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ORIENTATION DISTANCE GRAPHS REVISITED

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Abstract

The orientation distance graph $\mathcal{D}_o(G)$ of a graph G is defined as the graph whose vertex set is the pair-wise non-isomorphic orientations of G, and two orientations are adjacent iff the reversal of one edge in one orientation produces the other. Orientation distance graphs was introduced by Chartrand *et al.* in 2001. We provide new results about orientation distance graphs and simpler proofs to existing results, especially with regards to the bipartiteness of orientation distance graphs and the representation of orientation distance graphs using hypercubes. We provide results concerning the orientation distance graphs of paths, cycles and other common graphs.

Keywords: orientation, distance graph, arc reversal.

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1. Introduction

Often, one needs to compare objects from some family such as trees or DNA sequences. One approach is to define an elementary operation that transforms objects into each other. Then the distance between two objects is defined to be the minimum number of elementary operations that can transform one object into the other. Given this notion, one needs to know properties of this distance measure: for example, if two objects are at distance 10, does this mean they are close or far. One can represent such a situation as a graph where the vertices are the objects and two vertices are adjacent if one object can be transformed into the other by a single elementary operation. The graph-theoretical distance is the same as the distance mentioned before. This graph has become known as a distance graph.

Chartrand *et al.* [1] defined the *orientation distance graph* $\mathcal{D}_o(G)$ of a graph $G = (V, E)$. This has vertex set $\mathcal{O}(G)$, the collection of pair-wise non-isomorphic orientations of G. Adjacency is defined between two orientations iff the reversal of one arc in one orientation generates (an orientation isomorphic to) the other. If G has m edges, then the number of orientations of G is 2^m and so $|\mathcal{O}(G)| \leq 2^m$. The orientation distance between two orientations of G is the number of arcs that need to be reversed to obtain the one from the other. A similar notion of distance between distinct orientations was discussed earlier by Zelinka in [6].

For a nonempty subset S of $\mathcal{O}(G)$, the *orientation distance graph* is the induced subgraph $\langle S \rangle$ over $\mathcal{D}_o(G)$. Also, a graph H is said to be an orientation distance graph if there exists a graph G and a set $S \subseteq \mathcal{O}(G)$ such that $\mathcal{D}_o(G)$ restricted to S is isomorphic to H [1].

In this paper we observe a simple sufficient condition for the orientation distance graph to be bipartite, and show a connection with hypercubes. We then investigate the orientation distance graphs of common graphs such as paths, cycles, complete bipartite and complete graphs. This include work from the second author's thesis [2].

2. Bipartiteness and Hypercubes

We start with a simple sufficient condition for the orientation distance graph to be bipartite.

Theorem 1. *Let G be a connected bipartite graph. If the two bipartite sets have different sizes, or the number of edges is even, then $\mathcal{D}_o(G)$ is bipartite.*

Proof. Say the bipartition of G is (A, B). Since G is connected, this partition is unique. Suppose first the two partite sets have different sizes. Then any automorphism of G maps A to A, and so one can define an orientation D of G as even or odd depending on the parity of the number of arcs oriented from A to B: isomorphic orientations have the same parity. Further, reversing an arc changes the parity of the orientation. Thus every edge in $\mathcal{D}_o(G)$ joins an even and an odd orientation.

If G has an even number of edges, then the same argument works. An orientation is odd or even depending on the parity of the number of arcs oriented from one set to the other. Since the number of edges is even, the parity from A to B is the same as the parity from B to A. Again $\mathcal{D}_o(G)$ is bipartite. ■

Corollary 1.1. (a) [1] *$\mathcal{D}_o(P_n)$ is bipartite for n odd.*
(b) *$\mathcal{D}_o(C_n)$ is bipartite for n even.*
(c) *$\mathcal{D}_o(K_{a,b})$ is bipartite for $a \neq b$ and for $a = b$ even.*

We turn next to another approach for showing bipartiteness and other properties of the distance graph. We define the *raw orientation distance graph* $RO(G)$ of a graph G as the graph with all labeled (isomorphic or not) orientations of G as vertices and adjacency is defined between two orientations iff the reversal of exactly one arc generates the other.

The simple but powerful observation is that $RO(G)$ of G with m edges is the m-dimensional hypercube, denoted Q_m. (The m-dimensional hypercube is defined recursively by: $Q_m = Q_{m-1} \times K_2$, where Q_0 is the trivial graph.) For example, it follows that graphs that have no non-trivial automorphism have the hypercube as their orientation distance graph.

Theorem 2 [6]**.** *The orientation distance graph of an asymmetric graph with m edges is the m-dimensional hypercube.*

Proof. An asymmetric graph G has no non-trivial automorphism, and so no two orientations of G are isomorphic. That is, $\mathcal{D}_o(G) = RO(G) = Q_m$. ■

Corollary 2.1. *Every tree is an orientation distance graph.*

Proof. It is well known (see for example [3]) that any tree is isometrically embeddable into (is an induced subgraph of) a hypercube Q_n for some value of n. ■

Chartrand *et al.* [1] showed that in fact every tree is an orientation distance graph with respect to some path.

Now, suppose we identify the vertices of $RO(G)$ that correspond to isomorphic orientations, grouping them together as one vertex. Adjacency for the new graph exists between those vertices if any isomorphic form of one vertex and any isomorphic form of the other vertex were adjacent in $RO(G)$. From the definition, the graph so obtained is the orientation distance graph for G.

Theorem 3. *$\mathcal{D}_o(G)$ is obtained from $RO(G)$ by identification.*

We demonstrate the idea with P_4. Figure 1 shows the 3-cube with a bit-string labeling. Consider each bit-string as the representation of an orientation of P_4, where an edge oriented from left to right is labeled 1 and 0 otherwise. Those vertices of the 3-cube that are isomorphic with respect to P_4 are marked in Figure 1. By grouping the identified orientations as one vertex, one obtains $\mathcal{D}_o(P_4)$.

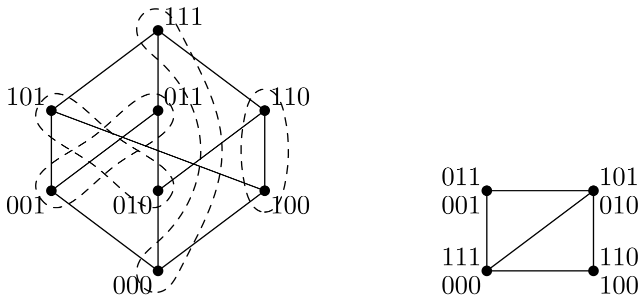

Figure 1: Obtaining $\mathcal{D}_o(P_4)$ from $RO(P_4)$.

3. Distance Graphs of Paths and Cycles

Paths were the focus of Chartrand *et al.* in [1].

3.1. Paths of odd order

Chartrand *et al.* showed the following:

Theorem 4 [1]**.** (a) *$\mathcal{D}_o(P_{2k+1})$ is bipartite.*

(b) *Further, 2^{2k-2} orientations have one color and $2^{2k-2} + 2^{k-1}$ have the other.*

(c) *Hence $\mathcal{D}_o(P_{2k+1})$ is not hamiltonian.*

They showed that the orientation distance graphs of paths are 2-connected for P_4 onwards. We can improve this. We utilize an idea similar to their discussion of the case where one graph is the disjoint union of two copies of another graph. We define the *pair graph* $G^{(2)}$ as having vertex set all unordered pairs of vertices of G, duplicates allowed. So if G has n vertices, then $G^{(2)}$ has $\binom{n+1}{2}$ vertices. Two pairs are defined to be adjacent iff they overlap in one vertex and their other vertices are adjacent in G.

(The pair graph of C_4 is the right-hand graph of Figure 3.) The following lemma is probably known.

Lemma 1. *If G is k-connected then so is $G^{(2)}$.*

Proof. It suffices to show that there are k internally disjoint paths between any two vertices x and y in $G^{(2)}$. For vertex $v \in V$ we use the notation G_v to denote the copy of G induced by the pairs of $G^{(2)}$ containing v.

There are two possibilities. Suppose pairs x and y overlap in one vertex of G: say $x = \{a, b\}$ and $y = \{a, c\}$ (where possibly $a = b$ or $a = c$). Then in G there are k internally disjoint b–c paths, and thus there are k internally disjoint x–y paths in G_a.

The second possibility is that x and y are disjoint: say $x = \{a, b\}$ and $y = \{c, d\}$. Assume first that $a \neq b$ and $c \neq d$.

Let R be $k-2$ vertices of G distinct from a and c; say $r_1, \ldots, r_{k-2}$. Let $\mathcal{T}_1$ be $k-1$ internally disjoint paths in G from a to $R \cup \{c\}$ avoiding d. Let $\mathcal{T}_2$ be $k-1$ internally disjoint paths in G from c to $R \cup \{a\}$ avoiding b.

Construct paths as follows. For $1 \leq i \leq k-2$, the path P_i has three segments: $\{a, b\}$–$\{r_i, b\}$–$\{r_i, d\}$–$\{c, d\}$. Paths Q_1 and Q_2 have two segments: Q_1 goes $\{a, b\}$–$\{a, d\}$–$\{c, d\}$ and Q_2 goes $\{a, b\}$–$\{c, b\}$–$\{c, d\}$. See Figure 2. The first segments of Q_2 and each P_i lie inside G_b and use the family $\mathcal{T}_1$. The middle segment of each P_i lies inside G_{r_i} and uses a b–d path disjoint from $R - \{r_i\} \cup \{a, c\}$. The first segment of Q_1 lies inside G_a and uses a b–d path disjoint from $R \cup \{c\}$; the final segment of Q_2 lies inside G_c and uses a b–d path disjoint from $R \cup \{a\}$. The final segments of Q_1 and each P_i lie inside G_d and use the family $\mathcal{T}_2$.

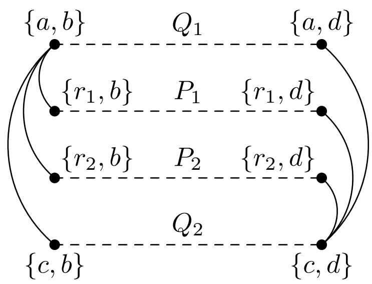

Figure 2: If G is 4-connected then so is $G^{(2)}$.

By construction, each r_i appears only on P_i. So the middle segments of P_i are disjoint from one another and from any other vertex used. By construction, a appears in the interior of only Q_1 and c in the interior of only Q_2. So the first segment of Q_1 and the final segment of Q_2 are disjoint from the rest of the vertices. The first segments of Q_2 and each P_i use $\mathcal{T}_1$ and contain b. But neither the final segment of Q_1 nor the final segment of any P_i contains b. And so the P_i together with the Q_j provide k internally disjoint x–y paths in $G^{(2)}$.

The case where $a = b$ and/or $c = d$ is handled similarly. ■

Theorem 5. *$\mathcal{D}_o(P_{2k+1})$ has connectivity k.*

Proof. The path P_{2k+1} can be obtained by taking two copies of P_{k+1} and identifying an end-vertex. Consider then $\mathcal{D}_o(P^M_{k+1})$, where P^M_{k+1} is P_{k+1} with a distinguished end-vertex. It follows that $\mathcal{D}_o(P_{2k+1})$ is the pair graph of $\mathcal{D}_o(P^M_{k+1})$: every orientation of P_{2k+1} corresponds to an unordered pair of orientations of P_{k+1}, and reversing one arc in P_{2k+1} is equivalent to reversing one arc in one of the orientations of P_{k+1}.

The case $k = 2$ is illustrated in Figure 3. There are four distinct orientations of the rooted P^M_3, say a, b, c and d. The vertices of $\mathcal{D}_o(P_5)$ correspond to the $\binom{5}{2}$ unordered pairs of a, b, c and d, repetitions allowed. ■

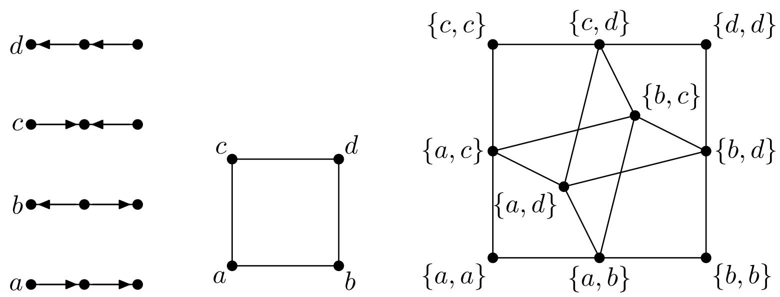

Figure 3: Obtaining $\mathcal{D}_o(P_5)$ from $\mathcal{D}_o(P^M_3)$.

This approach also gives another proof that $\mathcal{D}_o(P_{2k+1})$ is bipartite.

3.2. Paths of even order

The path has exactly one non-trivial automorphism: we call this a *flip*. It follows that $\mathcal{D}_o(P_n)$ is obtained from Q_{n-1} by identifying some pairs of vertices. This yields the following result:

Theorem 6. (a) [1] *$\mathcal{D}_o(P_{2k})$ has 2^{2k-2} vertices; and*
(b) *$\mathcal{D}_o(P_{2k})$ contains Q_{2k-2} as a spanning subgraph.*

Proof. Form the modified path P_{2k}^M by fixing the direction of the middle edge, say to 1. The mixed graph P_{2k}^M is asymmetric, and so $\mathcal{D}_o(P_{2k}^M) = Q_{k-2}$. Also, every orientation of P_{2k} is isomorphic to one of P_{2k}^M, and so $\mathcal{D}_o(P_{2k}^M)$ is a spanning subgraph of $\mathcal{D}_o(P_{2k})$. There may, however, be some new edges. ■

Since Q_m is hamiltonian and m-connected, it follows that:

Corollary 6.1. (a) [1] *$\mathcal{D}_o(P_{2k})$ is hamiltonian.*
(b) *$\mathcal{D}_o(P_{2k})$ is $(2k-2)$-connected.*

Chartrand *et al.* [1] showed that $\mathcal{D}_o(P_n)$ is 2-connected for all $n \geq 4$.

One idea used in the previous theorem is captured in the following:

Lemma 2. *If a mixed graph H is obtained from an undirected graph G by orienting some of the edges, then $\mathcal{D}_o(H)$ is a subgraph of $\mathcal{D}_o(G)$.*

For example, this shows the following:

Theorem 7. *$\mathcal{D}_o(P_{2k})$ is a subgraph of $\mathcal{D}_o(P_{2k+2})$.*

Proof. Define the modified path P_{2k+2}^M as P_{2k+2} with the first and last edges oriented toward the center. Then, $\mathcal{D}_o(P_{2k+2}^M) = \mathcal{D}_o(P_{2k})$. Hence $\mathcal{D}_o(P_{2k})$ is a subgraph of $\mathcal{D}_o(P_{2k+2})$. ■

(A similar result holds for odd-order paths.)

Corollary 7.1 [1]**.** *For $k \geq 2$, $\mathcal{D}_o(P_{2k})$ has a triangle.*

Proof. We have seen that $\mathcal{D}_o(P_4)$ contains a triangle. From the preceding theorem, $\mathcal{D}_o(P_4)$ is a subgraph of $\mathcal{D}_o(P_{2k})$ for all $k \geq 3$. ■

Chartrand *et al.* [1] showed that $\mathcal{D}_o(P_n)$ has no K_4.

3.3. Cycles

The orientation distance graphs of C_4, C_5 and C_6 are shown in Figure 4. The number of non-isomorphic orientations of cycles is listed in [5]: 1, 2, 2, 4, 4, 9, 10, 22, 30, ...

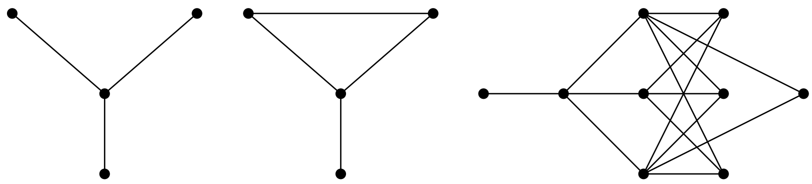

Figure 4: $\mathcal{D}_o(C_4)$, $\mathcal{D}_o(C_5)$ and $\mathcal{D}_o(C_6)$.

Theorem 8. *$\mathcal{D}_o(C_n)$ has a leaf and is therefore neither hamiltonian nor 2-connected.*

Proof. There is up to isomorphism exactly one transitive orientation of C_n (in which all edges are oriented in the same direction): call it u. Further, there is up to isomorphism exactly one orientation of C_n which has exactly one edge oriented differently from the other edges: call it v. Clearly, u and v are neighbors in $\mathcal{D}_o(C_n)$, with u having no other neighbor. ■

$\mathcal{D}_o(C_n)$ may or may not have a hamiltonian path: both $\mathcal{D}_o(C_4)$ and $\mathcal{D}_o(C_6)$ fail to have one, but $\mathcal{D}_o(C_5)$ does.

Some of the results mirror the situation for paths. We saw earlier that $\mathcal{D}_o(C_{2k})$ is bipartite. But $\mathcal{D}_o(C_{2k+1})$ is not.

Theorem 9. *$\mathcal{D}_o(C_{2k+1})$ contains a triangle.*

Proof. Let v be a vertex of C_{2k+1}. Then orient all edges towards v except for the three farthest from v, which remain undirected. Call the resultant mixed graph C^M_{2k+1}. One can then readily argue that $\mathcal{D}_o(C^M_{2k+1}) = \mathcal{D}_o(P_4)$. That graph contains a triangle. Hence $\mathcal{D}_o(C^M_{2k+1})$ and thus $\mathcal{D}_o(C_{2k+1})$ contains a triangle. Figure 5 shows these three mutually adjacent orientations in C_5. ■

We believe that the clique number of $\mathcal{D}_o(C_{2k+1})$ is 3.

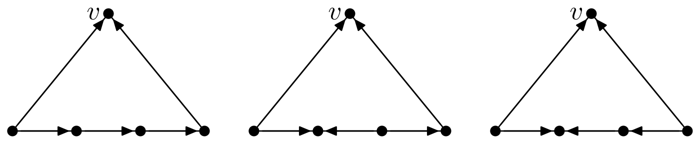

Figure 5: 3 mutually adjacent orientations of C_5.

4. Distance Graphs of Complete and Complete Bipartite Graphs

We start with the exact result for the star.

Theorem 10. *For $m \geq 2$, $\mathcal{D}_o(S_m)$ for the star graph S_m with m edges is P_{m+1}.*

Proof. The star graph S_m has $m+1$ non-isomorphic orientations: given by $0, 1, 2, \ldots, m$ arcs oriented towards the central vertex. ■

We saw earlier (Theorem 1) that the orientation distance graphs of most complete bipartite graphs are bipartite. However, $\mathcal{D}_o(K_{3,3})$ is not bipartite. Indeed, it has a clique of size 4—these are the unique orientations whose bi-degree sequences are

$$2,1,0:3,2,1 \qquad 3,1,0:2,2,1 \qquad 2,2,0:3,1,1 \qquad 3,2,0:2,1,1$$

$\mathcal{D}_o(K_{3,3})$ is shown in Figure 6. The orientation distance graph of a complete bipartite graph always has a leaf (the orientation with all arcs oriented from one side to the other).

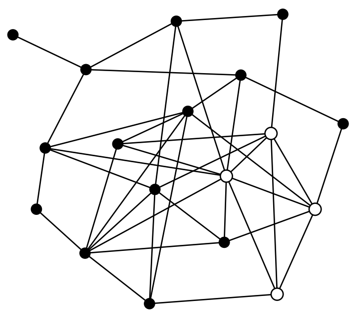

Figure 6: $\mathcal{D}_o(K_{3,3})$ (the hollow vertices form a clique).

For the complete graph, an orientation is commonly known as a tournament. The following sequence of number of non-isomorphic orientations of complete graphs is from [4]: 1, 2, 4, 12, 56, 456, 6880, ...

It is not hard to show that $\mathcal{D}_o(K_n)$ is a subgraph of $\mathcal{D}_o(K_{n+1})$. Figure 7 shows the orientation distance graphs of K_4 and K_5. The data for the latter was generated by computer.

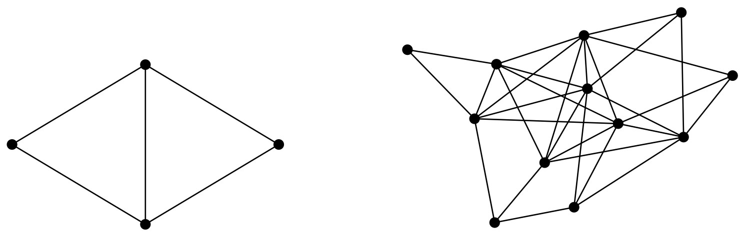

Figure 7: $\mathcal{D}_o(K_4)$ and $\mathcal{D}_o(K_5)$.

It may be noted that $\mathcal{D}_o(K_5)$ contains a clique of size 4. We conjecture that the clique number of $\mathcal{D}_o(K_n)$ tends to ∞ as n tends to ∞. (However, computer calculation shows that $\mathcal{D}_o(K_6)$ does not have a 5-clique.)

5. Other Results

Let us consider a special caterpillar graph: the caterpillar graph which is formed from a path by attaching to every vertex exactly one leaf. We denote this tree by I_n, where n is the number of vertices in the main path (see Figure 8 for I_4).

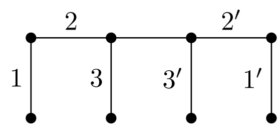

Figure 8: The caterpillar I_4.

The orientation distance graph of such caterpillars is related to the orientation distance graph of the path with the same number of edges.

Theorem 11. (a) *If n is even, $\mathcal{D}_o(I_n) = \mathcal{D}_o(P_{2n})$.*
(b) *If n is odd, $\mathcal{D}_o(I_n) = \mathcal{D}_o(P_{2n-1}) \times K_2$.*

Proof. (a) Consider the caterpillar graph I_4 shown in Figure 8. This graph has only one non-trivial automorphism: this maps 1 to $1'$, 2 to $2'$ and 3 to $3'$. The point is that this automorphism behaves exactly like the flip of P_8. We may clearly extend this argument for the case of I_n (n even), where the automorphism that maps the corresponding edges about the middle edge corresponds to the flip of P_{2n}.

(b) Fix the orientation of the leaf adjacent to the central vertex of I_n. Again the automorphism that maps the remaining edges around the central vertex corresponds to the flip in the path of the same size. Further, we have two such correspondences, one for each direction of the central leaf. It follows that $\mathcal{D}_o(I_n) = \mathcal{D}_o(P_{2n-1}) \times K_2$. ■

The last line of the above proof is the same argument used in the result about disjoint union. Chartrand *et al.* [1] showed that the orientation distance graph of a graph with non-isomorphic components is given by the cartesian product of the orientation distance graphs of the components.

Theorem 12 [1]**.** *If $G = G_1 \cup G_2$, where G_1 and G_2 are non-isomorphic and connected, then $\mathcal{D}_o(G) = \mathcal{D}_o(G_1) \times \mathcal{D}_o(G_2)$.*

We have seen several results that show that a graph has a bipartite orientation distance graph. We close with the following one about graphs with a pair of identical vertices.

Theorem 13. *Suppose graph G has a unique non-trivial automorphism and that automorphism maps one vertex u to v, and vice versa, leaving the other vertices fixed, with u and v not adjacent. Then, $\mathcal{D}_o(G)$ is bipartite.*

Proof. If two orientations are isomorphic, then the isomorphism is given by the unique non-trivial automorphism of G. The number of edges that the two orientations differ by must be even: if they differ on ux then they differ on vx and vice versa. Hence the orientations are an even distance apart in $RO(G)$ and have the same color there. ■

6. Conclusion

We studied orientation distance graphs of a wide variety of graphs. We observed a simple condition for bipartiteness, while the approach of obtaining the orientation distance graph from hypercubes proved useful in their understanding. Further, we studied the orientation distance graphs of cycles and complete graphs. We observed that the orientation distance graph of an even cycle is bipartite and that of an odd cycle has a triangle. We believe there are more interesting problems to study about orientation distance graphs and leave the following as open problems:

1. *Which graphs are distance graphs?* We know trees are distance graphs. Chartrand *et al.* [1] showed that every cycle C_n is an orientation distance graph with respect to the path P_{n+1}. We found K_4 in $\mathcal{D}_o(K_5)$ and in $\mathcal{D}_o(K_{3,3})$. We conjecture that all cliques can be found in $\mathcal{D}_o(K_n)$ as $n \to \infty$.
2. *What is the chromatic number of $\mathcal{D}_o(P_m)$ for m even, and the clique and chromatic numbers of $\mathcal{D}_o(C_m)$ for m odd?* We saw that the clique number of $\mathcal{D}_o(P_n)$ is 3 when n is odd: we conjecture that the chromatic number of $\mathcal{D}_o(P_n)$ is 3 when n is odd.
3. *Unique orientations.* Chartrand *et al.* claim in [1] (Theorem 2.5) that if D is an orientation of G that is isomorphic to no other orientation of G, then D lies on no odd cycle in $\mathcal{D}_o(G)$. The proof they provided is incomplete. But we were unable to provide a complete proof or a counter-example.

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