

Theorem 5 ([5]). *Let G be a graph. Let g and f map $V(G)$ into the non-negative integers such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod 2$ for every $v \in V(G)$. Then G contains a $[g, f]$-factor $\mathcal{F}$ such that $\deg_{\mathcal{F}}(v) \equiv f(v) \pmod 2$ for every $v \in V(G)$, if and only if, for every ordered pair (X, Y) of disjoint subsets of $V(G)$*

$$\sum_{y \in Y} \deg_G(y) - \sum_{y \in Y} g(y) + \sum_{x \in X} f(x) - h(X, Y) - e(X, Y) \geq 0.$$

Let $a, b \geq 2$ be even and let $g(v) = a$ and $f(v) = b$ for every $x \in V(G)$ in Theorem 5. Then we obtain

Corollary 6. *G contains an even $[a, b]$-factor if*

$$\sum_{y \in Y} \deg_G(y) - a|Y| + b|X| - h(X, Y) - e(X, Y) \geq 0 \tag{1}$$

for all ordered pairs (X, Y) of disjoint subsets of $V(G)$.

Now we prove the main result of this section.

Theorem 7. *Let $a, b \geq 2$ be even and let G be an a-edge connected graph with n vertices and with minimum degree $\delta(G) \geq \max\{a + 1, \frac{an}{a+b} + a - 2\}$. Then G contains an even $[a, b]$-factor.*

Proof. Let (X, Y) be any ordered pair of disjoint subsets of $V(G)$. Then

$$\sum_{y \in Y} \deg_G(y) \geq e(Y, V(G) \setminus Y) \geq e(X, Y) + h(X, Y)$$

and, hence,

$$\sum_{y \in Y} \deg_G(y) - a|Y| + b|X| - h(X, Y) - e(X, Y) \geq -a|Y| + b|X|. \tag{2}$$

Thus, if $-a|Y| + b|X| \geq 0$, then inequality (1) holds and the result follows. Now assume there is an ordered pair (X, Y) of disjoint subsets of $V(G)$ for which

$$-a|Y| + b|X| < 0. \tag{3}$$

If $|X| \geq \delta(G) - a + 2 = \delta - a + 2$, then (3) together with $|X| + |Y| \leq n$ imply

$$\delta - a + 2 \leq |X| < \frac{a}{b}|Y| \leq \frac{a}{b}(n - |X|) \leq \frac{a}{b}(n - \delta + a - 2)$$

and, hence, $\delta < \frac{an}{a+b} + a - 2$, a contradiction. Therefore, $|X| \leq \delta - a + 1$. Now we consider three cases.

Case 1. $|Y| \geq b + 1$.
Obviously, $e(X,Y) \leq |X||Y|$. The fact that each odd component of $G \setminus (X \cup Y)$ has at least one vertex leads to $h(X,Y) \leq n - |X| - |Y|$. Define

$$\tau = \sum_{y \in Y} \deg_G(y) - a|Y| + b|X| - h(X,Y) - e(X,Y).$$

Then

$$\begin{aligned}
\tau &\geq \delta|Y| - a|Y| + b|X| - n + |X| + |Y| - |X||Y| \\
&= (\delta - a + 1)|Y| + ((b+1) - |Y|)|X| - n \\
&\geq (\delta - a + 1)|Y| + (b + 1 - |Y|)(\delta - a + 1) - n \\
&= (b+1)(\delta - a + 1) - n \\
&\geq (b+1)\Big(\frac{an}{a+b} - 1\Big) - n = \frac{(a-1)bn}{a+b} - b - 1.
\end{aligned}$$

If $n \geq \frac{a+b+2}{a-1}$, then $\tau \geq \frac{-a+b}{a+b} \geq 0$ since $b \geq a$. Now let $n \leq \frac{a+b+2}{a-1}$. Then, since $\delta \geq a + 1$, we have

$$\tau \geq (b+1)(\delta - a + 1) - n \geq 2(b+1) - \frac{a+b+2}{a-1} \geq 0.$$

Therefore, (1) holds and, hence, the theorem is true.

Case 2. $|Y| \leq b$ and $|X|$ is even.
Notice that we still have $-a|Y| + b|X| < 0$ and $|X| \leq \delta - a + 1$. From $|X| < \frac{a}{b}|Y| \leq a$ we have $|X| \in \{0, 1, 2, \ldots, \theta\}$, where $\theta \leq \min\{\delta - a + 1, a - 2\}$. Let $h_1 = h_1(X,Y)$ be the number of odd components C of $G \setminus (X \cup Y)$ with $e(C,Y) \leq a - |X| - 1$, and let $h_2 = h_2(X,Y)$ be the number of odd components C of $G \setminus (X \cup Y)$ with $e(C,Y) \geq a - |X| + 1$. We consider two subcases.

Subcase 2.1. $|Y| \leq b$ and $|X| = 0$.
Since G is a-edge connected and $|X| = 0$ it follows that $e(X,Y) = 0$, $h_1 = 0$ and, hence, $h(X,Y) = h_2$. Furthermore, since $e(C,Y) \geq a+1-|X| = a+1$ for every odd component C of $G \setminus (X \cup Y) = G \setminus Y$, we have $\sum_{y \in Y} \deg_G(y) \geq (a+1)h_2$. So we have

$$\begin{aligned} \tau &= \sum_{y \in Y} \deg_G(y) - a|Y| + b|X| - h(X,Y) - e(X,Y) \\ &\geq (a+1)h_2 - a|Y| - h_2 \\ &= ah_2 - a|Y|. \end{aligned}$$

If $|Y| \leq h_2$ then (1) holds. Now let $|Y| > h_2$. Since $\delta \geq a+1$, we have

$$\tau \geq \delta|Y| - a|Y| - h_2 \geq (a+1)|Y| - a|Y| - h_2 > 0.$$

Therefore, (1) holds in this case.

Subcase 2.2. $|Y| \leq b$ and $2 \leq |X| \leq \theta$, where $\theta = \min\{\delta - a + 1, a - 1\}$. Notice that $|X| \leq a-2$ since $|X|$ is even. We have

$$\begin{aligned} \tau &= \sum_{y \in Y} \deg_G(y) - a|Y| + b|X| - h(X,Y) - e(X,Y) \\ &\geq [h_1 + (a - |X| + 1)h_2 + e(X,Y)] - a|Y| + b|X| - h_1 - h_2 - e(X,Y) \\ &= (a - |X|)h_2 - a|Y| + b|X|. \end{aligned}$$

If $|Y| \leq \frac{(a-|X|)h_2 + b|X|}{a}$ then (1) holds. Now let $|Y| > \frac{(a-|X|)h_2 + b|X|}{a}$. Then

$$h_2 < \frac{a|Y| - b|X|}{a - |X|} \leq \frac{ab - b|X|}{a - |X|} = b.$$

Since $b|X| - e(X,Y) \geq b|X| - |X||Y| = (b - |Y|)|X| \geq 0$ we obtain

$$\tau \geq \sum_{y \in Y} \deg_G(y) - a|Y| - h(X,Y) \geq (\delta - a)|Y| - h_1 - h_2. \tag{4}$$

By assumption we see that $\delta - |X| \geq a - 1$. We show that an h_1-component C of G has at least $\delta - |X| + 1$ vertices. Let $|C| = k$ and $c_1 \in C$. Then

$a+1 \le \delta \le \deg_G(c_1) \le (a-|X|-1)+|X|+(k-1)$, which implies $k \ge 3$. Moreover,

$$k|X| + (a-|X|-1) + k(k-1) \ge \sum_{c\in C} \deg_G(c) \ge k\delta.$$

This leads to $k(k-1) \ge (k-1)(\delta-|X|)+\delta-a+1$. Since k is an integer, we have

$$k \ge \delta - |X| + \frac{\delta-a+1}{k-1} \ge \delta - |X| + \frac{2}{k-1},$$

which implies $k \ge \delta - |X| + 1$. Therefore

$$h_1 \le \frac{n-|Y|-|X|-h_2}{\delta-|X|+1}. \tag{5}$$

By assumption $\delta - |X| + 1 > \delta - a + 2 \ge \frac{an}{a+b}$. This leads to

$$\frac{a+b}{a} > \frac{n}{\delta-|X|+1}. \tag{6}$$

Now by (4) we have

$$\begin{aligned}
\tau &\ge (\delta-a)|Y| - h_1 - h_2 \\
&\ge (\delta-a-1)|Y| + (|Y|-h_2) - \tfrac{n}{\delta-|X|+1} + \frac{|Y|+|X|+h_2}{\delta-|X|+1} \\
&\ge (\delta-a-1)|Y| + \left(\frac{(b-h_2)|X|}{a}\right) - \frac{n}{\delta-|X|+1} \\
&\ge (\delta-a-1)|Y| - \frac{a+b}{a}.
\end{aligned}$$

If $\delta \ge a+2$ then $\tau \ge |Y| - \frac{a+b}{a} \ge 0$ since $|Y| > \frac{(a-|X|)h_2+b|X|}{a} \ge \frac{2b}{a}$. Now let $\delta = a+1$. Then by assumption $|X| = \min\{\delta-a+1, a-1\} = 2$, $\delta-|X|+1 = a$, $a+b > n$ and $a \ge 4$. Since $h_1+h_2 \le n-|Y|-|X| \le a+b-|Y|-2$, we have

$$\begin{aligned}
\tau &= \sum_{y\in Y} \deg_G(y) - a|Y| + b|X| - h(X,Y) - e(X,Y) \\
&\ge \delta|Y| - a|Y| + 2b - (a+b-|Y|-2) - 2|Y| \\
&= (\delta-a-1)|Y| + b - a + 2 \ge 0.
\end{aligned}$$

Case 3. $|Y| \le b$ and $|X|$ is odd.
Notice that $1 \le |X| \le \theta$, where $\theta = \min\{\delta - a + 1, a - 1\}$. Let $h_1 = h_1(X,Y)$ be the number of odd components C of $G \setminus (X \cup Y)$ with $e(C,Y) \le a - |X|$, and let $h_2 = h_2(X,Y)$ be the number of odd components C of $G \setminus (X \cup Y)$ with $e(C,Y) \ge a - |X| + 2$. We have

$$\begin{aligned}
\tau &= \sum_{y \in Y} \deg_G(y) - a|Y| + b|X| - h(X,Y) - e(X,Y) \\
&\ge [h_1 + (a - |X| + 2)h_2 + e(X,Y)] - a|Y| + b|X| - h_1 - h_2 - e(X,Y) \\
&= (a - |X| + 1)h_2 - a|Y| + b|X|.
\end{aligned}$$

If $|Y| \le \frac{(a-|X|+1)h_2 + b|X|}{a}$ then (1) holds. Now let

$$|Y| > \frac{(a - |X| + 1)h_2 + b|X|}{a}. \tag{7}$$

Then

$$h_2 < \frac{a|Y| - b|X|}{a - |X| + 1} \le \frac{ab - b|X|}{a - |X| + 1} < b.$$

Since $b|X| - e(X,Y) \ge b|X| - |X||Y| = (b - |Y|)|X| \ge 0$ we obtain

$$\tau \ge \sum_{y \in Y} \deg_G(y) - a|Y| - h(X,Y) \ge (\delta - a)|Y| - h_1 - h_2. \tag{8}$$

By assumption we see that $\delta - |X| \ge a - 1$. We show that an h_1-component C of G has at least $\delta - |X| + 1$ vertices. Let $|C| = k$ and $c_1 \in C$. Then $a + 1 \le \delta \le \deg_G(c_1) \le (a - |X|) + |X| + (k - 1)$, which implies $k \ge 2$. Moreover,

$$k|X| + (a - |X|) + k(k - 1) \ge \sum_{c \in C} \deg_G(c) \ge k\delta.$$

This leads to $k(k-1) \ge (k-1)(\delta - |X|) + \delta - a$ and, hence, $k \ge \delta - |X| + 1$. Therefore $h_1 \le \frac{n - |Y| - |X| - h_2}{\delta - |X| + 1}$. By assumption

$$\delta - |X| + 1 \ge \delta - a + 2 \ge \frac{an}{a + b}.$$

This leads to $\frac{a+b}{a} \geq \frac{n}{\delta-|X|+1}$. Now by (8) we have

$$\begin{aligned}
\tau &\geq (\delta-a)|Y| - h_1 - h_2 \\
&\geq (\delta-a-1)|Y| + (|Y|-h_2) - \frac{n}{\delta-|X|+1} + \frac{|Y|+|X|+h_2}{\delta-|X|+1} \\
&\geq (\delta-a-1)|Y| + \left(\frac{(b-h_2)|X|+h_2}{a}\right) - \frac{n}{\delta-|X|+1} \\
&\geq (\delta-a-1)|Y| - \frac{a+b}{a}.
\end{aligned}$$

Let $\delta \geq a+2$. If $|X| \neq 1$ or $h_2 \neq 0$ then $\tau \geq |Y| - \frac{a+b}{a} \geq 0$ by (7). If $|X| = 1$ and $h_2 = 0$ then $|Y| > \frac{b}{a}$ and

$$\tau \geq (\delta-a)|Y| - h_1 - h_2 \geq 2|Y| - \frac{a+b}{a} \geq \frac{2b}{a} - \frac{a+b}{a} \geq 0.$$

Now let $\delta = a+1$. Then by assumption $|X| = 1$ and $|Y| > h_2 + \frac{b}{a}$. If $h_1 = 0$ then $\tau \geq (\delta-a)|Y| - h_1 - h_2 = |Y| - h_2 \geq 0$. Now assume $h_1 \neq 0$. Since $\delta h_1 + h_2 \leq n - |Y| - |X|$ we obtain

$$h_1+h_2 \leq n-|Y|-|X|-ah_1 \leq \frac{(a+1)(a+b)}{a} - |Y| - 1 - ah_1 = a+b+\frac{b}{a} - |Y| - ah_1.$$

Now we have

$$\begin{aligned}
\tau &= \sum_{y\in Y} \deg_G(y) - a|Y| + b|X| - h(X,Y) - e(X,Y) \\
&\geq \delta|Y| - a|Y| + b - a - b - \frac{b}{a} + |Y| + ah_1 - |Y| \\
&= |Y| - b/a + ah_1 - a > 0.
\end{aligned}$$

■

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