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IMPROVED UPPER BOUNDS FOR NEARLY ANTIPODAL CHROMATIC NUMBER OF PATHS*

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Abstract

For paths P_n, G. Chartrand, L. Nebeský and P. Zhang showed that $\mathrm{ac}'(P_n) \le \binom{n-2}{2} + 2$ for every positive integer n, where $\mathrm{ac}'(P_n)$ denotes the nearly antipodal chromatic number of P_n. In this paper we show that $\mathrm{ac}'(P_n) \le \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$ if n is even positive integer and $n \ge 10$, and $\mathrm{ac}'(P_n) \le \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$ if n is odd positive integer and $n \ge 13$. For all even positive integers $n \ge 10$ and all odd positive integers $n \ge 13$, these results improve the upper bounds for nearly antipodal chromatic number of P_n.

Keywords: radio colorings, nearly antipodal chromatic number, paths.

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1. Introduction

Radio k-colorings are generalizations of ordinary colorings of graphs, which were inspired by (FM Radio) Channel Assignments Problem (see [5, 7]) and introduced by G. Chartrand, D. Erwan, F. Harary and P. Zhang [1]. For a connected graph G of order n and diameter d and a integer k with $1 \leq k \leq d$, a radio k-coloring of G is a function c: $V(G) \to \mathbf{N}$, such that $d(u,v) + |c(u) - c(v)| \geq k+1$ for every pair u and v of distinct vertices of G, where $d(u,v)$ denotes the distance between u and v (the length of a shortest $u-v$ path) in G. Clearly, radio 1-colorings and ordinary colorings are synonymous. The *value* $\mathrm{rc}_k(c)$ of a radio k-coloring c of G is the maximum color assigned to a vertex of G; while the *radio k-chromatic number* $\mathrm{rc}_k(G)$ of G is $\min\{\mathrm{rc}_k(c)\}$ taken over all k-coloring c of G. In particular, radio d-colorings are referred to as *radio labelings* and the *radio d-chromatic number* is called the *radio number*. Radio $(d-1)$-colorings are referred to as *radio antipodal coloring* or, more simply, as an *antipodal coloring*, and the *radio $(d-1)$-chromatic number* is called the *antipodal chromatic number*, denoted by $\mathrm{ac}(G)$. Radio k-coloring and radio labeling of graphs were studied in [1, 2]. Radio antipodal coloring of paths were studied in [3, 4, 6].

Furthermore, G. Chartrand, L. Nebeský and P. Zhang gave the concepts of *nearly antipodal colorings* in [4]. For a connected graph G of diameter d, a nearly antipodal coloring of G is a function c: $V(G) \to \mathbf{N}$, such that $d(u,v) + |c(u) - c(v)| \geq d-1$ for every two distinct vertices u and v of G. The *value* $\mathrm{ac}'(c)$ of a nearly antipodal coloring c of G is the maximum color assigned to a vertex of G. The *nearly antipodal chromatic number* $\mathrm{ac}'(G)$ of G is $\min\{\mathrm{ac}'(c)\}$ taken over all nearly antipodal colorings of G (In fact, for $d \geq 3$, a nearly antipodal coloring is a radio $(d-2)$-coloring).

Clearly, if G is a connected graph of diameter 1 or 2, then $\mathrm{ac}'(G) = 1$; while if $\mathrm{diam}(G) = 3$, then $\mathrm{ac}'(G)$ is the chromatic number of G. Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. For this reason, the nearly antipodal chromatic number of paths P_n were investigated in [4] by G. Chartrand, L. Nebeský and P. Zhang. And they showed that $\mathrm{ac}'(P_5) = 5$, $\mathrm{ac}'(P_6) = 7$, $\mathrm{ac}'(P_7) = 11$ and $\mathrm{ac}'(P_8) = 16$. Moreover, they presented an upper bound for the nearly antipodal chromatic number of paths P_n for every positive integer n as follows.

Theorem 1.1 ([4]). *If n is a path of order $n \geq 1$, $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} + 2$.*

2. Our Rresults and the Idea of the Proof

In this paper we will provide an improved version for Theorem 1.1. We will show that

Theorem 2.1.

1. *If* P_n *is even and* $n \geq 10$*, then* $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$;
2. *If* n *is odd and* $n \geq 13$*, then* $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$.

Clearly, it holds that $-\frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7 \leq 1$ for all even integers $n \geq 10$, and $-\frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8 \leq 1$ for all odd integers $n \geq 13$. Thus, for all even integers $n \geq 10$ and all odd integers $n \geq 13$, Theorem 2.1 improves the upper bounds of $\mathrm{ac}'(P_n)$.

We will prove Theorem 2.1 in Section 3, and the proof will virtually provide a nearly antipodal coloring c for paths P_n with $\mathrm{ac}'(c)$ that is equal to the bound presented in Theorem 2.1. The idea of performing the coloring c is based on pseudo greedy algorithm: Let $V(P_n) = \{p_1, p_2, \ldots, p_n\}$. At first, we use the color $c_1 = 1$ to color some vertex $p_{n_1} \in \{p_1, p_2, \ldots, p_n\}$, where p_{n_1} is the (a) *central vertex* of P_n. Suppose that for $1 \leq i \leq n-1$ the vertices in $\{p_{n_1}, p_{n_2}, \ldots, p_{n_i}\} \subset \{p_1, p_2, \ldots, p_n\}$ have been colored with $c(p_{n_j}) = c_j$ for all $1 \leq j \leq i$, then we choose a color $c_{i+1} \in \mathbf{N}$ as small as possible to color one vertex $p_{n_{i+1}} \in V(P_n) \backslash \{p_{n_1}, p_{n_2}, \ldots, p_{n_i}\}$, such that $d(p_{n_{i+1}}, p_{n_j}) + |c(p_{n_{i+1}}) - c(p_{n_j})| \geq d - 1$ for all $1 \leq j \leq i$. And if there are two vertices can be chosen for $p_{n_{i+1}}$, then we take $p_{n_{i+1}}$ close to central vertices of P_n as near as possible. Finally, we obtain that $\mathrm{ac}'(c) = c(p_{n_n})$ and hence $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c)$. In Section 4 we will give some examples which present the nearly antipodal coloring c for some paths P_n with $\mathrm{ac}'(c)$ showed in Theorem 2.1 by our methods.

3. Proof of Theorem 2.1

Proof. 1. *n is even and $n \geq 10$.* Firstly, we let $n \geq 12$, note that $-\lfloor \frac{10}{n} \rfloor = 0$, it suffices to show that $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} + 7$. Write $n = 2k = 10 + 2(4p + q)$, where $p \in \{0, 1, 2, \ldots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 5 + (4p + q)$ and $d - 1 = \mathrm{diam}(P_n) - 1 = 2k - 2$.

We denote the vertices of P_n by x'_1, x'_2, x'_3; $v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}$; $w_1, w_2 \ldots, w_q$; $v_{2p}, v_{2p-1}, \ldots, v_2$, v_1; x_2, x_1; y_1, y_2; $u_1, u_2, \ldots, u_{2p-1}, u_{2p}$;

$z_q, \ldots, z_2, z_1$; $u'_{2p}, u'_{2p-1}, \ldots, u'_2, u'_1$; y'_3, y'_2, y'_1 (see Figure 1). And we write

$V_1 = \{x_1, x_2; y_1, y_2;\ x'_1, x'_2, x'_3; y'_1, y'_2, y'_3\}$,
$V_2 = \{v_1, u_2, v_3, u_4, \ldots, v_{2p-1}, u_{2p}; v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}; u'_1, u'_2, \ldots, u'_{2p-1}, u'_{2p}\}$,
$V_3 = \{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q; v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}$.

In the following we will present a coloring c for P_n by three steps, such that

$$d(u,v) + |c(u) - c(v)| \geq d - 1 = 2k - 2 \tag{1}$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $\mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$(note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

Step 1. Color the vertices in V_1 (see Figure 1).

Let

$c(x_1) = 1$ (x_1 is an central vertex of P_n);
$c(y'_1) = c(x_1) + (k-2) = k - 1$, $\quad c(x'_1) = c(x_1) + (k-1) = k$;
$c(y_1) = c(x'_1) + (k-2) = 2k - 2$;
$c(x'_2) = c(y_1) + k - 1 = 3k - 3$, $\quad c(y'_2) = c(x'_2) + 1 = 3k - 2$;
$c(x_2) = c(x'_2) + (k+1) = 4k - 2$;
$c(y'_3) = c(x_2) + (k-1) = 5k - 3$, $\quad c(x'_3) = c(y'_3) + 3 = 5k$;
$c(y_2) = c(x'_3) + (k-1) = 6k - 1$.

Then by the definition of c and the value of $d(u,v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

Claim 3.1. For all distinct vertices $u, v \in V_1$, the inequality (1) holds. At the same time, $\max_{v \in V_1} c(v) = c(y_2) = 6k - 1$ and $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_3) = 5k$.

Step 2. Color the vertices in V_2 (see Figure 1).

For $i = 1, 2, \ldots, p$, let

$$\begin{aligned}
c(v'_{2i-1}) &= c(y_2) + (2i-1)k + 3(2i-2) + 2[1+2+\ldots+(2i-2)] \\
&\quad +(2i-2)(k-1), \\
c(u'_{2i-1}) &= c(y_2) + (2i-1)k + 3(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad +(2i-2)(k-1); \\
c(v_{2i-1}) &= c(y_2) + (2i-1)k + 3(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad +(2i-1)(k-1); \\
c(u'_{2i}) &= c(y_2) + (2i)k + 3(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad +(2i-1)(k-1), \\
c(v'_{2i}) &= c(y_2) + (2i)k + 3(2i) + 2[1+2+\ldots+(2i)] + (2i-1)(k-1); \\
c(u_{2i}) &= c(y_2) + (2i)k + 3(2i) + 2[1+2+\ldots+(2i)] + (2i)(k-1).
\end{aligned}$$

Then we have the following claim.

Claim 3.2. For all distinct vertices $u, v \in V_1 \cup V_2$, the inequality (1) holds. At the same time, it holds that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 6k - 1 + 2p(2k + 2p + 3)$ and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 5k + 2p(2k + 2p + 3)$.

In fact, note $d - 1 = 2k - 2$. Since that $d(y_2, v'_1) = k - 2$, $d(y_2, u'_1) = k - 5$, $d(v'_1, u'_1) = 2k - 7$, $c(v'_1) = c(y_2) + k$ and $c(u'_1) = c(y_2) + k + 5$, then for all distinct vertices $u, v \in \{y_2, v'_1, u'_1\}$, the inequality (1) holds. As $\max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x'_3)$ by Claim 3.1, $c(v'_1) = c(y_2) + k = c(x'_3) + 2k - 1$ and $c(u'_1) > c(v'_1)$, we have that $c(v'_1) - c(x'_3) \geq d - 1$ and $c(u'_1) - c(x'_3) \geq d - 1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1\}$, the inequality (1) holds.

Since that $d(u'_1, v_1) = k - 1$, $d(v_1, v'_1) = k - 6$, and $c(v_1) = c(u'_1) + (k - 1)$ $= c(v'_1) + 5 + (k - 1)$, then for all distinct vertices $u, v \in \{v_1, v'_1, u'_1\}$, the inequality (1) holds. As $\max_{v \in V_1} c(v) = c(y_2)$ by Claim 3.1, and $c(v_1) = c(y_2) + k + 5 + (k - 1)$, we have that $c(v_1) - c(y_2) \geq d - 1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\}$, the inequality (1) holds.

Note the fact that $d(v_1, u'_2) = k - 2$, $d(v_1, v'_2) = k - 5 - 2$, $d(u'_2, v'_2) = 2k - 7 - 2$, $c(u'_2) = c(v_1) + k$, $c(v'_2) = c(v_1) + k + 5 + 2$; and $d(v'_2, u_2) = k - 1$, $d(u_2, u'_2) = k - 6 - 2$, $c(u_2) = c(v'_2) + (k - 1) = c(u'_2) + 5 + 2 + (k - 1)$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\}$, the inequality (1) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\} \cup \ldots \cup \{v'_{2p-1}, u'_{2p-1}, v_{2p-1}\} \cup \{u'_{2p}, v'_{2p}, u_{2p}\} = V_1 \cup V_2$, the inequality (1) holds.

By the definition of c, it is easy to verify that $\max_{v\in V_1\cup V_2} c(v) = c(u_{2p}) = 6k-1+2p(2k+2p+3)$ and $\max_{v\in(V_1\cup V_2)\setminus\{u_{2p}\}} c(v) = c(v'_{2p}) = 5k+2p(2k+2p+3)$.

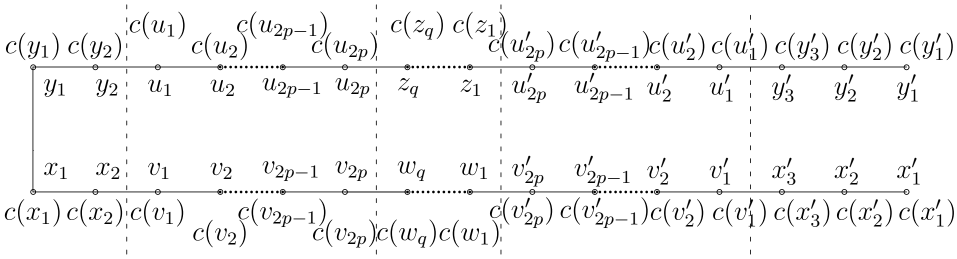

Figure 1: A nearly antipodal coloring for P_n ($n = 2k \geq 10$).

Step 3. Color the vertices in V_3 (see Figure 1).

Step 3.1. Color the vertices in $\{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q\}$.

According the value of q, there are four cases.

Case 1. $q = 1$. Let

$$c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),$$
$$c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5).$$

Case 2. $q = 2$. Let

$$c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),$$
$$c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5),$$
$$c(w_2) = c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5),$$
$$c(z_2) = c(w_2) + 3 + 2(2p + 2) = 8k + 10 + 2p(2k + 2p + 7).$$

Case 3. $q = 3$. Let

$$c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),$$
$$c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5),$$
$$c(w_3) = c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5),$$
$$c(z_2) = c(w_3) + k = 9k + 3 + 2p(2k + 2p + 5),$$
$$c(w_2) = c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7),$$
$$c(z_3) = c(w_2) + k = 10k + 10 + 2p(2k + 2p + 7).$$

Case 4. $q = 4$. Let

$$\begin{aligned}
c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3),\\
c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5),\\
c(w_4) &= c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5),\\
c(z_2) &= c(w_4) + k = 9k + 3 + 2p(2k + 2p + 5),\\
c(w_2) &= c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7),\\
c(z_3) &= c(w_2) + (k - 1) = 10k + 9 + 2p(2k + 2p + 7),\\
c(w_3) &= c(z_3) + 3 + 2(2p + 3) = 10k + 18 + 2p(2k + 2p + 9),\\
c(z_4) &= c(w_3) + (k + 1) = 11k + 19 + 2p(2k + 2p + 9).
\end{aligned}$$

Step 3.2. Color the vertices in $\{v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}$.

For any case above ($q = 1, 2, 3, 4$), we let

$$\begin{aligned}
&c(v_{2p}) = c(z_q) + [(k + q) - 1],\\
&c(u_{2p-1}) = c(v_{2p}) + [(k + q - 1) + 2],\\
&c(v_{2p-2}) = c(u_{2p-1}) + [(k + q - 1) + 2 \cdot 2],\\
&c(u_{2p-3}) = c(v_{2p-2}) + [(k + q - 1) + 2 \cdot 3],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\
&c(v_2) = c(u_3) + [(k + q - 1) + 2(2p - 2)],\\
&c(u_1) = c(v_2) + [(k + q - 1) + 2(2p - 1)]\\
&\qquad = c(z_q) + 2p(k + q - 1) + 2 \cdot \tfrac{2p(2p-1)}{2}\\
&\qquad = c(z_q) + 2p(k + q + 2p - 2).
\end{aligned}$$

Then by a similar method to prove Claim 3.2, we can obtain the following claim.

Claim 3.3. For all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, the inequality (1) holds. And $\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)$.

By Claim 3.3, we have shown that for all even integers $n \geq 12$, c is a nearly antipodal coloring for P_n. Therefore $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)$. To finish the proof of Theorem 2.1 for all even integers $n \geq 12$, it suffices to prove the following claim.

Claim 3.4. For any $p \in \{0, 1, 2, \ldots\}$ and any $q \in \{1, 2, 3, 4\}$, it holds that $c(u_1) = c(z_q) + 2p(k + q + 2p - 2) = \binom{n-2}{2} - \frac{n}{2} + 7$, where $n = 2k = 2(5 + 4p + q)$.

In fact, if $q = 1$, then $k = 4p + 6$, $2p = \frac{k-6}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_1) + 2p(k + q + 2p - 2) = 7k + 4 + 2p(2k + 2p + 5) \\ &\quad + 2p(k + 2p - 1) \\ &= 2k^2 - 6k + 10 = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 2$, then $k = 4p + 7$, $2p = \frac{k-7}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_2) + 2p(k + q + 2p - 2) = 8k + 10 + 2p(2k + 2p + 7) \\ &\quad + 2p(k + 2p) \\ &= 8k + 10 + 2p(3k + 4p + 7) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 3$, then $k = 4p + 8$, $2p = \frac{k-8}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_3) + 2p(k + q + 2p - 2) = 10k + 10 + 2p(2k + 2p + 7) \\ &\quad + 2p(k + 2p + 1) \\ &= 10k + 10 + 2p(3k + 4p + 8) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

If $q = 4$, then $k = 4p + 9$, $2p = \frac{k-9}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_4) + 2p(k + q + 2p - 2) = 11k + 19 + 2p(2k + 2p + 9) \\ &\quad + 2p(k + 2p + 2) \\ &= 11k + 19 + 2p(3k + 4p + 11) = \frac{n^2}{2} - 3n + 10 = \binom{n-2}{2} - \frac{n}{2} + 7. \end{aligned}$$

Thus Claim 3.4 holds and hence $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$ for all even integers $n \geq 12$.

Secondly, for $n = 10$, in the above proof we take $p = 0$ and $q = 0$. Namely, $V_2 = V_3 = \emptyset$, $V(P_{10}) = V_1 = \{x'_1, x'_2, x'_3; x_2, x_1; y_1, y_2; y'_3, y'_2, y'_1\}$ (also see Figure 1 and let $p = q = 0$). Then coloring $c|_{v \in V_1}(v)$ is a nearly antipodal coloring for P_{10}. Thus by Claim 3.1, $\mathrm{ac}'(P_{10}) \leq \mathrm{ac}'(c|_{v \in V_1}) = \max_{v \in V_1} c(v) = c(y_2) = (6k - 1)|_{k=5} = 29 = \binom{10-2}{2} + 1$. Since $-\lfloor \frac{10}{n} \rfloor = -1$ for $n = 10$, it follows that $\mathrm{ac}'(P_{10}) \leq \mathrm{ac}'(c|_{v \in V_1}) = \binom{10-2}{2} + 1 = \binom{10-2}{2} - \frac{10}{2} - \lfloor \frac{10}{10} \rfloor + 7$.

Thus we complete the proof of assertion 1 in Theorem 2.1.

2. *n is odd and $n \geq 13$.* Firstly, we let $n \geq 15$, note that $-\lfloor \frac{13}{n} \rfloor = 0$, it suffices to show that $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} + 8$. Write $n = 2k+1 = 13 + 2(4p+q)$, where $p \in \{0, 1, 2, \ldots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 6+(4p+q)$ and $d - 1 = \mathrm{diam}(P_n) - 1 = 2k - 1$.

We denote the vertices of P_n by x'_1, x'_2, x'_3, x'_4; $v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}$; $w_1, w_2, \ldots, w_q$; $v_{2p}, v_{2p-1}, \ldots, v_2, v_1$; x_2, x_1; x_0; y_1, y_2; $u_1, u_2, \ldots, u_{2p-1}, u_{2p}$; $z_q, \ldots, z_2, z_1$; $u'_{2p}, u'_{2p-1}, \ldots, u'_2, u'_1$; y'_4, y'_3, y'_2, y'_1 (see Figure 2). And we write

$$V_1 = \{x_0; x_1, x_2; y_1, y_2; x'_1, x'_2, x'_3, x'_4; y'_1, y'_2, y'_3, y'_4\},$$

$$V_2 = \{v_1, u_2, v_3, u_4, \ldots, v_{2p-1}, u_{2p}; v'_1, v'_2, \ldots, v'_{2p-1}, v'_{2p}; u'_1, u'_2, \ldots, u'_{2p-1}, u'_{2p}\},$$

$$V_3 = \{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q; v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}.$$

Similar to the method of proof assertion 1, we will present a coloring c for P_n by three steps, such that

$$d(u, v) + |c(u) - c(v)| \geq d - 1 = 2k - 1 \tag{2}$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $\mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n}{2} + 8$(note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

Step 1. Color the vertices in V_1 (see Figure 2).

Let

$c(x_0) = 1$ (x_0 is the central vertex of P_n);

$c(x'_1) = c(x_0) + (k-1) = k$, $\quad c(y'_1) = c(x_0) + (k-1) = k$;

$c(x_1) = c(x'_1) + k = 2k$;

$c(y'_2) = c(x_1) + (k-1) = 3k-1$, $\quad c(x'_2) = c(x_1) + (k+1) = 3k+1$;

$c(y_1) = c(y'_2) + (k+1) = 4k$;

$c(x'_3) = c(y_1) + k = 5k$, $\quad c(y'_3) = c(y'_3) + 3 = 5k+3$;

$c(x_2) = c(x'_3) + (k+3) = 6k+3$;

$c(y'_4) = c(x_2) + k = 7k+3$, $\quad c(x'_4) = c(y'_4) + 5 = 7k+8$;

$c(y_2) = c(x'_4) + k = 8k+8$.

Then by the definition of c and the value of $d(u, v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

Claim 3.5. For all distinct vertices $u, v \in V_1$, the inequality (2) holds. At the same time, $\max_{v\in V_1} c(v) = c(y_2) = 8k + 8$ and $\max_{v\in V_1\setminus\{y_2\}} c(v) = c(x'_4) = 7k + 8$.

Step 2. Color the vertices in V_2 (see Figure 2).

For $i = 1, 2, \ldots, p$, let

$$\begin{aligned}
c(v'_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-2) + 2[1+2+\ldots+(2i-2)] \\
&\quad + (2i-2)k, \\
c(u'_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad + (2i-2)k; \\
c(v_{2i-1}) &= c(y_2) + (2i-1)(k+1) + 5(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad + (2i-1)k; \\
c(u'_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i-1) + 2[1+2+\ldots+(2i-1)] \\
&\quad + (2i-1)k, \\
c(v'_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i) + 2[1+2+\ldots+(2i)] + (2i-1)k; \\
c(u_{2i}) &= c(y_2) + (2i)(k+1) + 5(2i) + 2[1+2+\ldots+(2i)] + (2i)k.
\end{aligned}$$

Then we have the following claim.

Claim 3.6. For all distinct vertices $u, v \in V_1 \cup V_2$, the inequality (2) holds. At the same time, it holds that $\max_{v\in V_1\cup V_2} c(v) = c(u_{2p}) = 8k+8+2p(2k+2p+7)$ and $\max_{v\in (V_1\cup V_2)\setminus\{u_{2p}\}} c(v) = c(v'_{2p}) = 7k + 8 + 2p(2k + 2p + 7)$.

In fact, note $d - 1 = 2k - 1$. Since that $d(y_2, v'_1) = k - 2$, $d(y_2, u'_1) = k - 6$, $d(v'_1, u'_1) = 2k - 8$, $c(v'_1) = c(y_2) + (k + 1)$ and $c(u'_1) = c(y_2) + (k + 1) + 7$, then for all distinct vertices $u, v \in \{y_2, v'_1, u'_1\}$, the inequality (2) holds. As $\max_{v\in V_1\setminus\{y_2\}} c(v) = c(x'_4)$ by Claim 3.5, $c(v'_1) = c(y_2)+(k+1) = c(x'_4)+2k+1$ and $c(u'_1) > c(v'_1)$, we have that $c(v'_1)-c(x'_4) \geq d-1$ and $c(u'_1)-c(x'_4) \geq d-1$. Therefore for all distinct vertices $u, v \in V_1\cup\{v'_1, u'_1\}$, the inequality (2) holds.

Since that $d(u'_1, v_1) = k - 1$, $d(v_1, v'_1) = k - 7$, and $c(v_1) = c(u'_1) + k = c(v'_1) + 7 + k$, then for all distinct vertices $u, v \in \{v_1, v'_1, u'_1\}$, the inequality (2) holds. As $\max_{v\in V_1} c(v) = c(y_2)$ by Claim 3.5, and $c(v_1) = c(y_2)+(k+1)+7+k$, we have that $c(v_1) - c(y_2) \geq d - 1$. Therefore for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\}$, the inequality (2) holds.

Note the fact that $d(v_1, u'_2) = k - 2$, $d(v_1, v'_2) = k - 6 - 2$, $d(u'_2, v'_2) = 2k - 8 - 2$, $c(u'_2) = c(v_1) + (k + 1)$, $c(v'_2) = c(v_1) + (k + 1) + 7 + 2$; and

$d(v_2', u_2) = k-1$, $d(u_2, u_2') = k-7-2$, $c(u_2) = c(v_2') + k = c(u_2') + 7 + 2 + k$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v_1', u_1', v_1\} \cup \{u_2', v_2', u_2\}$, the inequality (2) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v_1', u_1', v_1\} \cup \{u_2', v_2', u_2\} \cup \ldots \cup \{v_{2p-1}', u_{2p-1}', v_{2p-1}\} \cup \{u_{2p}', v_{2p}', u_{2p}\} = V_1 \cup V_2$, the inequality (2) holds.

By the definition of c, it is easy to see that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 8k + 8 + 2p(2k + 2p + 7)$, and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v_{2p}') = 7k + 8 + 2p(2k + 2p + 7)$.

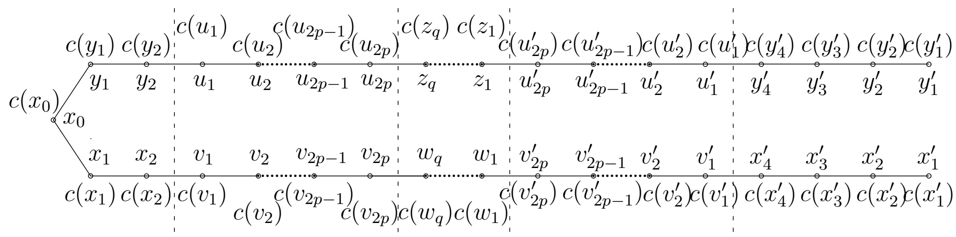

Figure 2. A nearly antipodal coloring for P_n ($n = 2k + 1 \geq 13$).

Step 3. Color the vertices in V_3 (see Figure 2).

Step 3.1. Color the vertices in $\{w_1, w_2, \ldots, w_q;\ z_1, z_2, \ldots, z_q\}$.

According the value of q, there are four cases.

Case 1. $q = 1$. Let

$$c(w_1) = c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k + 2p + 7),$$
$$c(z_1) = c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k + 2p + 9).$$

Case 2. $q = 2$. Let

$$c(w_1) = c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k + 2p + 7),$$
$$c(z_1) = c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k + 2p + 9),$$
$$c(w_2) = c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9),$$
$$c(z_2) = c(w_2) + 5 + 2(2p+2) = 10k + 25 + 2p(2k + 2p + 11).$$

Case 3. $q = 3$. Let

$$\begin{aligned}
c(w_1) &= c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k + 2p + 7),\\
c(z_1) &= c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k + 2p + 9),\\
c(w_3) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9),\\
c(z_2) &= c(w_3) + (k+1) = 11k + 17 + 2p(2k + 2p + 9),\\
c(w_2) &= c(z_2) + 5 + 2(2p+2) = 11k + 26 + 2p(2k + 2p + 11),\\
c(z_3) &= c(w_2) + (k+1) = 12k + 27 + 2p(2k + 2p + 11).
\end{aligned}$$

Case 4. $q = 4$. Let

$$\begin{aligned}
c(w_1) &= c(u_{2p}) + (k+1) = 9k + 9 + 2p(2k + 2p + 7),\\
c(z_1) &= c(w_1) + 5 + 2(2p+1) = 9k + 16 + 2p(2k + 2p + 9),\\
c(w_4) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9),\\
c(z_2) &= c(w_4) + (k+1) = 11k + 17 + 2p(2k + 2p + 9),\\
c(w_2) &= c(z_2) + 5 + 2(2p+2) = 11k + 26 + 2p(2k + 2p + 11),\\
c(z_3) &= c(w_2) + k = 12k + 26 + 2p(2k + 2p + 11),\\
c(w_3) &= c(z_3) + 5 + 2(2p+3) = 12k + 37 + 2p(2k + 2p + 13),\\
c(z_4) &= c(w_3) + (k+2) = 13k + 39 + 2p(2k + 2p + 13).
\end{aligned}$$

Step 3.2. Color the vertices in $\{v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}$.

For each case above ($q = 1, 2, 3, 4$), we let

$$\begin{aligned}
c(v_{2p}) &= c(z_q) + (k+q),\\
c(u_{2p-1}) &= c(v_{2p}) + [(k+q) + 2],\\
c(v_{2p-2}) &= c(u_{2p-1}) + [(k+q) + 2 \cdot 2],\\
c(u_{2p-3}) &= c(v_{2p-2}) + [(k+q) + 2 \cdot 3],\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
c(v_2) &= c(u_3) + [(k+q) + 2(2p-2)],\\
c(u_1) &= c(v_2) + [(k+q) + 2(2p-1)]\\
&= c(z_q) + 2p(k+q) + 2 \cdot \tfrac{2p(2p-1)}{2}\\
&= c(z_q) + 2p(k + q + 2p - 1).
\end{aligned}$$

Then by a similar method to prove Claim 3.6, we can obtain the following claim.

Claim 3.7. For all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, the inequality (2) holds. And $\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k+q+2p-1)$.

By Claim 3.7, we have shown that for all odd integers $n \geq 15$, c is a nearly antipodal coloring for P_n. Therefore $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k+q+2p-1)$. To finish the proof of Theorem 2.1 for all odd integers $n \geq 15$, it suffices to prove the following claim.

Claim 3.8. For any $p \in \{0, 1, 2, \ldots\}$ and any $q \in \{1, 2, 3, 4\}$, it holds that $c(u_1) = c(z_q) + 2p(k+q+2p-1) = \binom{n-2}{2} - \frac{n-1}{2} + 8$, where $n = 2k+1 = 13 + 2(4p+q)$.

In fact, if $q = 1$, then $k = 4p+7$, $4p = k-7$, $2p = \frac{k-7}{2}$. Thus

$$c(u_1) = c(z_1) + 2p(k+q+2p-1) = 9k + 16 + 2p(2k+2p+9) + 2p(k+2p)$$
$$= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8.$$

If $q = 2$, then $k = 4p+8$, $4p = k-8$, $p = \frac{k-8}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_2) + 2p(k+q+2p-1) \\ &= 10k + 25 + 2p(2k+2p+11) + 2p(k+2p+1) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 3$, then $k = 4p+9$, $4p = k-9$, $p = \frac{k-9}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_3) + 2p(k+q+2p-1) \\ &= 12k + 27 + 2p(2k+2p+11) + 2p(k+2p+2) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

If $q = 4$, then $k = 4p+10$, $4p = k-10$, $2p = \frac{k-10}{2}$. Thus

$$\begin{aligned} c(u_1) &= c(z_4) + 2p(k+q+2p-1) \\ &= 13k + 39 + 2p(2k+2p+13) + 2p(k+2p+3) \\ &= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n-2}{2} - \frac{n-1}{2} + 8. \end{aligned}$$

Thus Claim 3.8 holds and hence $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \binom{n-2}{2} - \frac{n-1}{2} + 8$ for all odd integers $n \geq 15$.

Secondly, for $n = 13$, in the above proof we take $p = 0$ and $q = 0$. Namely, $V_2 = V_3 = \emptyset$, $V(P_{13}) = V_1 = \{x'_1, x'_2, x'_3, x'_4; x_2, x_1; x_0; y_1, y_2; y'_4, y'_3, y'_2, y'_1\}$ (also see Figure 2 and let $p = q = 0$). Then coloring $c|_{v\in V_1}(v)$ is a nearly antipodal coloring for P_{13}. Thus by Claim 3.5, $\mathrm{ac}'(P_{13}) \leq \mathrm{ac}'(c|_{v\in V_1}) = \max_{v\in V_1} c(v) = c(y_2) = (8k+8)|_{k=6} = 56 = \binom{13-2}{2} + 1$. Since $-\lfloor \frac{13}{n} \rfloor = -1$ for $n = 13$, it follows that $\mathrm{ac}'(P_{13}) \leq \mathrm{ac}'(c|_{v\in V_1}) = \binom{13-2}{2} + 1 = \binom{13-2}{2} - \frac{13-1}{2} - \lfloor \frac{13}{13} \rfloor + 8$.

Thus the assertion 2 in Theorem 2.1 holds. ■

4. Examples

In this section we give some examples which present the nearly antipodal coloring c for some P_n with $\mathrm{ac}'(c)$ presented in Theorem 2.1 by our methods.

Example 4.1. A nearly antipodal coloring c for P_{10} with $ac'(c) = \binom{10-2}{2} - \frac{10}{2} - \lfloor \frac{10}{10} \rfloor + 7 = \binom{10-2}{2} + 1 = 29$ (see Figure 3).

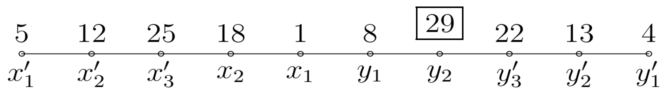

Figure 3. A nearly antipodal coloring for P_{10}.

Example 4.2. A nearly antipodal coloring c for P_{13} with $ac'(c) = \binom{13-2}{2} - \frac{13-1}{2} - \lfloor \frac{13}{13} \rfloor + 8 = \binom{13-2}{2} + 1 = 56$ (see Figure 4).

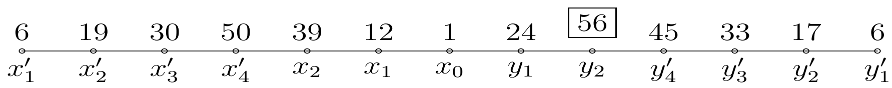

Figure 4. A nearly antipodal coloring c for P_{13}.

Example 4.3 A nearly antipodal coloring c for P_{32} with $ac'(c) = \binom{32-2}{2} - \frac{32}{2} + 7 = \binom{32-2}{2} - 9 = 426$ (see Figure 5).

Here $n = 2k = 10 + 2(4p + q) = 32$, then $k = 16$, $p = 2$ and $q = 3$.

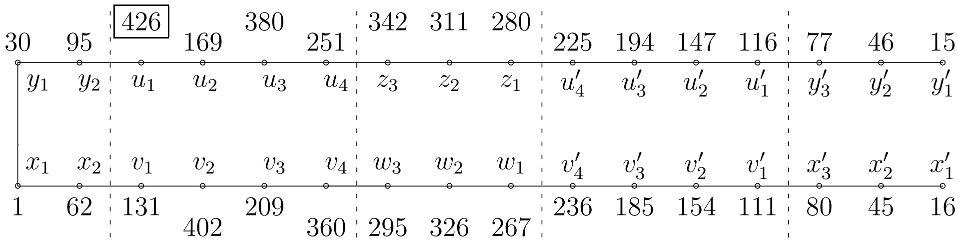

Figure 5. A nearly antipodal coloring for P_{32}.

Example 4.4. A nearly antipodal coloring c for P_{33} with $ac'(c) = \binom{33-2}{2} - \frac{33-1}{2} + 8 = \binom{33-2}{2} - 8 = 457$ (see Figure 6).

Here $n = 2k + 1 = 13 + 2(4p + q) = 33$, then $k = 16$, $p = 2$ and $q = 2$.

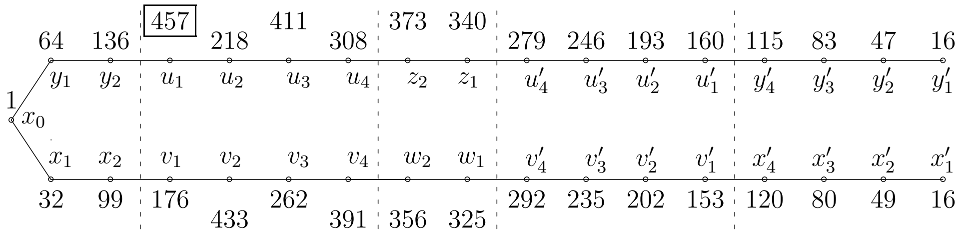

Figure 6. A nearly antipodal coloring c for P_{33}.

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