# IMPROVED UPPER BOUNDS FOR NEARLY ANTIPODAL CHROMATIC NUMBER OF PATHS* 

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#### Abstract

For paths $P_{n}$, G. Chartrand, L. Nebeský and P. Zhang showed that $\mathrm{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}+2$ for every positive integer $n$, where $\mathrm{ac}^{\prime}\left(P_{n}\right)$ denotes the nearly antipodal chromatic number of $P_{n}$. In this paper we show that $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}-\frac{n}{2}-\left\lfloor\frac{10}{n}\right\rfloor+7$ if $n$ is even positive integer and $n \geq 10$, and $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}-\frac{n-1}{2}-\left\lfloor\frac{13}{n}\right\rfloor+8$ if $n$ is odd positive integer and $n \geq 13$. For all even positive integers $n \geq 10$ and all odd positive integers $n \geq 13$, these results improve the upper bounds for nearly antipodal chromatic number of $P_{n}$.


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## 1. Introduction

Radio $k$-colorings are generalizations of ordinary colorings of graphs, which were inspired by (FM Radio) Channel Assignments Problem (see [5, 7]) and introduced by G. Chartrand, D. Erwan, F. Harary and P. Zhang [1]. For a connected graph $G$ of order $n$ and diameter $d$ and a integer $k$ with $1 \leq k \leq d$, a radio $k$-coloring of $G$ is a function $c: V(G) \rightarrow \mathbf{N}$, such that $d(u, v)+|c(u)-c(v)| \geq k+1$ for every pair $u$ and $v$ of distinct vertices of $G$, where $d(u, v)$ denotes the distance between $u$ and $v$ (the length of a shortest $u-v$ path) in $G$. Clearly, radio 1-colorings and ordinary colorings are synonymous. The value $\mathrm{rc}_{k}(c)$ of a radio $k$-coloring $c$ of $G$ is the maximum color assigned to a vertex of $G$; while the radio $k$-chromatic number $\operatorname{rc}_{k}(G)$ of $G$ is $\min \left\{\operatorname{rc}_{k}(c)\right\}$ taken over all $k$-coloring $c$ of $G$. In particular, radio $d$ colorings are referred to as radio labelings and the radio d-chromatic number is called the radio number. Radio $(d-1)$-colorings are referred to as radio antipodal coloring or, more simply, as an antipodal coloring, and the radio (d-1)-chromatic number is called the antipodal chromatic number, denoted by ac $(G)$. Radio $k$-coloring and radio labeling of graphs were studied in $[1,2]$. Radio antipodal coloring of paths were studied in $[3,4,6]$.

Furthermore, G. Chartrand, L. Nebeský and P. Zhang gave the concepts of nearly antipodal colorings in [4]. For a connected graph $G$ of diameter $d$, a nearly antipodal coloring of $G$ is a function $c: V(G) \rightarrow \mathbf{N}$, such that $d(u, v)+|c(u)-c(v)| \geq d-1$ for every two distinct vertices $u$ and $v$ of $G$. The value $\mathrm{ac}^{\prime}(c)$ of a nearly antipodal coloring $c$ of $G$ is the maximum color assigned to a vertex of $G$. The nearly antipodal chromatic number $\operatorname{ac}^{\prime}(G)$ of $G$ is $\min \left\{\mathrm{ac}^{\prime}(c)\right\}$ taken over all nearly antipodal colorings of $G$ (In fact, for $d \geq 3$, a nearly antipodal coloring is a radio ( $d-2$ )-coloring).

Clearly, if $G$ is a connected graph of diameter 1 or 2 , then $\operatorname{ac}^{\prime}(G)=1$; while if $\operatorname{diam}(G)=3$, then $\operatorname{ac}^{\prime}(G)$ is the chromatic number of $G$. Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. For this reason, the nearly antipodal chromatic number of paths $P_{n}$ were investigated in [4] by G. Chartrand, L. Nebeský and P. Zhang. And they showed that $\operatorname{ac}^{\prime}\left(P_{5}\right)=5, \operatorname{ac}^{\prime}\left(P_{6}\right)=7, \operatorname{ac}^{\prime}\left(P_{7}\right)=11$ and ac' $\left(P_{8}\right)=16$. Moreover, they presented an upper bound for the nearly antipodal chromatic number of paths $P_{n}$ for every positive integer $n$ as follows.

Theorem 1.1 ([4]). If $n$ is a path of order $n \geq 1, \operatorname{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}+2$.

## 2. Our Rresults and the Idea of the Proof

In this paper we will provide an improved version for Theorem 1.1. We will show that

## Theorem 2.1.

1. If $P_{n}$ is even and $n \geq 10$, then $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}-\frac{n}{2}-\left\lfloor\frac{10}{n}\right\rfloor+7$;
2. If $n$ is odd and $n \geq 13$, then $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}-\frac{n-1}{2}-\left\lfloor\frac{13}{n}\right\rfloor+8$.

Clearly, it holds that $-\frac{n}{2}-\left\lfloor\frac{10}{n}\right\rfloor+7 \leq 1$ for all even integers $n \geq 10$, and $-\frac{n-1}{2}-\left\lfloor\frac{13}{n}\right\rfloor+8 \leq 1$ for all odd integers $n \geq 13$. Thus, for all even integers $n \geq 10$ and all odd integers $n \geq 13$, Theorem 2.1 improves the upper bounds of $\mathrm{ac}^{\prime}\left(P_{n}\right)$.

We will prove Theorem 2.1 in Section 3, and the proof will virtually provide a nearly antipodal coloring $c$ for paths $P_{n}$ with $\operatorname{ac}^{\prime}(c)$ that is equal to the bound presented in Theorem 2.1. The idea of performing the coloring $c$ is based on pseudo greedy algorithm: Let $V\left(P_{n}\right)=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$. At first, we use the color $c_{1}=1$ to color some vertex $p_{n_{1}} \in\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$, where $p_{n_{1}}$ is the (a) central vertex of $P_{n}$. Suppose that for $1 \leq i \leq n-1$ the vertices in $\left\{p_{n_{1}}, p_{n_{2}}, \ldots, p_{n_{i}}\right\} \subset\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ have been colored with $c\left(p_{n_{j}}\right)=c_{j}$ for all $1 \leq j \leq i$, then we choose a color $c_{i+1} \in \mathbf{N}$ as small as possible to color one vertex $p_{n_{i+1}} \in V\left(P_{n}\right) \backslash\left\{p_{n_{1}}, p_{n_{2}}, \ldots, p_{n_{i}}\right\}$, such that $d\left(p_{n_{i+1}}, p_{n_{j}}\right)+\left|c\left(p_{n_{i+1}}\right)-c\left(p_{n_{j}}\right)\right| \geq d-1$ for all $1 \leq j \leq i$. And if there are two vertices can be chosen for $p_{n_{i+1}}$, then we take $p_{n_{i+1}}$ close to central vertices of $P_{n}$ as near as possible. Finally, we obtain that $\operatorname{ac}^{\prime}(c)=c\left(p_{n_{n}}\right)$ and hence $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq \operatorname{ac}^{\prime}(c)$. In Section 4 we will give some examples which present the nearly antipodal coloring $c$ for some paths $P_{n}$ with $\mathrm{ac}^{\prime}(c)$ showed in Theorem 2.1 by our methods.

## 3. Proof of Theorem 2.1

Proof. 1. $n$ is even and $n \geq 10$. Firstly, we let $n \geq 12$, note that $-\left\lfloor\frac{10}{n}\right\rfloor=0$, it suffices to show that $\mathrm{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}-\frac{n}{2}+7$. Write $n=2 k=$ $10+2(4 p+q)$, where $p \in\{0,1,2, \ldots\}$ and $q \in\{1,2,3,4\}$. Then we have that $k=5+(4 p+q)$ and $d-1=\operatorname{diam}\left(P_{n}\right)-1=2 k-2$.

We denote the vertices of $P_{n}$ by $x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime} ; v_{1}^{\prime}, v_{2}^{\prime}, \ldots, v_{2 p-1}^{\prime}, v_{2 p}^{\prime}$; $w_{1}, w_{2} \ldots, w_{q} ; v_{2 p}, v_{2 p-1}, \ldots, v_{2}, v_{1} ; x_{2}, x_{1} ; y_{1}, y_{2} ; u_{1}, u_{2}, \ldots, u_{2 p-1}, u_{2 p} ;$
$z_{q}, \ldots, z_{2}, z_{1} ; u_{2 p}^{\prime}, u_{2 p-1}^{\prime}, \ldots, u_{2}^{\prime}, u_{1}^{\prime} ; y_{3}^{\prime}, y_{2}^{\prime}, y_{1}^{\prime}$ (see Figure 1). And we write $V_{1}=\left\{x_{1}, x_{2} ; y_{1}, y_{2} ; x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime} ; y_{1}^{\prime}, y_{2}^{\prime}, y_{3}^{\prime}\right\}$,
$V_{2}=\left\{v_{1}, u_{2}, v_{3}, u_{4}, \ldots, v_{2 p-1}, u_{2 p} ; v_{1}^{\prime}, v_{2}^{\prime}, \ldots, v_{2 p-1}^{\prime}, v_{2 p}^{\prime} ; u_{1}^{\prime}, u_{2}^{\prime}, \ldots, u_{2 p-1}^{\prime}, u_{2 p}^{\prime}\right\}$, $V_{3}=\left\{w_{1}, w_{2}, \ldots, w_{q} ; z_{1}, z_{2}, \ldots, z_{q} ; v_{2 p}, u_{2 p-1}, \ldots, v_{4}, u_{3}, v_{2}, u_{1}\right\}$.

In the following we will present a coloring $c$ for $P_{n}$ by three steps, such that

$$
\begin{equation*}
d(u, v)+|c(u)-c(v)| \geq d-1=2 k-2 \tag{1}
\end{equation*}
$$

holds for all distinct vertices $u, v \in V_{1} \cup V_{2} \cup V_{3}=V\left(P_{n}\right)$, and $\operatorname{ac}^{\prime}(c)=$ $\binom{n-2}{2}-\frac{n}{2}+7$ (note that $V_{2}=\emptyset$ if $p=0$, and it is easy to see that the following proof is also suitable for $V_{2}=\emptyset$ ).

Step 1. Color the vertices in $V_{1}$ (see Figure 1).
Let

$$
\begin{array}{ll}
c\left(x_{1}\right)=1\left(x_{1} \text { is an central vertex of } P_{n}\right) ; \\
c\left(y_{1}^{\prime}\right)=c\left(x_{1}\right)+(k-2)=k-1, & c\left(x_{1}^{\prime}\right)=c\left(x_{1}\right)+(k-1)=k ; \\
c\left(y_{1}\right)=c\left(x_{1}^{\prime}\right)+(k-2)=2 k-2 ; & \\
c\left(x_{2}^{\prime}\right)=c\left(y_{1}\right)+k-1=3 k-3, & c\left(y_{2}^{\prime}\right)=c\left(x_{2}^{\prime}\right)+1=3 k-2 ; \\
c\left(x_{2}\right)=c\left(x_{2}^{\prime}\right)+(k+1)=4 k-2 ; & \\
c\left(y_{3}^{\prime}\right)=c\left(x_{2}\right)+(k-1)=5 k-3, & c\left(x_{3}^{\prime}\right)=c\left(y_{3}^{\prime}\right)+3=5 k ; \\
c\left(y_{2}\right)=c\left(x_{3}^{\prime}\right)+(k-1)=6 k-1 . &
\end{array}
$$

Then by the definition of $c$ and the value of $d(u, v)$ for $u, v \in V_{1}$, it is easy to verify that the following claim holds.

Claim 3.1. For all distinct vertices $u, v \in V_{1}$, the inequality (1) holds. At the same time, $\max _{v \in V_{1}} c(v)=c\left(y_{2}\right)=6 k-1$ and $\max _{v \in V_{1} \backslash\left\{y_{2}\right\}} c(v)=$ $c\left(x_{3}^{\prime}\right)=5 k$.

Step 2. Color the vertices in $V_{2}$ (see Figure 1).
For $i=1,2, \ldots, p$, let

$$
\begin{aligned}
c\left(v_{2 i-1}^{\prime}\right)= & c\left(y_{2}\right)+(2 i-1) k+3(2 i-2)+2[1+2+\ldots+(2 i-2)] \\
& +(2 i-2)(k-1), \\
c\left(u_{2 i-1}^{\prime}\right)= & c\left(y_{2}\right)+(2 i-1) k+3(2 i-1)+2[1+2+\ldots+(2 i-1)] \\
& +(2 i-2)(k-1) ; \\
c\left(v_{2 i-1}\right)= & c\left(y_{2}\right)+(2 i-1) k+3(2 i-1)+2[1+2+\ldots+(2 i-1)] \\
& +(2 i-1)(k-1) ; \\
c\left(u_{2 i}^{\prime}\right)= & c\left(y_{2}\right)+(2 i) k+3(2 i-1)+2[1+2+\ldots+(2 i-1)] \\
& +(2 i-1)(k-1) \\
c\left(v_{2 i}^{\prime}\right)= & c\left(y_{2}\right)+(2 i) k+3(2 i)+2[1+2+\ldots+(2 i)]+(2 i-1)(k-1) ; \\
c\left(u_{2 i}\right)= & c\left(y_{2}\right)+(2 i) k+3(2 i)+2[1+2+\ldots+(2 i)]+(2 i)(k-1) .
\end{aligned}
$$

Then we have the following claim.
Claim 3.2. For all distinct vertices $u, v \in V_{1} \cup V_{2}$, the inequality (1) holds. At the same time, it holds that $\max _{v \in V_{1} \cup V_{2}} c(v)=c\left(u_{2 p}\right)=6 k-1+$ $2 p(2 k+2 p+3)$ and $\max _{v \in\left(V_{1} \cup V_{2}\right) \backslash\left\{u_{2 p}\right\}} c(v)=c\left(v_{2 p}^{\prime}\right)=5 k+2 p(2 k+2 p+3)$.

In fact, note $d-1=2 k-2$. Since that $d\left(y_{2}, v_{1}^{\prime}\right)=k-2, d\left(y_{2}, u_{1}^{\prime}\right)=$ $k-5, d\left(v_{1}^{\prime}, u_{1}^{\prime}\right)=2 k-7, c\left(v_{1}^{\prime}\right)=c\left(y_{2}\right)+k$ and $c\left(u_{1}^{\prime}\right)=c\left(y_{2}\right)+k+5$, then for all distinct vertices $u, v \in\left\{y_{2}, v_{1}^{\prime}, u_{1}^{\prime}\right\}$, the inequality (1) holds. As $\max _{v \in V_{1} \backslash\left\{y_{2}\right\}} c(v)=c\left(x_{3}^{\prime}\right)$ by Claim 3.1, $c\left(v_{1}^{\prime}\right)=c\left(y_{2}\right)+k=c\left(x_{3}^{\prime}\right)+2 k-1$ and $c\left(u_{1}^{\prime}\right)>c\left(v_{1}^{\prime}\right)$, we have that $c\left(v_{1}^{\prime}\right)-c\left(x_{3}^{\prime}\right) \geq d-1$ and $c\left(u_{1}^{\prime}\right)-c\left(x_{3}^{\prime}\right) \geq d-1$. Therefore for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}\right\}$, the inequality (1) holds.

Since that $d\left(u_{1}^{\prime}, v_{1}\right)=k-1, d\left(v_{1}, v_{1}^{\prime}\right)=k-6$, and $c\left(v_{1}\right)=c\left(u_{1}^{\prime}\right)+(k-1)$ $=c\left(v_{1}^{\prime}\right)+5+(k-1)$, then for all distinct vertices $u, v \in\left\{v_{1}, v_{1}^{\prime}, u_{1}^{\prime}\right\}$, the inequality (1) holds. As $\max _{v \in V_{1}} c(v)=c\left(y_{2}\right)$ by Claim 3.1, and $c\left(v_{1}\right)=$ $c\left(y_{2}\right)+k+5+(k-1)$, we have that $c\left(v_{1}\right)-c\left(y_{2}\right) \geq d-1$. Therefore for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}, v_{1}\right\}$, the inequality (1) holds.

Note the fact that $d\left(v_{1}, u_{2}^{\prime}\right)=k-2, d\left(v_{1}, v_{2}^{\prime}\right)=k-5-2, d\left(u_{2}^{\prime}, v_{2}^{\prime}\right)=$ $2 k-7-2, c\left(u_{2}^{\prime}\right)=c\left(v_{1}\right)+k, c\left(v_{2}^{\prime}\right)=c\left(v_{1}\right)+k+5+2 ;$ and $d\left(v_{2}^{\prime}, u_{2}\right)=k-1$, $d\left(u_{2}, u_{2}^{\prime}\right)=k-6-2, c\left(u_{2}\right)=c\left(v_{2}^{\prime}\right)+(k-1)=c\left(u_{2}^{\prime}\right)+5+2+(k-1)$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}, v_{1}\right\} \cup\left\{u_{2}^{\prime}, v_{2}^{\prime}, u_{2}\right\}$, the inequality (1) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}, v_{1}\right\} \cup\left\{u_{2}^{\prime}, v_{2}^{\prime}, u_{2}\right\} \cup \ldots \cup\left\{v_{2 p-1}^{\prime}, u_{2 p-1}^{\prime}, v_{2 p-1}\right\} \cup$ $\left\{u_{2 p}^{\prime}, v_{2 p}^{\prime}, u_{2 p}\right\}=V_{1} \cup V_{2}$, the inequality (1) holds.

By the definition of $c$, it is easy to verify that $\max _{v \in V_{1} \cup V_{2}} c(v)=c\left(u_{2 p}\right)=$ $6 k-1+2 p(2 k+2 p+3)$ and $\max _{v \in\left(V_{1} \cup V_{2}\right) \backslash\left\{u_{2 p}\right\}} c(v)=c\left(v_{2 p}^{\prime}\right)=5 k+$ $2 p(2 k+2 p+3)$.

Figure 1: A nearly antipodal coloring for $P_{n}(n=2 k \geq 10)$.

Step 3. Color the vertices in $V_{3}$ (see Figure 1).
Step 3.1. Color the vertices in $\left\{w_{1}, w_{2}, \ldots, w_{q} ; z_{1}, z_{2}, \ldots, z_{q}\right\}$.
According the value of $q$, there are four cases.
Case 1. $q=1$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+k=7 k-1+2 p(2 k+2 p+3) \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+3+2(2 p+1)=7 k+4+2 p(2 k+2 p+5) .
\end{aligned}
$$

Case 2. $q=2$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+k=7 k-1+2 p(2 k+2 p+3) \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+3+2(2 p+1)=7 k+4+2 p(2 k+2 p+5), \\
& c\left(w_{2}\right)=c\left(z_{1}\right)+(k-1)=8 k+3+2 p(2 k+2 p+5) \\
& c\left(z_{2}\right)=c\left(w_{2}\right)+3+2(2 p+2)=8 k+10+2 p(2 k+2 p+7) .
\end{aligned}
$$

Case 3. $q=3$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+k=7 k-1+2 p(2 k+2 p+3), \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+3+2(2 p+1)=7 k+4+2 p(2 k+2 p+5), \\
& c\left(w_{3}\right)=c\left(z_{1}\right)+(k-1)=8 k+3+2 p(2 k+2 p+5), \\
& c\left(z_{2}\right)=c\left(w_{3}\right)+k=9 k+3+2 p(2 k+2 p+5), \\
& c\left(w_{2}\right)=c\left(z_{2}\right)+3+2(2 p+2)=9 k+10+2 p(2 k+2 p+7), \\
& c\left(z_{3}\right)=c\left(w_{2}\right)+k=10 k+10+2 p(2 k+2 p+7) .
\end{aligned}
$$

Case 4. $q=4$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+k=7 k-1+2 p(2 k+2 p+3), \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+3+2(2 p+1)=7 k+4+2 p(2 k+2 p+5), \\
& c\left(w_{4}\right)=c\left(z_{1}\right)+(k-1)=8 k+3+2 p(2 k+2 p+5), \\
& c\left(z_{2}\right)=c\left(w_{4}\right)+k=9 k+3+2 p(2 k+2 p+5), \\
& c\left(w_{2}\right)=c\left(z_{2}\right)+3+2(2 p+2)=9 k+10+2 p(2 k+2 p+7), \\
& c\left(z_{3}\right)=c\left(w_{2}\right)+(k-1)=10 k+9+2 p(2 k+2 p+7), \\
& c\left(w_{3}\right)=c\left(z_{3}\right)+3+2(2 p+3)=10 k+18+2 p(2 k+2 p+9), \\
& c\left(z_{4}\right)=c\left(w_{3}\right)+(k+1)=11 k+19+2 p(2 k+2 p+9) .
\end{aligned}
$$

Step 3.2. Color the vertices in $\left\{v_{2 p}, u_{2 p-1}, \ldots, v_{4}, u_{3}, v_{2}, u_{1}\right\}$.
For any case above ( $q=1,2,3,4$ ), we let

$$
\begin{aligned}
& c\left(v_{2 p}\right)=c\left(z_{q}\right)+[(k+q)-1], \\
& c\left(u_{2 p-1}\right)=c\left(v_{2 p}\right)+[(k+q-1)+2], \\
& c\left(v_{2 p-2}\right)=c\left(u_{2 p-1}\right)+[(k+q-1)+2 \cdot 2], \\
& c\left(u_{2 p-3}\right)=c\left(v_{2 p-2}\right)+[(k+q-1)+2 \cdot 3], \\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots, \\
& c\left(v_{2}\right)=c\left(u_{3}\right)+[(k+q-1)+2(2 p-2)], \\
& c\left(u_{1}\right)=c\left(v_{2}\right)+[(k+q-1)+2(2 p-1)] \\
& \quad=c\left(z_{q}\right)+2 p(k+q-1)+2 \cdot \frac{2 p(2 p-1)}{2} \\
& \quad=c\left(z_{q}\right)+2 p(k+q+2 p-2) .
\end{aligned}
$$

Then by a similar method to prove Claim 3.2, we can obtain the following claim.

Claim 3.3. For all distinct vertices $u, v \in V_{1} \cup V_{2} \cup V_{3}=V\left(P_{n}\right)$, the inequality (1) holds. And $\max _{v \in V\left(P_{n}\right)} c(v)=c\left(u_{1}\right)=c\left(z_{q}\right)+2 p(k+q+2 p-2)$.

By Claim 3.3, we have shown that for all even integers $n \geq 12, c$ is a nearly antipodal coloring for $P_{n}$. Therefore $\mathrm{ac}^{\prime}\left(P_{n}\right) \leq \mathrm{ac}^{\prime}(c)=$ $\max _{v \in V\left(P_{n}\right)} c(v)=c\left(u_{1}\right)=c\left(z_{q}\right)+2 p(k+q+2 p-2)$. To finish the proof of Theorem 2.1 for all even integers $n \geq 12$, it suffices to prove the following claim.

Claim 3.4. For any $p \in\{0,1,2, \ldots\}$ and any $q \in\{1,2,3,4\}$, it holds that $c\left(u_{1}\right)=c\left(z_{q}\right)+2 p(k+q+2 p-2)=\binom{n-2}{2}-\frac{n}{2}+7$, where $n=2 k=2(5+4 p+q)$.

In fact, if $q=1$, then $k=4 p+6,2 p=\frac{k-6}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right)= & c\left(z_{1}\right)+2 p(k+q+2 p-2)=7 k+4+2 p(2 k+2 p+5) \\
& +2 p(k+2 p-1) \\
= & 2 k^{2}-6 k+10=\frac{n^{2}}{2}-3 n+10=\binom{n-2}{2}-\frac{n}{2}+7
\end{aligned}
$$

If $q=2$, then $k=4 p+7,2 p=\frac{k-7}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right)= & c\left(z_{2}\right)+2 p(k+q+2 p-2)=8 k+10+2 p(2 k+2 p+7) \\
& +2 p(k+2 p) \\
= & 8 k+10+2 p(3 k+4 p+7)=\frac{n^{2}}{2}-3 n+10=\binom{n-2}{2}-\frac{n}{2}+7 .
\end{aligned}
$$

If $q=3$, then $k=4 p+8,2 p=\frac{k-8}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right)= & c\left(z_{3}\right)+2 p(k+q+2 p-2)=10 k+10+2 p(2 k+2 p+7) \\
& +2 p(k+2 p+1) \\
= & 10 k+10+2 p(3 k+4 p+8)=\frac{n^{2}}{2}-3 n+10=\binom{n-2}{2}-\frac{n}{2}+7
\end{aligned}
$$

If $q=4$, then $k=4 p+9,2 p=\frac{k-9}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right)= & c\left(z_{4}\right)+2 p(k+q+2 p-2)=11 k+19+2 p(2 k+2 p+9) \\
& +2 p(k+2 p+2) \\
= & 11 k+19+2 p(3 k+4 p+11)=\frac{n^{2}}{2}-3 n+10=\binom{n-2}{2}-\frac{n}{2}+7 .
\end{aligned}
$$

Thus Claim 3.4 holds and hence $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq \operatorname{ac}^{\prime}(c)=\binom{n-2}{2}-\frac{n}{2}+7$ for all even integers $n \geq 12$.

Secondly, for $n=10$, in the above proof we take $p=0$ and $q=0$. Namely, $V_{2}=V_{3}=\emptyset, V\left(P_{10}\right)=V_{1}=\left\{x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime} ; x_{2}, x_{1} ; y_{1}, y_{2} ; y_{3}^{\prime}, y_{2}^{\prime}, y_{1}^{\prime}\right\}$ (also see Figure 1 and let $p=q=0$ ). Then coloring $\left.c\right|_{v \in V_{1}}(v)$ is a nearly antipodal coloring for $P_{10}$. Thus by Claim 3.1, $\operatorname{ac}^{\prime}\left(P_{10}\right) \leq \operatorname{ac}^{\prime}\left(\left.c\right|_{v \in V_{1}}\right)=$ $\max _{v \in V_{1}} c(v)=c\left(y_{2}\right)=\left.(6 k-1)\right|_{k=5}=29=\binom{10-2}{2}+1$. Since $-\left\lfloor\frac{10}{n}\right\rfloor=-1$ for $n=10$, it follows that $\operatorname{ac}^{\prime}\left(P_{10}\right) \leq \operatorname{ac}^{\prime}\left(\left.c\right|_{v \in V_{1}}\right)=\binom{10-2}{2}+1=\binom{10-2}{2}-\frac{10}{2}-$ $\left\lfloor\frac{10}{10}\right\rfloor+7$.

Thus we complete the proof of assertion 1 in Theorem 2.1.
2. $n$ is odd and $n \geq 13$. Firstly, we let $n \geq 15$, note that $-\left\lfloor\frac{13}{n}\right\rfloor=0$, it suffices to show that $\mathrm{ac}^{\prime}\left(P_{n}\right) \leq\binom{ n-2}{2}-\frac{n}{2}+8$. Write $n=2 k+1=13+2(4 p+q)$, where $p \in\{0,1,2, \ldots\}$ and $q \in\{1,2,3,4\}$. Then we have that $k=6+(4 p+q)$ and $d-1=\operatorname{diam}\left(P_{n}\right)-1=2 k-1$.

We denote the vertices of $P_{n}$ by $x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}, x_{4}^{\prime} ; v_{1}^{\prime}, v_{2}^{\prime}, \ldots, v_{2 p-1}^{\prime}, v_{2 p}^{\prime}$; $w_{1}, w_{2}, \ldots, w_{q} ; v_{2 p}, v_{2 p-1}, \ldots, v_{2}, v_{1} ; x_{2}, x_{1} ; x_{0} ; y_{1}, y_{2} ; u_{1}, u_{2}, \ldots, u_{2 p-1}, u_{2 p} ;$ $z_{q}, \ldots, z_{2}, z_{1} ; u_{2 p}^{\prime}, u_{2 p-1}^{\prime}, \ldots, u_{2}^{\prime}, u_{1}^{\prime} ; y_{4}^{\prime}, y_{3}^{\prime}, y_{2}^{\prime}, y_{1}^{\prime}$ (see Figure 2). And we write $V_{1}=\left\{x_{0} ; x_{1}, x_{2} ; y_{1}, y_{2} ; x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}, x_{4}^{\prime} ; y_{1}^{\prime}, y_{2}^{\prime}, y_{3}^{\prime}, y_{4}^{\prime}\right\}$,
$V_{2}=\left\{v_{1}, u_{2}, v_{3}, u_{4}, \ldots, v_{2 p-1}, u_{2 p} ; v_{1}^{\prime}, v_{2}^{\prime}, \ldots, v_{2 p-1}^{\prime}, v_{2 p}^{\prime} ; u_{1}^{\prime}, u_{2}^{\prime}, \ldots, u_{2 p-1}^{\prime}, u_{2 p}^{\prime}\right\}$, $V_{3}=\left\{w_{1}, w_{2}, \ldots, w_{q} ; z_{1}, z_{2}, \ldots, z_{q} ; v_{2 p}, u_{2 p-1}, \ldots, v_{4}, u_{3}, v_{2}, u_{1}\right\}$.

Similar to the method of proof assertion 1, we will present a coloring $c$ for $P_{n}$ by three steps, such that

$$
\begin{equation*}
d(u, v)+|c(u)-c(v)| \geq d-1=2 k-1 \tag{2}
\end{equation*}
$$

holds for all distinct vertices $u, v \in V_{1} \cup V_{2} \cup V_{3}=V\left(P_{n}\right)$, and $\operatorname{ac}^{\prime}(c)=$ $\binom{n-2}{2}-\frac{n}{2}+8$ (note that $V_{2}=\emptyset$ if $p=0$, and it is easy to see that the following proof is also suitable for $V_{2}=\emptyset$ ).

Step 1. Color the vertices in $V_{1}$ (see Figure 2).
Let

$$
\begin{aligned}
& c\left(x_{0}\right)=1\left(x_{0} \text { is the central vertex of } P_{n}\right) ; \\
& c\left(x_{1}^{\prime}\right)=c\left(x_{0}\right)+(k-1)=k, \quad c\left(y_{1}^{\prime}\right)=c\left(x_{0}\right)+(k-1)=k ; \\
& c\left(x_{1}\right)=c\left(x_{1}^{\prime}\right)+k=2 k ; \\
& c\left(y_{2}^{\prime}\right)=c\left(x_{1}\right)+(k-1)=3 k-1, \\
& c\left(y_{1}\right)=c\left(y_{2}^{\prime}\right)+(k+1)=4 k ; \\
& c\left(x_{2}^{\prime}\right)=c\left(x_{1}^{\prime}\right)=c\left(x_{1}\right)+(k+1)=3 k+1 ; \\
& c\left(x_{2}\right)=c\left(x_{3}^{\prime}\right)+(k+3)=6 k+3 ; \\
& c\left(y_{4}^{\prime}\right)=c\left(x_{2}\right)+k=7 k+3, \quad c\left(y_{3}^{\prime}\right)=c\left(y_{3}^{\prime}\right)+3=5 k+3 ; \\
& c\left(y_{2}\right)=c\left(x_{4}^{\prime}\right)+k=8 k+8 .
\end{aligned}
$$

Then by the definition of $c$ and the value of $d(u, v)$ for $u, v \in V_{1}$, it is easy to verify that the following claim holds.

Claim 3.5. For all distinct vertices $u, v \in V_{1}$, the inequality (2) holds. At the same time, $\max _{v \in V_{1}} c(v)=c\left(y_{2}\right)=8 k+8$ and $\max _{v \in V_{1} \backslash\left\{y_{2}\right\}} c(v)=$ $c\left(x_{4}^{\prime}\right)=7 k+8$.

Step 2. Color the vertices in $V_{2}$ (see Figure 2).
For $i=1,2, \ldots, p$, let

$$
\begin{aligned}
c\left(v_{2 i-1}^{\prime}\right)= & c\left(y_{2}\right)+(2 i-1)(k+1)+5(2 i-2)+2[1+2+\ldots+(2 i-2)] \\
& +(2 i-2) k, \\
c\left(u_{2 i-1}^{\prime}\right)= & c\left(y_{2}\right)+(2 i-1)(k+1)+5(2 i-1)+2[1+2+\ldots+(2 i-1)] \\
& +(2 i-2) k ; \\
c\left(v_{2 i-1}\right)= & c\left(y_{2}\right)+(2 i-1)(k+1)+5(2 i-1)+2[1+2+\ldots+(2 i-1)] \\
& +(2 i-1) k ; \\
c\left(u_{2 i}^{\prime}\right)= & c\left(y_{2}\right)+(2 i)(k+1)+5(2 i-1)+2[1+2+\ldots+(2 i-1)] \\
& +(2 i-1) k \\
c\left(v_{2 i}^{\prime}\right)= & c\left(y_{2}\right)+(2 i)(k+1)+5(2 i)+2[1+2+\ldots+(2 i)]+(2 i-1) k ; \\
c\left(u_{2 i}\right)= & c\left(y_{2}\right)+(2 i)(k+1)+5(2 i)+2[1+2+\ldots+(2 i)]+(2 i) k .
\end{aligned}
$$

Then we have the following claim.
Claim 3.6. For all distinct vertices $u, v \in V_{1} \cup V_{2}$, the inequality (2) holds. At the same time, it holds that $\max _{v \in V_{1} \cup V_{2}} c(v)=c\left(u_{2 p}\right)=8 k+8+2 p(2 k+$ $2 p+7)$ and $\max _{v \in\left(V_{1} \cup V_{2}\right) \backslash\left\{u_{2 p}\right\}} c(v)=c\left(v_{2 p}^{\prime}\right)=7 k+8+2 p(2 k+2 p+7)$.

In fact, note $d-1=2 k-1$. Since that $d\left(y_{2}, v_{1}^{\prime}\right)=k-2, d\left(y_{2}, u_{1}^{\prime}\right)=k-6$, $d\left(v_{1}^{\prime}, u_{1}^{\prime}\right)=2 k-8, c\left(v_{1}^{\prime}\right)=c\left(y_{2}\right)+(k+1)$ and $c\left(u_{1}^{\prime}\right)=c\left(y_{2}\right)+(k+1)+7$, then for all distinct vertices $u, v \in\left\{y_{2}, v_{1}^{\prime}, u_{1}^{\prime}\right\}$, the inequality (2) holds. As $\max _{v \in V_{1} \backslash\left\{y_{2}\right\}} c(v)=c\left(x_{4}^{\prime}\right)$ by Claim 3.5, $c\left(v_{1}^{\prime}\right)=c\left(y_{2}\right)+(k+1)=c\left(x_{4}^{\prime}\right)+2 k+1$ and $c\left(u_{1}^{\prime}\right)>c\left(v_{1}^{\prime}\right)$, we have that $c\left(v_{1}^{\prime}\right)-c\left(x_{4}^{\prime}\right) \geq d-1$ and $c\left(u_{1}^{\prime}\right)-c\left(x_{4}^{\prime}\right) \geq d-1$. Therefore for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}\right\}$, the inequality (2) holds.

Since that $d\left(u_{1}^{\prime}, v_{1}\right)=k-1, d\left(v_{1}, v_{1}^{\prime}\right)=k-7$, and $c\left(v_{1}\right)=c\left(u_{1}^{\prime}\right)+k=$ $c\left(v_{1}^{\prime}\right)+7+k$, then for all distinct vertices $u, v \in\left\{v_{1}, v_{1}^{\prime}, u_{1}^{\prime}\right\}$, the inequality (2) holds. As $\max _{v \in V_{1}} c(v)=c\left(y_{2}\right)$ by Claim 3.5, and $c\left(v_{1}\right)=c\left(y_{2}\right)+$ $(k+1)+7+k$, we have that $c\left(v_{1}\right)-c\left(y_{2}\right) \geq d-1$. Therefore for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}, v_{1}\right\}$, the inequality (2) holds.

Note the fact that $d\left(v_{1}, u_{2}^{\prime}\right)=k-2, d\left(v_{1}, v_{2}^{\prime}\right)=k-6-2, d\left(u_{2}^{\prime}, v_{2}^{\prime}\right)=$ $2 k-8-2, c\left(u_{2}^{\prime}\right)=c\left(v_{1}\right)+(k+1), c\left(v_{2}^{\prime}\right)=c\left(v_{1}\right)+(k+1)+7+2 ;$ and
$d\left(v_{2}^{\prime}, u_{2}\right)=k-1, d\left(u_{2}, u_{2}^{\prime}\right)=k-7-2, c\left(u_{2}\right)=c\left(v_{2}^{\prime}\right)+k=c\left(u_{2}^{\prime}\right)+7+2+k$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}, v_{1}\right\} \cup\left\{u_{2}^{\prime}, v_{2}^{\prime}, u_{2}\right\}$, the inequality (2) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_{1} \cup\left\{v_{1}^{\prime}, u_{1}^{\prime}, v_{1}\right\} \cup\left\{u_{2}^{\prime}, v_{2}^{\prime}, u_{2}\right\} \cup \ldots \cup\left\{v_{2 p-1}^{\prime}, u_{2 p-1}^{\prime}, v_{2 p-1}\right\} \cup$ $\left\{u_{2 p}^{\prime}, v_{2 p}^{\prime}, u_{2 p}\right\}=V_{1} \cup V_{2}$, the inequality (2) holds.
By the definition of $c$, it is easy to see that $\max _{v \in V_{1} \cup V_{2}} c(v)=c\left(u_{2 p}\right)=$ $8 k+8+2 p(2 k+2 p+7)$, and $\max _{v \in\left(V_{1} \cup V_{2}\right) \backslash\left\{u_{2 p}\right\}} c(v)=c\left(v_{2 p}^{\prime}\right)=7 k+8+$ $2 p(2 k+2 p+7)$.

Figure 2. A nearly antipodal coloring for $P_{n}(n=2 k+1 \geq 13)$.

Step 3. Color the vertices in $V_{3}$ (see Figure 2).
Step 3.1. Color the vertices in $\left\{w_{1}, w_{2}, \ldots, w_{q} ; z_{1}, z_{2}, \ldots, z_{q}\right\}$.
According the value of $q$, there are four cases.
Case 1. $q=1$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+(k+1)=9 k+9+2 p(2 k+2 p+7), \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+5+2(2 p+1)=9 k+16+2 p(2 k+2 p+9) .
\end{aligned}
$$

Case 2. $q=2$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+(k+1)=9 k+9+2 p(2 k+2 p+7) \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+5+2(2 p+1)=9 k+16+2 p(2 k+2 p+9) \\
& c\left(w_{2}\right)=c\left(z_{1}\right)+k=10 k+16+2 p(2 k+2 p+9) \\
& c\left(z_{2}\right)=c\left(w_{2}\right)+5+2(2 p+2)=10 k+25+2 p(2 k+2 p+11) .
\end{aligned}
$$

Case 3. $q=$ 3. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+(k+1)=9 k+9+2 p(2 k+2 p+7), \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+5+2(2 p+1)=9 k+16+2 p(2 k+2 p+9), \\
& c\left(w_{3}\right)=c\left(z_{1}\right)+k=10 k+16+2 p(2 k+2 p+9), \\
& c\left(z_{2}\right)=c\left(w_{3}\right)+(k+1)=11 k+17+2 p(2 k+2 p+9), \\
& c\left(w_{2}\right)=c\left(z_{2}\right)+5+2(2 p+2)=11 k+26+2 p(2 k+2 p+11), \\
& c\left(z_{3}\right)=c\left(w_{2}\right)+(k+1)=12 k+27+2 p(2 k+2 p+11) .
\end{aligned}
$$

Case 4. $q=4$. Let

$$
\begin{aligned}
& c\left(w_{1}\right)=c\left(u_{2 p}\right)+(k+1)=9 k+9+2 p(2 k+2 p+7), \\
& c\left(z_{1}\right)=c\left(w_{1}\right)+5+2(2 p+1)=9 k+16+2 p(2 k+2 p+9), \\
& c\left(w_{4}\right)=c\left(z_{1}\right)+k=10 k+16+2 p(2 k+2 p+9), \\
& c\left(z_{2}\right)=c\left(w_{4}\right)+(k+1)=11 k+17+2 p(2 k+2 p+9), \\
& c\left(w_{2}\right)=c\left(z_{2}\right)+5+2(2 p+2)=11 k+26+2 p(2 k+2 p+11), \\
& c\left(z_{3}\right)=c\left(w_{2}\right)+k=12 k+26+2 p(2 k+2 p+11), \\
& c\left(w_{3}\right)=c\left(z_{3}\right)+5+2(2 p+3)=12 k+37+2 p(2 k+2 p+13), \\
& c\left(z_{4}\right)=c\left(w_{3}\right)+(k+2)=13 k+39+2 p(2 k+2 p+13) .
\end{aligned}
$$

Step 3.2. Color the vertices in $\left\{v_{2 p}, u_{2 p-1}, \ldots, v_{4}, u_{3}, v_{2}, u_{1}\right\}$.
For each case above ( $q=1,2,3,4$ ), we let

$$
\begin{aligned}
& c\left(v_{2 p}\right)=c\left(z_{q}\right)+(k+q), \\
& c\left(u_{2 p-1}\right)=c\left(v_{2 p}\right)+[(k+q)+2], \\
& c\left(v_{2 p-2}\right)=c\left(u_{2 p-1}\right)+[(k+q)+2 \cdot 2], \\
& c\left(u_{2 p-3}\right)=c\left(v_{2 p-2}\right)+[(k+q)+2 \cdot 3], \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots, \\
& c\left(v_{2}\right)=c\left(u_{3}\right)+[(k+q)+2(2 p-2)], \\
& c\left(u_{1}\right)=c\left(v_{2}\right)+[(k+q)+2(2 p-1)] \\
& \quad=c\left(z_{q}\right)+2 p(k+q)+2 \cdot \frac{2 p(2 p-1)}{2} \\
& \quad=c\left(z_{q}\right)+2 p(k+q+2 p-1) .
\end{aligned}
$$

Then by a similar method to prove Claim 3.6, we can obtain the following claim.

Claim 3.7. For all distinct vertices $u, v \in V_{1} \cup V_{2} \cup V_{3}=V\left(P_{n}\right)$, the inequality (2) holds. And $\max _{v \in V\left(P_{n}\right)} c(v)=c\left(u_{1}\right)=c\left(z_{q}\right)+2 p(k+q+2 p-1)$.

By Claim 3.7, we have shown that for all odd integers $n \geq 15, c$ is a nearly antipodal coloring for $P_{n}$. Therefore $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq \operatorname{ac}^{\prime}(c)=\max _{v \in V\left(P_{n}\right)} c(v)=$ $c\left(u_{1}\right)=c\left(z_{q}\right)+2 p(k+q+2 p-1)$. To finish the proof of Theorem 2.1 for all odd integers $n \geq 15$, it suffices to prove the following claim.

Claim 3.8. For any $p \in\{0,1,2, \ldots\}$ and any $q \in\{1,2,3,4\}$, it holds that $c\left(u_{1}\right)=c\left(z_{q}\right)+2 p(k+q+2 p-1)=\binom{n-2}{2}-\frac{n-1}{2}+8$, where $n=2 k+1=$ $13+2(4 p+q)$.

In fact, if $q=1$, then $k=4 p+7,4 p=k-7,2 p=\frac{k-7}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right) & =c\left(z_{1}\right)+2 p(k+q+2 p-1)=9 k+16+2 p(2 k+2 p+9)+2 p(k+2 p) \\
& =2 k^{2}-4 k+9=\frac{n^{2}}{2}-3 n+\frac{23}{2}=\binom{n-2}{2}-\frac{n-1}{2}+8 .
\end{aligned}
$$

If $q=2$, then $k=4 p+8,4 p=k-8, p=\frac{k-8}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right) & =c\left(z_{2}\right)+2 p(k+q+2 p-1) \\
& =10 k+25+2 p(2 k+2 p+11)+2 p(k+2 p+1) \\
& =2 k^{2}-4 k+9=\frac{n^{2}}{2}-3 n+\frac{23}{2}=\binom{n-2}{2}-\frac{n-1}{2}+8
\end{aligned}
$$

If $q=3$, then $k=4 p+9,4 p=k-9, p=\frac{k-9}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right) & =c\left(z_{3}\right)+2 p(k+q+2 p-1) \\
& =12 k+27+2 p(2 k+2 p+11)+2 p(k+2 p+2) \\
& =2 k^{2}-4 k+9=\frac{n^{2}}{2}-3 n+\frac{23}{2}=\binom{n-2}{2}-\frac{n-1}{2}+8 .
\end{aligned}
$$

If $q=4$, then $k=4 p+10,4 p=k-10,2 p=\frac{k-10}{2}$. Thus

$$
\begin{aligned}
c\left(u_{1}\right) & =c\left(z_{4}\right)+2 p(k+q+2 p-1) \\
& =13 k+39+2 p(2 k+2 p+13)+2 p(k+2 p+3) \\
& =2 k^{2}-4 k+9=\frac{n^{2}}{2}-3 n+\frac{23}{2}=\binom{n-2}{2}-\frac{n-1}{2}+8
\end{aligned}
$$

Thus Claim 3.8 holds and hence $\operatorname{ac}^{\prime}\left(P_{n}\right) \leq \operatorname{ac}^{\prime}(c)=\binom{n-2}{2}-\frac{n-1}{2}+8$ for all odd integers $n \geq 15$.

Secondly, for $n=13$, in the above proof we take $p=0$ and $q=0$. Namely, $V_{2}=V_{3}=\emptyset, V\left(P_{13}\right)=V_{1}=\left\{x_{1}^{\prime}, x_{2}^{\prime}, x_{3}^{\prime}, x_{4}^{\prime} ; x_{2}, x_{1} ; x_{0} ; y_{1}, y_{2} ; y_{4}^{\prime}, y_{3}^{\prime}\right.$, $\left.y_{2}^{\prime}, y_{1}^{\prime}\right\}$ (also see Figure 2 and let $p=q=0$ ). Then coloring $\left.c\right|_{v \in V_{1}}(v)$ is a nearly antipodal coloring for $P_{13}$. Thus by Claim $3.5, \operatorname{ac}^{\prime}\left(P_{13}\right) \leq \operatorname{ac}^{\prime}\left(\left.c\right|_{v \in V_{1}}\right)=$ $\max _{v \in V_{1}} c(v)=c\left(y_{2}\right)=\left.(8 k+8)\right|_{k=6}=56=\binom{13-2}{2}+1$. Since $-\left\lfloor\frac{13}{n}\right\rfloor=-1$ for $n=13$, it follows that $\operatorname{ac}^{\prime}\left(P_{13}\right) \leq \operatorname{ac}^{\prime}\left(\left.c\right|_{v \in V_{1}}\right)=\binom{13-2}{2}+1=\binom{13-2}{2}-$ $\frac{13-1}{2}-\left\lfloor\frac{13}{13}\right\rfloor+8$.

Thus the assertion 2 in Theorem 2.1 holds.

## 4. Examples

In this section we give some examples which present the nearly antipodal coloring $c$ for some $P_{n}$ with $\mathrm{ac}^{\prime}(c)$ presented in Theorem 2.1 by our methods.

Example 4.1. A nearly antipodal coloring $c$ for $P_{10}$ with $a c^{\prime}(c)=\binom{10-2}{2}-$ $\frac{10}{2}-\left\lfloor\frac{10}{10}\right\rfloor+7=\binom{10-2}{2}+1=29$ (see Figure 3)


Figure 3. A nearly antipodal coloring for $P_{10}$.

Example 4.2. A nearly antipodal coloring $c$ for $P_{13}$ with $a c^{\prime}(c)=\binom{13-2}{2}-$ $\frac{13-1}{2}-\left\lfloor\frac{13}{13}\right\rfloor+8=\binom{13-2}{2}+1=56$ (see Figure 4 ).


Figure 4. A nearly antipodal coloring $c$ for $P_{13}$.
Example 4.3 A nearly antipodal coloring $c$ for $P_{32}$ with $a c^{\prime}(c)=\binom{32-2}{2}-$ $\frac{32}{2}+7=\binom{32-2}{2}-9=426$ (see Figure 5) .

Here $n=2 k=10+2(4 p+q)=32$, then $k=16, p=2$ and $q=3$.


Figure 5. A nearly antipodal coloring for $P_{32}$.
Example 4.4. A nearly antipodal coloring $c$ for $P_{33}$ with $a c^{\prime}(c)=\binom{33-2}{2}-$ $\frac{33-1}{2}+8=\binom{33-2}{2}-8=457$ (see Figure 6) .

Here $n=2 k+1=13+2(4 p+q)=33$, then $k=16, p=2$ and $q=2$.


Figure 6. A nearly antipodal coloring $c$ for $P_{33}$.

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