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TOTAL DOMINATION OF CARTESIAN PRODUCTS OF GRAPHS*

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Abstract

Let $\gamma_t(G)$ and $\gamma_{pr}(G)$ denote the total domination and the paired domination numbers of graph G, respectively, and let $G \,\Box\, H$ denote the Cartesian product of graphs G and H. In this paper, we show that $\gamma_t(G)\gamma_t(H) \leq 5\gamma_t(G \,\Box\, H)$, which improves the known result $\gamma_t(G)\gamma_t(H) \leq 6\gamma_t(G \,\Box\, H)$ given by Henning and Rall.

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Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$, the set of vertices adjacent to v. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V$, the open neighborhood of S is defined by $N_G(S) = \cup_{v \in S} N_G(v)$, and the closed neighborhood of S by $N_G[S] = N_G(S) \cup S$. The subgraph of G induced by the vertices in S is denoted by $G[S]$.

A set of vertices or of edges is independent if no two of its elements are adjacent. A matching in a graph G is a set of independent edges in G. A perfect matching M in G is a matching in G such that every vertex of G is incident with an edge of M.

For $S \subseteq V(G)$, the set S is a dominating set if $N[S] = V$, a total dominating set, denoted TDS, if $N(S) = V$, and a paired dominating set,

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denoted PDS, if $N(S) = V$ and $G[S]$ contains at least one perfect matching. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. The total domination number $\gamma_t(G)$ (resp. paired domination number $\gamma_{pr}(G)$) is the minimum cardinality of a total dominating set (resp. a paired dominating set) of G. For all graphs G without isolated vertices, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$. For a detailed treatment of the total domination and paired domination in graphs, the reader is referred to [1] and [4].

For graphs G and H, the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. The most famous open problem involving domination in graphs is Vizing's conjecture which states that

Conjecture 1 (Vizing's Conjecture [5]). For any graphs G and H,

$$\gamma(G)\gamma(H) \leq \gamma(G \square H).$$

The best general upper bound to date on $\gamma(G)\gamma(H)$ in terms of $\gamma(G \square H)$ is due to Clark and Suen. They proved in [2] that: For any graphs G and H, $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$.

The unability of proving or disproving Vizing's conjecture lead authors to pose different variations of the original problem. The total domination version of Vizing's conjecture has been studied by Henning and Rall [3]. They proved that $\gamma_t(G)\gamma_t(H) \leq 6\gamma_t(G \square H)$ and proposed an open question: For any graphs G and H without isolated vertices, is it true that $\gamma_t(G)\gamma_t(H) \leq 2\gamma_t(G \square H)$?

In this note, we prove that $\gamma_t(G)\gamma_t(H) \leq 5\gamma_t(G \square H)$, which improves the general upper bound of $\gamma_t(G)\gamma_t(H)$ given by Henning and Rall [3].

For any vertex (x, u) of $G \square H$, the vertex u of H is the H-projection of (x, u), denoted $u = \phi_H(x, u)$. For any subset $A = \{(x_1, u_1), \ldots, (x_k, u_k)\}$ of $V(G \square H)$, the H-projection of A, denoted $\phi_H(A)$, is defined by $\phi_H(A) = \cup_{i=1}^{k}\{\phi_H(x_i, u_i)\}$, which is a subset of $V(H)$. For simplicity, we denote $N_{G\square H}(A) = N(A)$ and $N_{G\square H}[A] = N[A]$, where A is a subset of $V(G \square H)$. For any vertex $\theta = (x, u)$ of $G \square H$, a neighbor $(x, v) \in N(\theta)$ is called an H-neighbor of (x, u). Similarly, a neighbor $(y, u) \in N(\theta)$ is called a G-neighbor of (x, u).

Theorem 1. *For any graphs G and H without isolated vertices,*

$$\gamma_t(G)\gamma_t(H) \leq 5\gamma_t(G \,\square\, H).$$

Proof. Let D be a minimum TDS of $G\square H$. Let $D = D_G \cup D_H$, where D_G is the set of vertices in D which have no H-neighbors in D and $D_H = D - D_G$. By symmetry of G and H in Cartesian product $G \,\square\, H$, we may assume that $|D_G| \leq |D_H|$. For each vertex $(x, u) \in D_G$, add exactly one H-neighbor (x, v) of (x, u) to D_G (note that $(x, v) \notin D_H$). The resulting set is denoted by $\bar{D}_G$. Let $\bar{D} = \bar{D}_G \cup D_H$. Then $|\bar{D}| = |\bar{D}_G| + |D_H| \leq 2|D_G| + |D_H| \leq \frac{3}{2}|D|$ since $|D_G| \leq |D_H|$ and $|D| = |D_G| + |D_H|$.

Let $A = \{x_1, y_1, \ldots, x_k, y_k\}$ be a minimum PDS of G where for each i, x_i is adjacent to y_i in G, and so $\gamma_{pr}(G) = 2k$. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be a partition of $V(G)$ such that $\{x_i, y_i\} \subseteq \Pi_i \subseteq N(\{x_i, y_i\})$ for each i, $1 \leq i \leq k$. For each $w \in V(H)$, let $V_w = V(G) \times \{w\}$ and G_w be the subgraph of $G\square H$ induced by V_w, let $D_w = D \cap V_w$ and $\bar{D}_w = \bar{D} \cap V_w$. For each $i = 1, 2, \ldots, k$, let $H_i = \Pi_i \times V(H)$, let $D_i = D \cap H_i$ and $\bar{D}_i = \bar{D} \cap H_i$. Let

$$\bar{L}_i = \{(i, w) | (\Pi_i \times \{w\}) \cap N[\bar{D}_i] = \varnothing, w \in V(H)\}.$$

Corresponding to each set $\bar{D}_i$, we construct a TDS of H as follows. Let $\bar{D}'_i = \phi_H(\bar{D}_i)$, then $H[\bar{D}'_i]$ contains no isolated vertices in H. If $|\bar{L}_i| \geq 1$, then let F_i denote the subgraph of H induced by the set of vertices w that correspond to elements (i, w) in $\bar{L}_i$. For each isolated vertex w in F_i, add exactly one neighbor w' of w in H to the set $\bar{D}'_i$ (note that neither w nor w' belong to the set $\bar{D}'_i$, but since $(\Pi_i \times \{w'\}) \cap N[\bar{D}_i] \neq \varnothing$, w' is adjacent to a vertex of $\bar{D}'_i$). For each nontrivial component of F_i, add every vertex from that component to the set $\bar{D}'_i$. By construction, the resulting set is a TDS of H, and so $\gamma_t(H) \leq |\bar{D}'_i| + |\bar{L}_i| \leq |\bar{D}_i| + |\bar{L}_i|$. Summing over all i,

$$\begin{aligned}
\frac{1}{2}\gamma_t(G)\gamma_t(H) \leq \frac{1}{2}\gamma_{pr}(G)\gamma_t(H) &= \sum_{i=1}^{k} \gamma_t(H) \leq \sum_{i=1}^{k}(|\bar{D}_i| + |\bar{L}_i|) \\
&= |\bar{D}| + \sum_{i=1}^{k} |\bar{L}_i| \leq \frac{3}{2}|D| + \sum_{i=1}^{k} |\bar{L}_i|.
\end{aligned} \tag{1}$$

For each $w \in V(H)$, let $\bar{M}_w = \{(i, w) | (\Pi_i \times \{w\}) \cap N[\bar{D}_i] = \varnothing, 1 \leq i \leq k\}$. Then $\sum_{i=1}^{k} |\bar{L}_i| = \sum_{w \in V(H)} |\bar{M}_w|$. Let $M_w = \{(i, w) | (\Pi_i \times \{w\}) \cap N[D_i] = \varnothing, 1 \leq i \leq k\}$. By construction of $\bar{D}_i$, $D_i \subseteq \bar{D}_i$, so $N[D_i] \subseteq N[\bar{D}_i]$. Hence, for any i, if $(\Pi_i \times \{w\}) \cap N[\bar{D}_i] = \varnothing$ then $(\Pi_i \times \{w\}) \cap N[D_i] = \varnothing$. So, if $(i, w) \in \bar{M}_w$ then $(i, w) \in M_w$. Therefore, $|\bar{M}_w| \leq |M_w|$.

We claim that $|M_w| \leq |D_w|$. In fact, for any i, if $(i, w) \in M_w$ then $(\Pi_i \times \{w\}) \cap N[D_i] = \varnothing$. Since D is a TDS of $G \,\Box\, H$, $(\Pi_i \times \{w\}) \subseteq N[D_w]$. Hence each vertex in $\Pi_i \times \{w\}$ is dominated by D_w. Note that each vertex in $\Pi_j \times \{w\}$ is totally dominated by $\{x_j, y_j\} \times \{w\}$. To complete the proof of this claim, we give the following claim which has been proved by Henning and Rall:

Claim 1 ([3])**.**

$$\gamma_{pr}(G) \leq 2(k - |M_w|) + 2|D_w|.$$

Our claim is an immediate consequence of Claim 1. Hence

$$\sum_{i=1}^{k} |\bar{L}_i| = \sum_{w \in V(H)} |\bar{M}_w| \leq \sum_{w \in V(H)} |M_w| \leq \sum_{w \in V(H)} |D_w| = |D|.$$

Thus, by (1), we have

$$\gamma_t(G)\gamma_t(H) \leq 5|D| = 5\gamma_t(G \,\Box\, H).$$

■

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