

## CYCLES THROUGH SPECIFIED VERTICES IN TRIANGLE-FREE GRAPHS

DANIEL PAULUSMA

*Department of Computer Science, Durham University*  
*Science Laboratories, South Road, Durham DH1 3LE, England*

**e-mail:** daniel.paulusma@durham.ac.uk

AND

KIYOSHI YOSHIMOTO\*

*Department of Mathematics*  
*College of Science and Technology*  
*Nihon University, Tokyo 101-8308, Japan*

**e-mail:** yosimoto@math.cst.nihon-u.ac.jp

### Abstract

Let  $G$  be a triangle-free graph with  $\delta(G) \geq 2$  and  $\sigma_4(G) \geq |V(G)| + 2$ . Let  $S \subset V(G)$  consist of less than  $\sigma_4/4 + 1$  vertices. We prove the following. If all vertices of  $S$  have degree at least three, then there exists a cycle  $C$  containing  $S$ . Both the upper bound on  $|S|$  and the lower bound on  $\sigma_4$  are best possible.

**Keywords:** cycle, path, triangle-free graph.

**2000 Mathematics Subject Classification:** 05C38, 05C45.

## 1. Introduction

Let  $G = (V(G), E(G))$  be a graph, where  $V(G)$  is a finite set of vertices and  $E(G)$  is a set of unordered pairs of two different vertices, called edges.

---

\*Supported by JSPS. KAKENHI (14740087).

All notation and terminology not explained is given in [6]. For simplicity, the order of a graph is denoted by  $n$  and  $G - V(H)$  by  $G - H$ . Let

$$\sigma_k(G) = \min \left\{ \sum_{i=1}^k d_G(x_i) \mid x_1, x_2, \dots, x_k \text{ are independent} \right\},$$

where  $d_G(x_i)$  is the degree of a vertex  $x_i$ . If the independence number of  $G$  is less than  $k$ , then we define  $\sigma_k(G) = \infty$ .

Ore [11] showed that a graph  $G$  with  $\sigma_2 \geq n$  is hamiltonian and Bondy [3] proved that if  $G$  is a 2-connected graph with  $\sigma_3 \geq n+2$ , then for any longest cycle  $C$ ,  $E(G - C) = \emptyset$ . Enomoto *et al.* [9] generalized this theorem as follows: if  $G$  is a 2-connected graph with  $\sigma_3 \geq n+2$ , then  $p(G) - c(G) \leq 1$ , where  $p(G)$  and  $c(G)$  are the order of longest paths and the circumference, respectively.

In this paper we study triangle-free graphs. For triangle-free graphs with  $\sigma_2 \geq (n+1)/2$ , all longest cycles are dominating [16]. This lower bound is almost best possible by the examples due to Ash and Jackson [1]. Corresponding to the theorem by Enomoto *et al.*, the following result has been proven.

**Theorem 1** ([13]). *Let  $G$  be a triangle-free graph with  $\delta \geq 2$ . If  $\sigma_4 \geq n+2$ , then for any path  $P$ , there exists a cycle  $C$  such that  $|V(P - C)| \leq 1$  or  $G$  is isomorphic to the graph in Figure 1.*

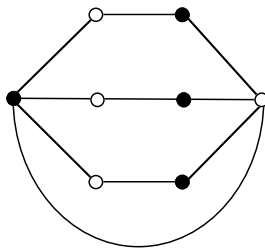


Figure 1

In the literature the question has been studied whether for a given graph  $G$  any subset  $S$  of vertices of restricted size has some cycle passing through it. Many results on general graphs and graph classes are known (see, e.g., [2, 4, 5, 7, 8, 10, 12, 14, 15, 17]). For triangle-free graphs the following result has been proven.

**Theorem 2** ([13]). *Let  $G$  be a triangle-free graph with  $\delta \geq 2$ . If  $\sigma_4 \geq n+2$ , then for any set  $S$  of at most  $\delta$  vertices, there exists a cycle  $C$  containing  $S$ .*

In this paper, we show the following related theorem.

**Theorem 3.** *Let  $G$  be a triangle-free graph with  $\delta \geq 2$  and  $\sigma_4 \geq n+2$ . Let  $S \subset V(G)$  consist of less than  $\sigma_4/4 + 1$  vertices. If all vertices of  $S$  have degree at least three, then there exists a cycle  $C$  containing  $S$ .*

The several bounds in these theorems are all tight. We show this by a number of counter examples. For these counter examples we use the following notations. We denote the *complement* of graph  $G = (V, E)$  by  $\overline{G} = (V, (V \times V) \setminus E)$ . For two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , we denote their *union* by  $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$  and their *join* by  $G_1 * G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup (V_1 \times V_2))$ . A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on  $n$  vertices is denoted by  $K_n$ . The *complete bipartite* graph  $\overline{K_k} * \overline{K_\ell}$  is denoted by  $K_{k,\ell}$ .

- Consider the graph  $\overline{K_{k-1}} * \overline{K_k} * K_1 * \overline{K_k} * \overline{K_{k-1}}$  with  $\delta = (n+1)/4$  and  $\sigma_4 = n+1$ . If we choose two vertices from each  $\overline{K_k}$ , obviously there is no cycle containing the vertices. See Figure 2(i). Hence, in Theorem 2 and Theorem 3, the lower bound on  $\sigma_4$  is best possible.

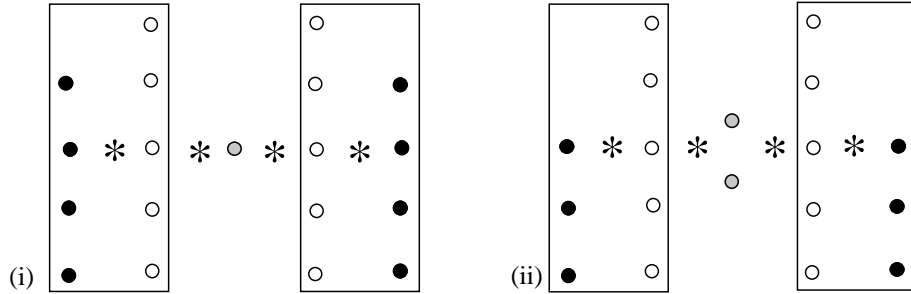


Figure 2

- Consider the graph  $\overline{K_{k-2}} * \overline{K_k} * \overline{K_2} * \overline{K_k} * \overline{K_{k-2}}$  with  $\delta = (n+2)/4$  and  $\sigma_4 = n+2$ . There is no cycle containing all  $k = (n+2)/4$  vertices of the left  $\overline{K_k}$  and a vertex in the right  $\overline{K_{k-2}}$ . See Figure 2(ii). Hence, in Theorem 2 and Theorem 3, the upper bound on  $|S|$  is best possible.

We cannot relax the degree condition of vertices in  $S$  in Theorem 3 into “all vertices of  $S$  have degree at least two”. For example in the graph in Figure 1,  $\sigma_4/4 + 1 = 10/4 + 1$ . So we can choose three vertices. However, if we choose the three white vertices of degree two in the graph, obviously there is no desired cycle. There is even a class of counter examples of large order as follows. Consider the graph  $K_{k,k}$  for any  $k \geq 3$ . Let  $x, x'$  be two vertices in the same partite set of this graph. Add two extra vertices  $w, w'$  and add all edges between  $\{x, x'\}$  and  $\{w, w'\}$ . This way we obtain a graph  $G_k$  with  $\sigma_4 = 2k + 4 = n + 2 \geq 10$ . Now let  $S \subset V(G_k)$  consist of the vertices  $w, w'$  and some vertex  $u$  not in  $\{x, x'\}$ . Then  $|S| = 3 < 10/4 + 1 \leq \sigma_4/4 + 1$ . However, the only cycle in  $G_k$  that contains both  $w$  and  $w'$  is the cycle on the four vertices  $x, x', w, w'$ . This means that  $G_k$  does not contain a cycle passing through  $S$ . We note that  $S$  contains two vertices of degree two. The following conjecture seems to hold.

**Conjecture 4.** Let  $G$  be a triangle-free graph with  $\delta \geq 2$  and  $\sigma_4 \geq n + 2$ . Let  $S \subset V(G)$  consist of less than  $\sigma_4/4 + 1$  vertices. If  $S$  contains at most one vertex of degree 2, then there exists a cycle  $C$  containing  $S$ .

Finally, we give some additional definitions and notations. The set of all the neighbours of a vertex  $x \in V(G)$  is denoted by  $N_G(x)$  or simply  $N(x)$ , and its cardinality by  $d_G(x)$  or  $d(x)$ . For a subgraph  $H$  of  $G$ , we denote  $N_G(x) \cap V(H)$  by  $N_H(x)$  and its cardinality by  $d_H(x)$ . For simplicity, we denote  $|V(H)|$  by  $|H|$  and “ $u_i \in V(H)$ ” by “ $u_i \in H$ ”. The set of neighbours  $\bigcup_{v \in H} N_G(v) \setminus V(H)$  is written by  $N_G(H)$  or  $N(H)$ , and for a subgraph  $F \subset G$ ,  $N_G(H) \cap V(F)$  is denoted by  $N_F(H)$ . Especially, for an edge  $e = xy$ , we denote  $N(e) = (N(x) \cup N(y)) \setminus \{x, y\}$  and  $d(e) = |N(e)|$ .

Let  $C = v_1 v_2 \dots v_p v_1$  be a cycle with a fixed orientation. The segment  $v_i v_{i+1} \dots v_j$  is written by  $v_i \overrightarrow{C} v_j$  where the subscripts are to be taken modulo  $|C|$ . The converse segment  $v_j v_{j-1} \dots v_i$  is written by  $v_j \overleftarrow{C} v_i$ . The successor of  $u_i$  is denoted by  $u_i^+$  and the predecessor by  $u_i^-$ . For a subset  $A \subseteq V(C)$ , we write  $\{u_i^+ \mid u_i \in A\}$  and  $\{u_i^- \mid u_i \in A\}$  by  $A^+$  and  $A^-$ , respectively.

## 2. The Proof of Theorem 3

In the proof we make use of the following lemma. A cycle  $C$  in a graph  $G$  is called a *swaying* cycle of a subset  $S \subseteq V(G)$  if  $|C \cap S|$  is maximum in all cycles of  $G$ .

**Lemma 5.** *Let  $G$  be a connected graph such that for any path  $P$ , there exists a cycle  $C$  such that  $|P - C| \leq 1$ . Let  $S \subset V(G)$ . Then for any longest swaying cycle  $C$  of  $S$ ,  $S \subset V(C)$  or  $N(x) \subset C$  for any  $x \in S - C$ .*

**Proof.** Let  $S \subset V(G)$  and  $C$  a longest swaying cycle of  $S$ . Suppose  $S - C \neq \emptyset$ . For any vertex  $x \in S - C$ , there is a path  $Q$  joining  $x$  and  $C$ . Let  $P$  be a longest path containing  $V(C \cup Q)$ . Then there exists a cycle  $D$  such that  $|P - D| \leq 1$ . If  $x$  has neighbours in  $G - C$ , then  $|P| \geq |C| + 2$  and so  $|D| \geq |C| + 1$ . Because  $|D \cap S| \geq |C \cap S|$ , this contradicts the assumption that  $C$  is a longest swaying cycle. Hence  $N_{G-C}(x) = \emptyset$ . ■

Now let  $G$  be a graph with  $\delta \geq 2$  and  $\sigma_4 \geq n + 2$ . Let  $S \subset V(G)$  be a set of less than  $\sigma_4/4 + 1$  vertices that all have degree at least three. Let  $\mathcal{C}$  be the set of all longest swaying cycles of  $S$ . Suppose a cycle in  $\mathcal{C}$  does not contain all vertices in  $S$ .

**Claim 1.** If there exists a swaying cycle  $D$  of  $S$  and  $v \in S - D$  such that  $N(v) \subset V(D)$ , then  $d(v) \leq |D \cap S|$ , and so  $d(v) < \sigma_4/4$ .

**Proof.** If  $d(v) > |D \cap S|$ , then there exist  $y, z \in N(v)$  such that  $y^+ = z$  or  $y^+ \vec{D} z^- \cap S = \emptyset$  because  $N(v) \subset V(D)$ . Then the cycle  $yvz\vec{D}y$  contains  $|D \cap S| + 1$  vertices in  $S$ . This contradicts the assumption that  $D$  is a swaying cycle. Hence  $d(v) \leq |D \cap S| \leq |S| - 1 < \sigma_4/4$ . ■

Note that our statement holds if  $G$  is isomorphic to the graph in Figure 1. Hence Claim 1 together with Theorem 1 and Lemma 5 implies that

$$(1) \quad d(v) < \sigma_4/4 \text{ for any } D \in \mathcal{C} \text{ and } v \in S - D.$$

Let  $C = u_1 u_2 \cdots u_{|C|} \in \mathcal{C}$  such that  $\max\{d(v) \mid v \in S - C\}$  is maximum in  $\mathcal{C}$ , and let  $x \in S - C$  such that  $d(x)$  is maximum in  $S - C$ . Then  $d(x) < \sigma_4/4$  by (1). Let  $N(x) = \{u_{\tau(1)}, u_{\tau(2)}, \dots, u_{\tau(d(x))}\}$  which occur on  $C$  in the order of their indices. Then clearly

$$(2) \quad N(x)^+ \text{ is an independent set;}$$

otherwise there is a cycle containing  $|C \cap S| + 1$  vertices of  $S$ . As  $G$  is triangle-free, a vertex  $u_{\tau(l)}^+ \in N(x)^+$  is not adjacent to  $x$ . If  $u_{\tau(l)}^+$  is adjacent to a vertex  $y \in G - (C \cup x)$ , then the order of the path  $yu_{\tau(l)}^+ \vec{C} u_{\tau(l)} x$  is

$|C| + 2$ . By Theorem 1, there is a cycle  $D'$  such that  $|D' \cap S| \geq |C \cap S|$  and  $|D'| \geq |C| + 1$ . This is a contradiction. Therefore

$$(3) \quad N(u_{\tau(l)}^+) \subset V(C) \text{ for } u_{\tau(l)}^+ \in N(x)^+.$$

Let  $I_l = u_{\tau(l)}^+ \overrightarrow{C} u_{\tau(l+1)}$  and  $J_l = u_{\tau(l+1)}^+ \overrightarrow{C} u_{\tau(l)}$ , and

$$L = \{u_{\tau(i)}^+ \mid d(u_{\tau(i)}^+) \text{ is maximum in } N(x)^+\}.$$

Because  $\sigma_4/4 > d(x) \geq 3$  and  $N(x)^+ \cup x$  is an independent set, there is a vertex in  $N(x)^+$  whose degree is at least  $\sigma_4/4$ . Hence the degree of a vertex in  $L$  is greater than  $\sigma_4/4$ . If  $u_{\tau(i)}^{++} \in L^+$  is adjacent to  $u_{\tau(j)}^+ \in (N(x) \setminus u_{\tau(i)}^+)^+$ , then the cycle  $u_{\tau(i)}^{++} u_{\tau(j)}^+ \overrightarrow{C} u_{\tau(i)} x u_{\tau(j)} \overleftarrow{C} u_{\tau(i)}^{++}$  and  $u_{\tau(i)}^+ \in S$  contradict (1). If  $u_{\tau(i)}^{++} x \in E(G)$ , then the cycle  $u_{\tau(i)} x u_{\tau(i)}^{++} \overrightarrow{C} u_{\tau(i)}$  and  $u_{\tau(i)}^+$  contradict (1). Hence

$$(4) \quad u_{\tau(i)}^{++} \in L^+ \text{ is adjacent to none of } (N(x) \setminus u_{\tau(i)}^+)^+ \cup x.$$

For each  $u_{\tau(l)}^+ \in N(x)^+$ , we denote the edge  $u_{\tau(l)}^+ u_{\tau(l)}^{++}$  by  $e_l$ .

**Claim 2.** For any  $u_{\tau(i)}^+ \in L$ , it holds that:

1.  $N_{I_i}(e_i)^- \cap N_{I_i}(u_{\tau(i+1)}^+) = \emptyset$ .
2.  $N_{J_i}(x)^+ \cap N_{J_i}(e_i) = \emptyset$ .
3.  $N_{J_i}(e_i) \cap N_{J_i}(u_{\tau(i+1)}^+)^- = \emptyset$ .

**Proof.** Suppose there is a vertex  $u_l \in N_{I_i}(e_i)^- \cap N_{I_i}(u_{\tau(i+1)}^+)$ , and let  $y \in V(e_i) \cap N(u_l^+)$ . Then the cycle

$$D = y \overrightarrow{C} u_l u_{\tau(i+1)}^+ \overrightarrow{C} u_{\tau(i)} x u_{\tau(i+1)} \overleftarrow{C} u_l^+ y$$

contains all vertices of  $V(C) \cup x$  if  $y = u_{\tau(i)}^+$ , i.e.,  $|D| = |C \cap S| + 1$ . See Figure 3(i). This contradicts the assumption that  $C \in \mathcal{C}$ . If  $y = u_{\tau(i)}^{++}$ , then  $D \in \mathcal{C}$  and  $d(u_{\tau(i)}^+) \geq \sigma_4/4$ . This contradicts (1). Hence  $N_{I_i}(e_i)^- \cap N_{I_i}(u_{\tau(i+1)}^+) = \emptyset$ . Similarly, we can show the other statements. See Figure 3(ii)–(iii). ■

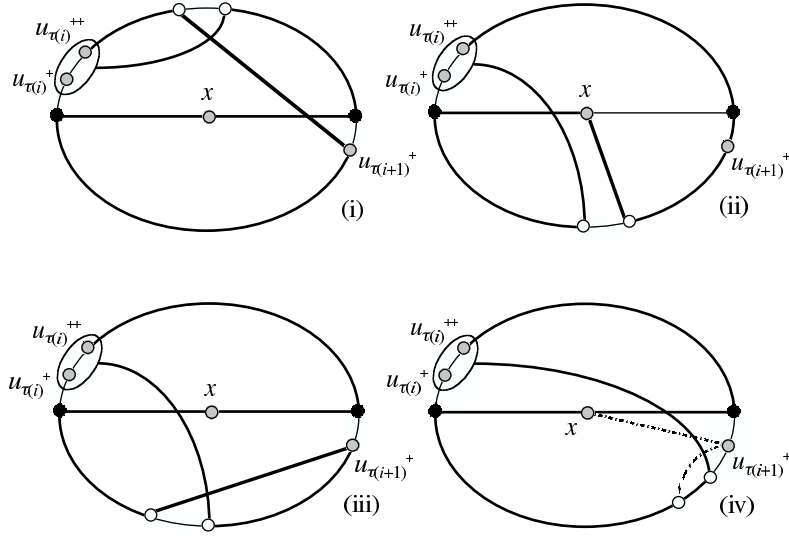


Figure 3

Let  $\alpha_i = |N_{J_i}(x)^+ \cap N_{J_i}(u_{\tau(i+1)}^+)^-|$ . By this number, we will divide our argument into three cases, and in each case, the following claim will be used.

**Claim 3.** For any  $u_{\tau(i)}^+ \in L$ ,  $n \geq d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 2 - \alpha_i$ . Especially if the equality holds, then  $u_{\tau(i+1)}^{++} \in N(e_i)$  and  $u_{\tau(i+1)}^+ \in S$ , and

$$J_i = (N_{J_i}(x) \setminus u_{\tau(i)})^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u_{\tau(i+1)}^+)^-.$$

**Proof.** By the previous claim, we have:

$$\begin{aligned}
 |I_i| &\geq |N_{I_i}(e_i)^- \cup N_{I_i}(u_{\tau(i+1)}^+) \cup \{u_{\tau(i)}^+\}| \\
 &\geq |N_{I_i}(e_i)^-| + |N_{I_i}(u_{\tau(i+1)}^+)| + |\{u_{\tau(i)}^+\}| \\
 &= d_{I_i}(e_i) + d_{I_i}(u_{\tau(i+1)}^+) + 1, \\
 |J_i| &\geq |(N_{J_i}(x) \setminus u_{\tau(i)})^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u_{\tau(i+1)}^+)^-| \\
 &\geq |(N_{J_i}(x) \setminus u_{\tau(i)})^+| + |N_{J_i}(e_i)| + |N_{J_i}(u_{\tau(i+1)}^+)^-| - \alpha_i \\
 &= d_{J_i}(x) - 1 + d_{J_i}(e_i) + d_{J_i}(u_{\tau(i+1)}^+) - \alpha_i.
 \end{aligned}$$

Therefore

$$\begin{aligned}
n &\geq |C| + d_{G-C}(e_i) + |\{x\}| = |I_i| + |J_i| + d_{G-C}(e_i) + 1 \\
&\geq (d_{I_i}(e_i) + d_{I_i}(u_{\tau(i+1)}^+)) + (d_{J_i}(x) + d_{J_i}(e_i) + d_{J_i}(u_{\tau(i+1)}^+) - \alpha_i) \\
&\quad + d_{G-C}(e_i) + 1 \\
&= (d_{I_i}(e_i) + d_{J_i}(e_i) + d_{G-C}(e_i)) + (d_{I_i}(u_{\tau(i+1)}^+) + d_{J_i}(u_{\tau(i+1)}^+)) \\
&\quad + (d_{J_i}(x) + 1) - \alpha_i \\
&= d(e_i) + d(u_{\tau(i+1)}^+) + d(x) - \alpha_i \\
&= d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 2 - \alpha_i.
\end{aligned}$$

If equalities hold in the above inequalities, then

$$|J_i| = |(N_{J_i}(x) \setminus u_{\tau(i)}^+)^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u_{\tau(i+1)}^+)^-|$$

also holds and so

$$J_i = (N_{J_i}(x) \setminus u_{\tau(i)}^+)^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u_{\tau(i+1)}^+)^-.$$

Because  $G$  is triangle-free,  $u_{\tau(i+1)}^{++} \notin N(u_{\tau(i+1)}^+)^-$  and  $u_{\tau(i+1)}^{++} \notin N(x)^+$ , and so  $u_{\tau(i+1)}^{++} \in N(e_i)$ .

Let  $y = V(e_i) \cap N(u_{\tau(i+1)}^{++})$  and

$$C' = y \overrightarrow{C} u_{\tau(i+1)} x u_{\tau(i)} \overleftarrow{C} u_{\tau(i+1)}^{++} y.$$

Suppose  $u_{\tau(i+1)}^+ \notin S$ . Because  $C'$  does not contain  $|C \cap S| + 1$  vertices of  $S$ , we have  $y \neq u_{\tau(i)}^+$  and  $u_{\tau(i)}^+ \in S$ . Therefore  $N(u_{\tau(i)}^+) \subset V(C')$  by (2) and (3). This contradicts Claim 1 as  $u_{\tau(i)}^+ \in L$ . See Figure 3(iv). Hence  $u_{\tau(i+1)}^+ \in S$ . ■

For  $u_{\tau(i)}^+ \in L$ , if the vertex  $u_{\tau(i+1)}^+$  is adjacent to  $u_{\tau(s)}^{++} \in (N_{J_i}(x) \setminus u_{\tau(i)}^+)^{++}$ , then the cycle

$$C' = u_{\tau(i+1)}^+ u_{\tau(s)}^{++} \overrightarrow{C} u_{\tau(i+1)} x u_{\tau(s)} \overleftarrow{C} u_{\tau(i+1)}^+$$



is a longest swaying cycle of  $S$ . Hence  $u_{\tau(s)}^+ \in S$ ; otherwise  $|C' \cap S| \geq |C \cap S| + 1$ . Therefore from (1), it holds that

$$(5) \quad u_{\tau(s)}^+ \in S \text{ and } d(u_{\tau(s)}^+) < \sigma_4/4 \text{ for all } u_{\tau(s)}^+ \in N_{J_i}(x)^+ \cap N_{J_i}(u_{\tau(i+1)}^+)^-.$$

If there are three vertices in  $N_{J_i}(x)^+ \cap N_{J_i}(u_{\tau(i+1)}^+)^-$ , then the three vertices and  $x$  are independent by (2), however, the sum of these degrees are less than  $\sigma_4$ . Therefore,  $\alpha_i \leq 2$ . Now we divide our argument.

*Case 1.* There is  $u_{\tau(i)}^+ \in L$  such that  $\alpha_i = 1$ .

Let  $\{u_{\tau(s)}^+\} = N_{J_i}(x)^+ \cap N_{J_i}(u_{\tau(i+1)}^+)^-$ . By (5),  $d(u_{\tau(s)}^+) < \sigma_4/4 \leq d(u_{\tau(i)}^+)$ , and by (2) and (4),  $\{u_{\tau(i)}^{++}, u_{\tau(i+1)}^+, u_{\tau(s)}^+, x\}$  is an independent set. Hence by Claim 3, it holds that

$$\begin{aligned} n &\geq d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 2 - 1 \\ &\geq d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(u_{\tau(s)}^+) + d(x) + (d(u_{\tau(i)}^+) - d(u_{\tau(s)}^+)) - 3 \\ &\geq \sigma_4 + (d(u_{\tau(i)}^+) - d(u_{\tau(s)}^+)) - 3 \\ &\geq (n + 2) + 1 - 3 = n. \end{aligned}$$

Therefore all equalities have to hold in the above inequalities, and so we have

$$(6) \quad n = d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 3,$$

$$(7) \quad d(u_{\tau(i)}^+) = d(u_{\tau(s)}^+) + 1.$$

Because  $u_{\tau(i+1)}^+ u_{\tau(s)}^{++} \overrightarrow{C} u_{\tau(i+1)} x u_{\tau(s)} \overleftarrow{C} u_{\tau(i+1)}^+ \in \mathcal{C}$ , we have  $d(x) \geq d(u_{\tau(s)}^+)$  by the maximality of  $d(x)$ . Then  $d(x) + 1 \geq d(u_{\tau(s)}^+) + 1 = d(u_{\tau(i)}^+) \geq \sigma_4/4$  by (7). On the other hand,  $d(x) + 1 \leq |C \cap S| + 1 \leq |S| < \sigma_4/4 + 1$  by Claim 1. Thus

$$\frac{\sigma_4}{4} \leq d(x) + 1 \leq |S| < \frac{\sigma_4}{4} + 1,$$

i.e.,  $|S| = d(x) + 1$ . Therefore  $|u_{\tau(l)}^+ \overrightarrow{C} u_{\tau(l+1)}^- \cap S| = 1$  for all  $l \leq d(x)$ ; otherwise we can easily obtain a cycle containing  $|C \cap S| + 1$  vertices of  $S$

as in the proof of Claim 1. However, by (6) and Claim 3,  $u_{\tau(i+1)}^+ \in S$ , and by (5),  $u_{\tau(s)}^+ \in S$ , and hence

$$u_{\tau(i+1)}^{++} \vec{C} u_{\tau(i+2)}^- \cap S = u_{\tau(s)}^{++} \vec{C} u_{\tau(s+1)}^- \cap S = \emptyset.$$

Then, the cycle  $u_{\tau(i+1)}^+ u_{\tau(s)}^{++} \overleftarrow{C} u_{\tau(i+2)}^- x u_{\tau(s+1)}^- \vec{C} u_{\tau(i+1)}^+$  contains  $|C \cap S| + 1$  vertices in  $S$ . See Figure 4. This contradicts the assumption that  $C$  is a swaying cycle.

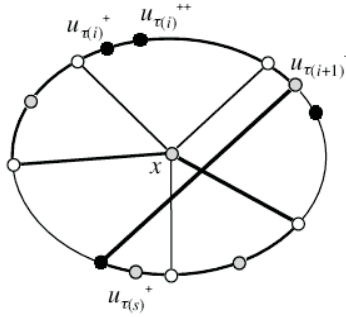


Figure 4

*Case 2.* There exists  $u_{\tau(i)}^+ \in L$  such that  $\alpha_i = 2$ .

Let  $\{u_{\tau(s)}^{++}, u_{\tau(t)}^{++}\} = N_{J_i}(x)^+ \cap N_{J_i}(u_{\tau(i+1)}^+)^-$ . By (2),  $\{u_{\tau(s)}^+, u_{\tau(t)}^+, x, u_{\tau(i+1)}^+\}$  is an independent set. By (5), both of the degrees of  $u_{\tau(s)}^+$  and  $u_{\tau(t)}^+$  are less than  $\sigma_4/4$ , and so  $d(u_{\tau(i+1)}^+) \geq \sigma_4/4$ . Thus, it holds that

$$d(u_{\tau(s)}^+) < \sigma_4/4 \leq d(u_{\tau(i)}^+) \text{ and } d(u_{\tau(t)}^+) < \sigma_4/4 \leq d(u_{\tau(i+1)}^+).$$

Therefore by Claim 3,

$$\begin{aligned} n &\geq d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 2 - 2 \\ &\geq d(u_{\tau(i)}^{++}) + d(u_{\tau(s)}^+) + d(u_{\tau(t)}^+) + d(x) \\ &\quad + (d(u_{\tau(i)}^+) - d(u_{\tau(s)}^+)) + (d(u_{\tau(i+1)}^+) - d(u_{\tau(t)}^+)) - 4 \end{aligned}$$

$$\begin{aligned}
&\geq \sigma_4 + (d(u_{\tau(i)}^+) - d(u_{\tau(s)}^+)) + (d(u_{\tau(i+1)}^+) - d(u_{\tau(t)}^+)) - 4 \\
&\geq (n+2) + 1 + 1 - 4 = n
\end{aligned}$$

because  $\{u_{\tau(i)}^{++}, u_{\tau(s)}^+, u_{\tau(t)}^+, x\}$  is an independent set by (2) and (4). Thus all equalities hold in the above inequalities, and we can use the same arguments as in Case 1.

*Case 3.*  $\alpha_i = 0$  for any  $u_{\tau(i)+} \in L$ .

For any  $u_{\tau(s)}^+ \in (N(x) \setminus \{u_{\tau(i)}, u_{\tau(i+1)}\})^+$ ,

$$\begin{aligned}
n &\geq d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 2 \\
&\geq d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(u_{\tau(s)}^+) + d(x) + (d(u_{\tau(i)}^+) - d(u_{\tau(s)}^+)) - 2 \\
&\geq \sigma_4 + (d(u_{\tau(i)}^+) - d(u_{\tau(s)}^+)) - 2 \\
&\geq (n+2) - 2 = n
\end{aligned}$$

by Claim 3 because  $\{u_{\tau(i)}^{++}, u_{\tau(i+1)}^+, u_{\tau(s)}^+, x\}$  is an independent set from (2) and (4). Therefore all equalities hold in the above inequalities, and so we have:

$$(8) \quad n = d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) - 2 = \sigma_4 - 2,$$

$$(9) \quad d(u_{\tau(i)}^+) = d(u_{\tau(s)}^+).$$

From (9), we obtain  $u_{\tau(s)}^+ \in L$ , and so, by symmetry,  $N(x)^+ \subset L$ .

**Claim 4.**  $u_{\tau(i)}^{++}$  is adjacent to all of  $\{u_{\tau(s)}^{++} \mid s \neq i\}$ .

**Proof.** By (8) and Claim 3,  $u_{\tau(i+1)}^{++} \in N(e_i)$ . Because  $u_{\tau(i+1)}^+ \in L$ ,  $u_{\tau(i+1)}^{++}$  is not adjacent to  $u_{\tau(i)}^+$  by (4). Hence  $u_{\tau(i)}^{++}u_{\tau(i+1)}^{++} \in E(G)$ .

Suppose the vertex  $u_{\tau(i)}^{++}$  is not adjacent to  $u_{\tau(s)}^{++}$  ( $s \neq i, i+1$ ). If  $u_{\tau(i+1)}^+u_{\tau(s)}^{++} \notin E(G)$ , i.e.,  $u_{\tau(s)}^{++} \notin N(u_{\tau(i+1)}^+)$ , then  $u_{\tau(s)}^{++} \in N(e_i)$  by (8) and Claim 3, and so  $u_{\tau(i)}^+u_{\tau(s)}^{++} \in E(G)$ . This contradicts (4) because  $u_{\tau(s)}^+ \in L$ .

Assume  $u_{\tau(i+1)}^+ u_{\tau(s)}^{+++} \in E(G)$ . By (4), (8) and (9) we have

$$\begin{aligned} & d(u_{\tau(s)}^{++}) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) \\ & \geq \sigma_4 \\ & = d(u_{\tau(i)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x) \\ & = d(u_{\tau(s)}^+) + d(u_{\tau(i)}^{++}) + d(u_{\tau(i+1)}^+) + d(x). \end{aligned}$$

Hence  $d(u_{\tau(s)}^{++}) \geq d(u_{\tau(s)}^+) \geq \sigma_4/4$ .

Let

$$D = u_{\tau(i+1)}^+ \overrightarrow{C} u_{\tau(s)} x u_{\tau(i+1)} \overleftarrow{C} u_{\tau(s)}^{+++} u_{\tau(i+1)}^+.$$

By (3),  $N(u_{\tau(s)}^+) \subset V(C)$ . As  $u_{\tau(s)}^+ \in L$ , the vertex  $u_{\tau(s)}^{+++}$  is not adjacent to  $x$ . If  $u_{\tau(s)}^{+++}$  is adjacent to the vertex  $y \in G - C$ , then the order of the path  $yu_{\tau(s)}^{+++} \overleftarrow{C} u_{\tau(i+1)}^+ u_{\tau(s)}^{+++} \overrightarrow{C} u_{\tau(i+1)} x$  is  $|C| + 2$ . As in the proof of (3), this contradicts the assumption that  $C \in \mathcal{C}$  by Theorem 1. Hence, we obtain  $N(u_{\tau(s)}^{+++}) \subset V(C)$ . Thus  $N(e_s) \subset V(D)$ . Because  $|D \cap S| \leq |C \cap S| < \sigma_4/4$ ,

$$d(e_s) \geq \sigma_4/2 - 2 \geq \sigma_4/4 > |S \cap D|.$$

Therefore, there exist vertices  $y, z \in D \cap N(e_s)$  such that  $y^+ = z$  or  $y^+ \overrightarrow{D} z^- \cap S = \emptyset$  and  $y^+ \overrightarrow{D} z^- \cap N(e_s) = \emptyset$ . If  $y$  and  $z$  are adjacent to distinct ends of  $e_s$ , say  $yu_{\tau(s)}^+, zu_{\tau(s)}^{+++} \in E(G)$ , then  $yu_{\tau(s)}^+ u_{\tau(s)}^{+++} z \overrightarrow{D} y$  contains  $|C \cap S| + 1$  vertices of  $S$ . Hence, by symmetry, we may assume  $u_{\tau(s)}^+$  is adjacent to both  $y$  and  $z$ . Then the cycle  $D' = yu_{\tau(s)}^+ z \overrightarrow{D} y$  is a swaying cycle and  $N(u_{\tau(s)}^{+++}) \subset N(D')$ . This contradicts Claim 1 because  $d(u_{\tau(s)}^{+++}) \geq \sigma_4/4$ . ■

By symmetry, the vertex  $u_{\tau(i+1)}^{+++}$  is adjacent to  $u_{\tau(s)}^{+++}$ , and so there is the triangle  $u_{\tau(i)}^{+++} u_{\tau(i+1)}^{+++} u_{\tau(s)}^{+++}$ . This is a contradiction.

## References

- [1] P. Ash and B. Jackson, *Dominating cycles in bipartite graphs*, in: Progress in Graph Theory, J.A. Bondy, U.S.R. Murty, eds., (Academic Press, 1984), 81–87.

- [2] B. Bollobás and G. Brightwell, *Cycles through specified vertices*, *Combinatorica* **13** (1993) 137–155.
- [3] J.A. Bondy, *Longest Paths and Cycles in Graphs of High Degree*, Research Report CORR 80-16 (1980).
- [4] J.A. Bondy and L. Lovász, *Cycles through specified vertices of a graph*, *Combinatorica* **1** (1981) 117–140.
- [5] H. Broersma, H. Li, J. Li, F. Tian and H.J. Veldman, *Cycles through subsets with large degree sums*, *Discrete Math.* **171** (1997) 43–54.
- [6] R. Diestel, *Graph Theory*, Second edition, Graduate Texts in Mathematics **173**, Springer (2000).
- [7] Y. Egawa, R. Glas and S.C. Locke, *Cycles and paths through specified vertices in  $k$ -connected graphs*, *J. Combin. Theory (B)* **52** (1991) 20–29.
- [8] J. Harant, *On paths and cycles through specified vertices*, *Discrete Math.* **286** (2004) 95–98.
- [9] H. Enomoto, J. van den Heuvel, A. Kaneko and A. Saito, *Relative length of long paths and cycles in graphs with large degree sums*, *J. Graph Theory* **20** (1995) 213–225.
- [10] D.A. Holton, *Cycles through specified vertices in  $k$ -connected regular graphs*, *Ars Combin.* **13** (1982) 129–143.
- [11] O. Ore, *Note on hamiltonian circuits*, *American Mathematical Monthly* **67** (1960) 55.
- [12] K. Ota, *Cycles through prescribed vertices with large degree sum*, *Discrete Math.* **145** (1995) 201–210.
- [13] D. Paulusma and K. Yoshimoto, *Relative length of longest paths and longest cycles in triangle-free graphs*, submitted,  
[http://www.math.cst.nihon-u.ac.jp/~yosimoto/paper/related\\_length1\\_sub.pdf](http://www.math.cst.nihon-u.ac.jp/~yosimoto/paper/related_length1_sub.pdf).
- [14] A. Saito, *Long cycles through specified vertices in a graph*, *J. Combin. Theory (B)* **47** (1989) 220–230.
- [15] L. Stacho, *Cycles through specified vertices in 1-tough graphs*, *Ars Combin.* **56** (2000) 263–269.
- [16] K. Yoshimoto, *Edge degree conditions and all longest cycles which are dominating*, submitted.
- [17] S.J. Zheng, *Cycles and paths through specified vertices*, *Journal of Nanjing Normal University, Natural Science Edition*, Nanjing Shida Xuebao, Ziran Kexue Ban **23** (2000) 9–13.

Received 8 March 2006

Revised 31 October 2006