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# A NEW UPPER BOUND FOR THE CHROMATIC NUMBER OF A GRAPH*

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**Abstract**

Let $G$ be a graph of order $n$ with clique number $\omega(G)$, chromatic number $\chi(G)$ and independence number $\alpha(G)$. We show that $\chi(G) \leq \frac{n+\omega+1-\alpha}{2}$. Moreover, $\chi(G) \leq \frac{n+\omega-\alpha}{2}$, if either $\omega + \alpha = n + 1$ and $G$ is not a split graph or $\alpha+\omega = n-1$ and $G$ contains no induced $K_{\omega+3}-C_5$.

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## 1. Introduction

We consider [10] for terminology and notation not defined here and consider finite, simple and undirected graphs only. A *k-colouring* of a graph $G$ is an assignment of $k$ different colours to the vertices of $G$ such that adjacent vertices receive different colours. The minimum cardinality $k$ for which $G$ has a $k$-colouring is called the *chromatic number* of $G$ and is denoted by $\chi(G)$ or briefly $\chi$ if no ambiguity can arise.

An obvious lower bound for $\chi$ is the size of a largest clique in a graph $G$. This number is called the *clique number* of $G$ and denoted by $\omega(G)$ or briefly $\omega$. Unfortunately, the computations of $\chi$ and $\omega$ are both NP-hard.

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By a classical result of Erdős [9] we know that the difference $\chi(G) - \omega(G)$ can be arbitrarily large. On the other hand the graphs, where $\chi$ attains the lower bound $\omega$, form a graph class of great variety, even if we impose the equality on all induced subgraphs of a graph. A graph $G$ is called *perfect* if the chromatic number $\chi(H)$ equals the clique number $\omega(H)$ for every induced subgraph $H$ of $G$. More than four decades ago Berge [2] introduced the concept of perfect graphs.

Berge [3] conjectured that a graph $G$ is perfect if and only if neither $G$ nor its complement $\bar{G}$ contains an induced odd cycle of order at least five. In honor of Berge the graphs defined by the righthand side of the conjecture are known as *Berge graphs.* This famous longstanding conjecture known as *Strong Perfect Graph Conjecture* has recently been solved by Chudnovsky, Robertson, Seymour and Thomas [7]. Polynomial time recognition algorithms for Berge graphs have recently be announced by Chudnovsky and Seymour and Cornuéjols, Liu and Vušković (see [8, **?**, **?**]).

Upper bounds for $\chi$ can be obtained by studying the degrees of the vertices of a graph $G$. In particular, we are interested in the *maximum degree* of $G$, which is denoted by $\Delta(G)$ or simply $\Delta$. Obviously, the chromatic number of $G$ is at most $\Delta + 1$. In fact, there is a simple recursive greedy algorithm for colouring $G$ with at most $\Delta+1$ colours. Having coloured $G-v$, we just colour the vertex $v$ of $G$ with one of the colours not appearing on any of the at most $\Delta$ neighbours of $v$.

Hence, for a given graph $G$, the clique number $\omega(G)$, the chromatic number $\chi(G)$ and the maximum degree $\Delta(G)$ satisfy

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1.$$

In 1941 Brooks [5] determined for connected graphs $G$ the families of graphs attaining the upper bound $\Delta(G)+1$, namely complete graphs and odd cycles. This characterization leads to an improvement of the upper bound.

**Theorem 1** [5]**.** *If a connected graph $G = (V, E)$ is neither complete nor an odd cycle, then $G$ has a $\Delta(G)$-colouring.*

Based on Lovász algorithmic proof [11] of Brooks Theorem it is possible to design a linear time algorithm (see for instance [1] for an implementation in time $O(|V| + |E|)$).

## 2. A New Upper Bound

**Theorem 2.** *Let $G$ be a connected graph of order $n$ with clique number $\omega(G)$, chromatic number $\chi(G)$ and independence number $\alpha(G)$. Then $\chi(G) \leq \frac{n+\omega+1-\alpha}{2}$. Moreover, $\chi(G) \leq \frac{n+\omega-\alpha}{2}$, if either $\omega + \alpha = n + 1$ and $G$ is not a split graph or $\alpha + \omega = n - 1$ and $G$ contains no induced $K_{\omega+3} - C_5$.*

**Corollary 1** (Brigham and Dutton, 1985, [4])**.** *Let $G$ be a connected graph of order $n$ with clique number $\omega(G)$, chromatic number $\chi(G)$ and independence number $\alpha(G)$. Then $\chi(G) \leq \frac{n+\omega+1-\alpha}{2}$.*

Applying this upper bound both to $G$ and its complement $\overline{G}$, we obtain the following result of Nordhaus and Gaddum [12].

**Corollary 2** (Nordhaus and Gaddum, 1956, [12])**.** *Let $G$ be a graph of order $n$ with clique number $\omega(G)$, chromatic number $\chi(G)$ and independence number $\alpha(G)$. Then $\chi(G) + \chi(\overline{G}) \leq n + 1$.*

Combining Theorem 1 and Theorem 2 we obtain the following improved upper bound for the chromatic number of a graph.

**Theorem 3.** *Let $G$ be a connected graph of order $n$ with clique number $\omega(G)$, chromatic number $\chi(G)$, maximum degree $\Delta(G)$ and independence number $\alpha(G)$. Then $\chi(G) \leq min\{\Delta(G)+1, \frac{n+\omega+1-\alpha}{2}\}$. Moreover, if $G$ contains no induced $(K_{\omega+3} - C_5)$ and is neither a split graph nor an odd cycle, then $\chi(G) \leq min\{\Delta(G), \frac{n+\omega-\alpha}{2}\}$.*

For the proof of Theorem 2 we will make use of the following lemma.

**Lemma 1.** *Let $G$ be a $K_3$-free graph. Then $\chi(G) \leq \lfloor \frac{n+4}{3} \rfloor$*

***Proof.*** We generate $t = \lceil \frac{n-(r(K_3,K_3)-1)}{3} \rceil = \lceil \frac{n-5}{3} \rceil$ independent sets $I_1$, $I_2, \ldots, I_t$ of order three using the Ramsey number $r(K_3, K_3) = 6$. Let $H = G - (\cup_{i=1}^{t} I_i)$. If $|V(H)| = 3$, then $\chi(H) \leq 2$ and thus $\chi(G) \leq \frac{n-3}{3} + 2 = \frac{n+3}{3}$. If $|V(H)| = 4$, then $\chi(H) \leq 2$ and thus $\chi(G) \leq \frac{n-4}{3} + 2 = \frac{n+2}{3}$. If $|V(H)| = 5$, then $\chi(H) \leq 3$ and thus $\chi(G) \leq \frac{n-5}{3} + 3 = \frac{n+4}{3}$. ■

***Proof of Theorem* 2.** Let $I$ be a maximum independent set and $F = G - I$. Compute a maximum matching with vertex set $M$ in $\overline{F}$. Let $H = F - M$.

Then $\overline{H}$ is independent and $H$ is complete. Let $p = |V(H)|$. Then $\chi(G) \leq 1 + \frac{|M|}{2} + p = p + \frac{n-\alpha-p}{2} + 1 = \frac{n+p+2-\alpha}{2} \leq \frac{n+\omega+2-\alpha}{2}$.

If $\omega = p \geq 2$, then $d_H(v) \leq p-1$ for all vertices $v \in I$. Hence each vertex of $I$ can be coloured with a colour used for $H$. Hence $\chi(G) \leq \frac{n-\alpha-p}{2} + p = \frac{n+p-\alpha}{2} = \frac{n+\omega-\alpha}{2}$. If $\omega \geq p+2$, then $\chi(G) \leq \frac{n+p+2-\alpha}{2} \leq \frac{n+\omega-\alpha}{2}$. Therefore, if $\omega \neq p+1$, then $\chi(G) \leq \frac{n+\omega-\alpha}{2}$. So assume $\omega = p+1$.

*Case* 1. $p = 1$, $\omega = 2$

Applying Lemma 1 to the graph $G - I$, we get $\chi(G) \leq 1 + \lfloor \frac{n-\alpha+4}{3} \rfloor \leq \frac{n-\alpha+7}{3} \leq \frac{n+2-\alpha}{2}$ for $\alpha \leq n-8$. Hence we may assume $\alpha \geq n-7$.

If $|M| = 6$, then $|V(F)| = 7$. If $\Delta(F) \geq 4$, then $\chi(F) \leq 3$. If $\Delta(F) = 3$, then $\chi(F) \leq 3$ by Brooks' Theorem (1). And if $\Delta(F) \leq 2$, then $\chi(F) \leq \Delta + 1 \leq 3$. Therefore $\chi(G) \leq 1 + 3 = 4 < \frac{n+2-\alpha}{2}$.

If $|M| = 4$, then $|V(F)| = 5$. If $\Delta(F) \geq 3$, then $\chi(F) \leq 2$. If $\Delta(F) = 2$, then $\chi(F) = 2$ or $H \cong C_5$. And if $\Delta(F) = 1$, then $\chi(F) \leq \Delta + 1 = 2$. Therefore $\chi(G) \leq 1 + 2 = 3 < \frac{n+2-\alpha}{2}$, if $F \not\cong C_5$. Suppose $F \cong C_5$. Since $G$ is $K_3$-free, we have $d_F(v) \leq 2$ for all vertices $v \in I$. Then any 3-colouring of the $C_5$ can be extended to a 3-colouring of $G$. Therefore $\chi(G) \leq 3 < \frac{n+2-\alpha}{2}$.

If $|M| = 2$, then $|V(F)| = 3$. If $G$ is bipartite, then $\chi(G) \leq 2$. Else $G$ contains a $C_5$, since $I$ is an independent set and $|V(F)| = 3$. We may assume that $V(F) = \{w_1, w_2, w_4\}$ and $I$ contains two vertices $w_3, w_5$ such that $G[\{w_1, w_2, w_3, w_4, w_5\}] \cong C_5$ with edges $w_i w_{i+1}$ (mod 5). Since $G$ is $K_3$-free, we have $|N(v) \cap \{w_1, w_2, w_3, w_4, w_5\}| \leq 2$ for all vertices $v \in I - \{w_3, w_5\}$. Then any 3-colouring of the $C_5$ can be extended to a 3-colouring of $G$. Hence $\chi(G) = 3 = \frac{n+3-\alpha}{2} = \frac{n+\omega+1-\alpha}{2}$. Note that $C_5 \cong K_{\omega+3} - C_5$ for $\omega = 2$.

If $|M| = 0$, then $|V(F)| = 1$. Thus $G$ is a split graph. Since $G$ is connected, $G \cong K_{1,n-1}$. Therefore, $\chi(G) = 2 = \frac{n+\omega+1-\alpha}{2}$.

*Case* 2. $p \geq 2$

Let $M = U \cup W = \{u_1, u_2, \ldots, u_q\} \cup \{w_1, w_2, \ldots, w_q\}$ such that $u_i w_i \in E(\overline{G})$ for $1 \leq i \leq q$. If $u_i v, w_i v \notin E(G)$ for some $i$ and a vertex $v \in V(H)$, then $u_i, w_i$ can receive the same colour as $v$. Hence $\chi(G) \leq p + \frac{n-\alpha-p}{2} - 1 + 1 = \frac{n+p-\alpha}{2} < \frac{n+\omega-\alpha}{2}$.

If $u_i v_1, w_i v_2 \notin E(G)$ for four vertices $v_1, v_2 \in V(H)$ and $u_i, w_i \in M$, then $M$ is not a maximum matching, since $u_i w_i \in E(\overline{G})$ could be replaced by $u_i v_1, w_i v_2 \in E(\overline{G})$, a contradiction. Hence we may assume that $d_H(u_i) = p$ for $1 \leq i \leq q$. Then $G[U]$ is independent, since $\omega = p+1$. If $d_H(v) \leq p-1$

for all vertices $v \in I$, then every vertex $v \in I$ can be coloured with a colour from $H$. Then $\chi(G) \leq \frac{n+p-\alpha}{2} < \frac{n+\omega-\alpha}{2}$.

So let $I_0 \subset I$ contain all vertices of $I$ such that $d_H(v) = p$. Then $I_0 \cup U$ is independent, since $\omega = p+1$. If $\chi(G[W]) \leq q-1$, then $\chi(G) \leq \chi(G[V(H) \cup (I - I_0)]) + \chi(G[W]) + \chi(G[U \cup I_0]) \leq p + (q-1) + 1 = \frac{n+p-\alpha}{2} < \frac{n+\omega-\alpha}{2}$ using one colour for all vertices of $I_0 \cup U$. If $\chi(G[W]) = q$, then $G[W] \cong K_q$. Let $d_H(w_1) \leq d_H(w_2) \leq \ldots \leq d_H(w_q)$. If $v_1w_i, v_2w_j \notin E(G)$ for four vertices $v_1, v_2 \in V(H)$ and $w_i, w_j \in W$, then $M$ is not a maximum matching, since $u_iw_i, u_jw_j \in E(\overline{G})$ could be replaced by $v_1w_i, v_2w_j, u_iu_j \in E(\overline{G})$, a contradiction. Therefore we may assume $N_H(w_1) \subset N_H(w_2) \subset \ldots \subset N_H(w_q)$. This implies that either $d_H(w_i) = p$ for all $i \geq 2$ or $d_H(w_i) \geq p-1$ for all $i \geq 1$. In both cases, one can deduce that $p+1 \geq q+p-1$, and therefore $q \leq 2$.

*Subcase* 2.1. $q = 2$

Suppose $w_1v, w_2v \in E(G)$ for a vertex $v \in I_0$. If $d_H(w_2) = p$, then $G[H \cup \{w_2, v\}]$ is complete. Hence $\omega(G) \geq p+2$, a contradiction. If $d_H(w_2) = p-1$, then $d_H(w_1) = p-1$ and $N_H(w_1) = N_H(w_2)$. Then $\omega(G) \geq (p-1)+3 = p+2$, a contradiction. Therefore $d_W(v) \leq 1$ for all vertices $v \in I_0$. Since $I_0 \cup U$ is independent we obtain $d(v) \leq p+1$ for all vertices $v \in I_0$. Now $\chi(G - I_0) \leq \frac{n+p-\alpha}{2} = \frac{n+\omega-1-\alpha}{2} = p+2$. Then any $(p+2)$-colouring of $G - I_0$ can be extended to a $(p+2)$-colouring of $G$ and hence $\chi(G) \leq \frac{n+\omega-1-\alpha}{2}$.

*Subcase* 2.2. $q = 1$

We have $\alpha(G) + \omega(G) = n-1$ and $\omega(G) = p+1 \leq \chi(G) \leq p+2$. We will now show that $\chi(G) = p+1$, if $G$ contains no $K_{p+4} - C_5$. Suppose that $\chi(G) = p+1$.. We may assume that the vertices of $H$ receive colours $1, 2, \ldots, p$ and that $c(v) = p+1$ for all vertices $v \in I_0 \cup \{u_1\}$. If $d_H(w_1) = p$, then $I_0 \cup \{u_1, w_1\}$ is independent and we can choose $c(w_1) = p+1$. Since $d_H(v) \leq p-1$ for all vertices $v \in I - I_0$, we can choose $c(v) \in \{1, \ldots, p\}$ for all vertices $v \in I - I_0$. Suppose now $d_H(w_1) \leq p-1$. If $I_0 \cup \{u_1, w_1\}$ is independent, then the same colouring as above can be used. Hence assume that $w_1x \in E(G)$ for a vertex $x \in I_0$. Choose $c(w_1) = i$ for a proper colour $i \in \{1, \ldots, p\}$. Then we can find $c(v) \in \{1, \ldots, p+1\}$ for a vertex $v \in I - I_0$ unless $vu_1 \in E(G)$ and $N_H(w_1) = N_H(v)$ with $d_H(w_1) = d_H(v) = p-1$. But then $G[H \cup \{u_1, w_1, x, v\}] \cong K_{p+4} - C_5$, a contradiction, since $\chi(K_{p+4} - C_5) = p+2$.

*Subcase* 2.3. $q = 0$

Then $G$ is a split graph with $\omega(G) + \alpha(G) = n + 1$ and $\chi(G) = \omega(G) = p + 1 = \frac{n+\omega+1-\alpha}{2}$.

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