Discussiones Mathematicae
Graph Theory 27 (2007) 93–103

CHARACTERIZATION OF BLOCK GRAPHS WITH EQUAL 2-DOMINATION NUMBER AND DOMINATION NUMBER PLUS ONE

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Abstract

Let G be a simple graph, and let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if every vertex $v \in V(G) - D$ is adjacent with at least p vertices of D. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$.

If G is a nontrivial connected block graph, then we show that $\gamma_2(G) \geq \gamma(G) + 1$, and we characterize all connected block graphs with $\gamma_2(G) = \gamma(G) + 1$. Our results generalize those of Volkmann [12] for trees.

Keywords: domination, 2-domination, multiple domination, block graph.

2000 Mathematics Subject Classification: 05C69.

1. Terminology and Introduction

We consider finite, undirected, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*. Let $L(G)$ be the set of leaves of a graph G. For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$, $N[S] = N_G[S] = N(S) \cup S$, and $G[S]$ is the subgraph induced by S.

A *block* of a graph G is maximal subgraph of G without a cutvertex. If every block of a graph is complete, then we speak of a *block graph*. We write K_n for the *complete graph* of order n, and $K_{p,q}$ for the the *complete bipartite graph* with bipartition X, Y such that $|X| = p$ and $|Y| = q$.

The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw. In the case that G is the trivial graph, we define $S(G) = G$. Let SS_t be the subdivision graph of the star $K_{1,t}$. A tree is a *double star* if it contains exactly two vertices of degree at least two. A double star with respectively s and t leaves attached at each support vertex is denoted by $S_{s,t}$. Instead of $S(S_{s,t})$ we write $SS_{s,t}$.

The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

A vertex and an edge are said to *cover* each other if they are incident. A *vertex cover* in a graph G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the *covering number* of G and is denoted by $\beta(G) = \beta$. A set of pairwise non-adjacent vertices of G is an *independent* set of G. The cardinality of a maximum independent set is called the *independence number* $\alpha(G)$ of the graph G.

Let p be a positive integer. A subset $D \subseteq V(G)$ is a *p-dominating set* of the graph G, if $|N_G(v) \cap D| \geq p$ for every $v \in V(G) - D$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A p-dominating set of minimum cardinality of a graph G is called a $\gamma_p(G)$-set.

In [2, 3], Fink and Jacobson introduced the concept of p-domination. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [6, 7].

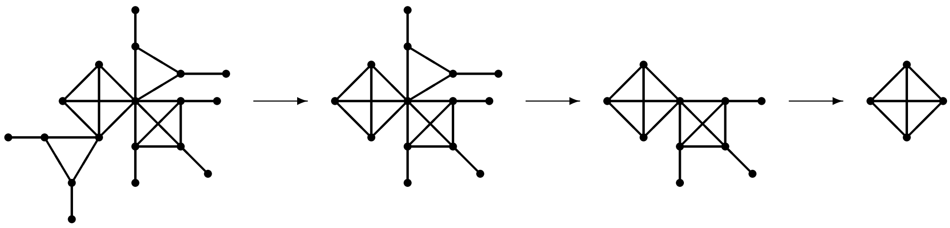
If T is a nontrivial tree, then it is easy to see that $\gamma_2(T) \geq \gamma(T) + 1$. Recently, Volkmann has proved the following result.

Theorem 1.1 (Volkmann [12]). *A nontrivial tree T satisfies $\gamma_2(T) = \gamma(T) + 1$ if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$.*

In this paper we show that $\gamma_2(G) \geq \gamma(G) + 1$ for every nontrivial connected block graph G, and as an extension of Theorem 1.1, we characterize all block graphs G with $\gamma_2(G) = \gamma(G) + 1$.

The procedure to achieve this objective is to classify all connected block graphs with $\gamma_2 = \gamma + 1$ in a finite number of determined family classes. The family classes are given by a reduction method, in which every graph is assigned to a certain subgraph.

If G is a connected block graph with $\gamma_2(G) = \gamma(G) + 1$, we will show that, if there is an endblock B of $G - L(G)$ with cutvertex u in $G - L(G)$ and with $N_G(B - u) \cap L(G) \neq \emptyset$, then the graph $G' = G - (N_G[B - u] - u)$ satisfies again the property $\gamma_2(G') = \gamma(G') + 1$. If we repeat this reduction process until it is not possible anymore, we obtain a subgraph that belongs to the set of graphs that represent the family class of this particular block graph. As an example, regard following reduction of a block graph G with $\gamma_2(G) = \gamma(G) + 1$:



The resulted graph is the block K_4. The graph G will belong to the family of block graphs with $\gamma_2 = \gamma + 1$ which can be reduced to a K_p for an integer $p \geq 3$.

We consider this reduction method to be important concerning graph characterization problems and therefore it could be in some way attractive for other graph theoretical investigations.

2. Preliminary Results

The following well known results play an important role in our investigations.

Theorem 2.1 (Gallai [5], 1959). *If G is a graph, then $\alpha(G)+\beta(G) = n(G)$.*

Theorem 2.2 (Blidia, Chellali, Volkmann [1], 2006). *If G is block graph, then $\gamma_2(G) \geq \alpha(G)$.*

Theorem 2.3 (Topp, Volkmann [10] 1990). *If G is a block graph, then $\gamma(G) = \alpha(G)$ if and only if every vertex belongs to exactly one simplex.*

Theorem 2.4 (Payan, Xuong [8] 1982, Fink Jacobson, Kinch, Roberts [4] 1985). *For a graph G with even order n and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of G consist of the cycle C_4 or the corona graph $H \circ K_1$ for any connected graph H.*

Proofs of Theorems 2.1, 2.3 and 2.4 can also be found in the book of Volkmann [11], pp. 193, 223 and 228. In 1998, Randerath and Volkmann [9] and independently, in 2000, Xu, Cockayne, Haynes, Hedetniemi and Zhou [13] (cf. also [6], pp. 42–48) characterized the odd order graphs G for which $\gamma(G) = \lfloor n/2 \rfloor$. In the next theorem we only note the part of this characterization which we will use in the next section

Theorem 2.5 (Randerath, Volkmann [9] 1998). *Let G be a nontrivial connected block graph of odd order n with $\delta(G) = 1$, $\gamma(G) = \lfloor n/2 \rfloor$ and $\gamma(G) = \beta(G)$. Then the following cases are possible:*

(1) *$|N_G(L(G))| = |L(G)| - 1$ and $G - N_G[L(G)] = \emptyset$.*

(2) *$|N_G(L(G))| = |L(G)|$ and $G - N_G[L(G)]$ is an isolated vertex.*

(3) *$|N_G(L(G))| = |L(G)|$ and $G - N_G[L(G)]$ is a star of order three such that the center of the star has degree two in G.*

3. Main Results

Theorem 3.1. *If G is a nontrivial connected block graph, then $\gamma_2(G) \geq \gamma(G) + 1$.*

Proof. Since every maximal independent set is also a domination set, we deduce that $\alpha(G) \geq \gamma(G)$. Combining this with Theorem 2.2, we obtain $\gamma_2(G) \geq \alpha(G) \geq \gamma(G)$. In view of Theorem 2.3, we have $\gamma(G) = \alpha(G)$ if and only if every vertex belongs to exactly one simplex. If $S_1, S_2, \ldots, S_q$ are the simplexes of G, then it is clear that $\gamma(G) \leq q < \gamma_2(G)$ or $\gamma(G) = 1 = \gamma_2(G)$ and G is the trivial graph. ■

Lemma 3.2. *If G is a connected block graph with $\gamma_2(G) = \gamma(G) + 1$, then either $|N_G(L(G))| = |L(G)|$ or $G = K_{1,2}$.*

Proof. If $n(G) = 2$, then the statement is valid. Therefore let $n(G) \geq 3$ in the following. Assume that there exists a vertex $v \in V(G)$ with $|N_G(v) \cap L(G)| \geq 2$. Let $N_G(v) \cap L(G) = \{x_1, x_2, \ldots, x_p\}$ with $p \geq 2$, and let $G' = G - \{x_1, x_2, \ldots, x_p\}$. If $V(G) = \{v, x_1, x_2, \ldots, x_p\}$, then it follows from the hypothesis $\gamma_2(G) = \gamma(G) + 1$ that $G = K_{1,2}$. Hence we assume in the following that $|V(G)| \geq p + 2$ and thus, since $|N_G(v) \cap L(G)| = p$, $|V(G)| \geq p + 3$. If D_2 is a minimum 2-dominating set of G, then we distinguish two cases.

Case 1. Assume that $v \in D_2$. It follows that $D_2 - \{x_1, x_2, \ldots, x_p\}$ is a 2-dominating set of G', and the hypothesis $\gamma_2(G) = \gamma(G) + 1$ leads to

$$\gamma_2(G') \leq \gamma_2(G) - p = \gamma(G) - p + 1 \leq \gamma(G') - p + 2.$$

In the case $p \geq 3$, we obtain the contradiction $\gamma_2(G') < \gamma(G')$. In the remaining case $p = 2$, Theorem 3.1 implies that G' is the trivial graph, a contradiction to $|V(G)| \geq p + 3$.

Case 2. Assume that $v \notin D_2$. It follows that $D_2 - \{x_1, x_2, \ldots, x_p\}$ is a 2-dominating set of $G' - v$, and we observe that all the components of the block graph $G' - v$ are of order at least 2. The hypothesis $\gamma_2(G) = \gamma(G) + 1$ leads to

$$\gamma_2(G' - v) \leq \gamma_2(G) - p = \gamma(G) - p + 1 \leq \gamma(G' - v) - p + 2.$$

Like above, we obtain the contradiction $\gamma_2(G' - v) < \gamma(G' - v)$ when $p \geq 3$, and if $p = 2$, then Theorem 3.1 implies the contradiction that all the components of $G' - v$ are trivial graphs. ■

Lemma 3.3. *Let G be a connected block graph with $\gamma_2(G) = \gamma(G) + 1$, and let B be an endblock of $G - L(G)$ with a cutvertex s. Then*

(1) *Either $|N_G(v) \cap L(G)| = 1$ for all vertices $v \in V(B - s)$ or $|N_G(v) \cap L(G)| = 0$ for all vertices $v \in V(B - s)$.*

(2) *The block graph* $G' = G - (N_G[V(B-s)] - s)$ *satisfies* $\gamma_2(G') = \gamma(G') + 1$.

(3) *There is at most one endblock* B *in* $G - L(G)$ *with* $|N_G(v) \cap L(G)| = 0$ *for all vertices* $v \in V(B - s)$.

Proof. (1) Assume that there is a vertex $w \in V(B-s)$ such that $|N_G(w) \cap L(G)| \geq 1$. Then Lemma 3.2 implies that $|N_G(w) \cap L(G)| = 1$. If $n(B) = 2$, then we are done. Now let $n(B) \geq 3$ and suppose that there is a vertex $v \in V(B-s)$ such that $|N_G(v) \cap L(G)| = 0$. Let t be the number of vertices in $B-s$ which are adjacent with a leaf in G, and let $G' = G - (N_G[V(B-s)] - s)$. If D_2 is a minimum 2-dominating set of G, then we distinguish two cases.

Case 1. Assume that $s \in D_2$. Then $D_2 \cap V(G')$ is a 2-dominating set of G'. Since $|D_2 \cap (N_G[V(B-s)] - s)| = t + 1$, it follows that

$$\gamma_2(G') \leq \gamma_2(G) - t - 1 = \gamma(G) - t \leq \gamma(G'),$$

a contradiction to Theorem 3.1.

Case 2. Assume that $s \notin D_2$. It follows that $D_2 \cap V(G' - s)$ is a 2-dominating set of $G' - s$. Since $|D_2 \cap N_G[V(B-s)]| \geq t + 1$, it follows that

$$\gamma_2(G' - s) \leq \gamma_2(G) - t - 1 = \gamma(G) - t \leq \gamma(G' - s).$$

In view of Theorem 3.1, we deduce that the components of $G' - s$ are trivial graphs. However, this is a contradiction to the fact that s is a cutvertex of $G - L(G)$.

(2) In the case that $|N_G(v) \cap L(G)| = 0$ for all vertices $v \in V(B - s)$, it follows that $n(B) \geq 3$ and hence

$$\gamma_2(G') \leq \gamma_2(G) - 1 = \gamma(G) \leq \gamma(G') + 1.$$

Now Theorem 3.1 yields the identity $\gamma_2(G') = \gamma(G') + 1$. In the remaining case that $|N_G(v) \cap L(G)| = 1$ for all vertices $v \in V(B - s)$, we obtain

$$\gamma_2(G') \leq \gamma_2(G) - (n(B) - 1) = \gamma(G) - n(B) + 2 \leq \gamma(G') + 1.$$

Again Theorem 3.1 leads to the desired result.

(3) Suppose that there are two endblocks B_1 and B_2 in $G - L(G)$ with $N_G(v) \cap L(G) = \emptyset$ for all vertices $v \in V(B_i - s_i)$, where $s_i \in V(B_i)$ is the

cutvertex of $G - L(G)$ for $i = 1, 2$. It follows that $n(B_i) \geq 3$ for $i = 1, 2$. Now let $G'' = G - (V(B_1 - s_1) \cup V(B_2 - s_2))$, and let D_2 be a minimum 2-dominating set of G. We can assume, without loss of generality, that $s_1, s_2 \in D_2$. Then $D_2 \cap V(G'')$ is a 2-dominating set of G'' and so a dominating set of G''. Because of $s_1, s_2 \in (D_2 \cap V(G''))$, we observe that $D_2 \cap V(G'')$ is also a dominating set of G. The property $|D_2 \cap V(B_i)| \geq 2$ for $i = 1, 2$ leads to

$$\gamma_2(G) = |D_2| = |D_2 \cap V(G'')| + 2 \geq \gamma(G) + 2.$$

This is a contradiction to our hypothesis $\gamma_2(G) = \gamma(G) + 1$, and the proof is complete. ■

Corollary 3.4. *Let G be a connected block graph with $\gamma_2(G) = \gamma(G) + 1$. If we extract, like in Lemma* 3.3 (2), *the vertex set $N_G[V(B - s)] - s$ from G for every endblock B of $G - L(G)$ with cutvertex s and $|N_G(v) \cap L(G)| = 1$ for all $v \in V(B - s)$, and if we repeat this process again and again until there is no more such endblock, then the remaining block graph G_0 is isomorphic to K_p, to $K_p \circ K_1$ or to $(K_p \circ K_1) - w$ for a vertex $w \in L(K_p \circ K_1)$, where $p \geq 1$ is an integer.*

Proof. It follows from Lemma 3.3 (2) and (3) that $\gamma_2(G_0) = \gamma(G_0) + 1$ and $G_0 - L(G_0) = K_p$ for some integer $p \geq 1$. Now it is easy to see that G_0 is isomorphic to K_p, to $K_p \circ K_1$ or to $(K_p \circ K_1) - w$ for a vertex $w \in L(K_p \circ K_1)$. ■

Theorem 3.5. *Let G be a nontrivial connected block graph. Then G satisfies $\gamma_2(G) = \gamma(G) + 1$ if and only if*

(a) *$G = H \circ K_1$, where H is a connected block graph with at most one cutvertex.*

(b) *$G = (H \circ K_1) - w$, where H is either a connected block graph with exactly one cutvertex s and w is the leaf adjacent to s in $H \circ K_1$ or it is isomorphic to K_p for an integer $p \geq 2$ and w is an arbitrary leaf of $H \circ K_1$.*

(c) *$G = (H_1 \circ K_1) \cup (H_2 \circ K_1)$, where H_1 and H_2 are connected block graphs with at most one cutvertex such that there is a vertex $v \in V(G)$ with*

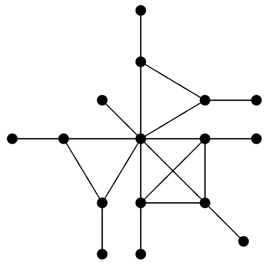
$$V(H_1 \circ K_1) \cap V(H_2 \circ K_1) = \{v\} = N_{H_i \circ K_1}(s_i) \cap L(H_i \circ K_1),$$

where s_i is the cutvertex of H_i or, if does not exist, some vertex in $V(H_i)$ for $i = 1, 2$.

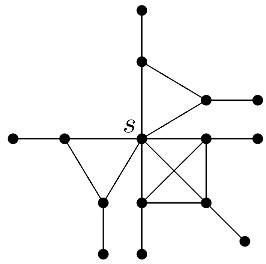
(d) *G consists of a block B isomorphic to K_p for some $p \geq 3$ and of two graphs $G_1 = (H_1 \circ K_1) - w_1$ and $G_2 = (H_2 \circ K_1) - w_2$ of the form as in* (b), *where $N_{H_i \circ K_1}(w_i) = \{s_i\} = V(G_i) \cap V(B)$ for $i = 1, 2$ and $s_1 \neq s_2$* (G_1 *and* G_2 *can also be trivial*).

In order to illustrate the different types of block graphs of this theorem, we want to give some example graphs for each case (a)–(d).

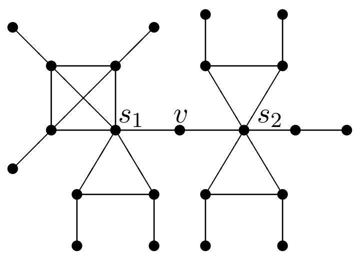
(a)



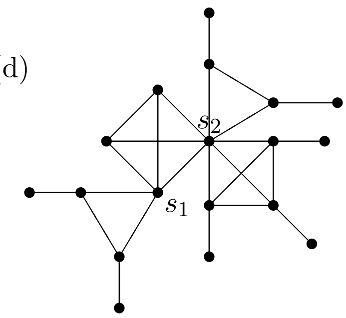
(b)



(c)



(d)



Proof. It is straightforward to verify that the graphs of the families (a)–(d) satisfy the identity $\gamma_2(G) = \gamma(G) + 1$.

Conversely, assume that G is a nontrivial connected block graph such that $\gamma_2(G) = \gamma(G) + 1$. Let G_0 be one of the graphs resulting from the reducing process described in Corollary 3.4. Assume that $G - L(G)$ has an end block B with cutvertex s such that $N_G(V(B - s)) \cap L(G) \neq \emptyset$. It follows from Lemma 3.3(1) that $|N_G(v) \cap L(G)| = 1$ for every $v \in V(B - s)$. If U is a minimum covering of G, then we can assume, without loss of generality, that $V(B - s)$ is contained in U. Hence $U - V(B - s)$ is a covering of $G' = G - (N_G[V(B - s)] - s)$, and it is easy to see that $U - V(B - s)$ is even a minimum covering of G'. Thus $\beta(G') = \beta(G) - n(B - s)$ and the order of

G and G' are of the same parity. The condition $|N_G(v) \cap L(G)| = 1$ for all vertices $v \in V(B-s)$ leads to

$$\gamma_2(G') \leq \gamma_2(G) - (n(B)-1) = \gamma(G) - n(B) + 2 \leq \gamma(G') + 1.$$

Applying the identity $\gamma_2(G') = \gamma(G')+1$ in Lemma 3.3(2), we conclude that $\gamma(G') = \gamma(G) - n(B-s)$. If we continue this process we finally arrive at $\beta(G_0) = \beta(G) - k$ and $\gamma(G_0) = \gamma(G) - k$ for an integer $k \geq 0$.

Case 1. Assume that G_0 is isomorphic to $K_p \circ K_1$ or to $(K_p \circ K_1) - w$, where w is a leaf of $K_p \circ K_1$. Because of $\gamma(G_0) = \beta(G_0)$, we conclude that $\gamma(G) = \beta(G)$. Applying Theorem 2.2, we obtain $\gamma(G)+1 = \gamma_2(G) \geq \alpha(G) \geq \gamma(G)$ and therefore $\alpha(G) = \gamma(G)$ or $\alpha(G) = \gamma(G)+1$. This implies together with Theorem 2.1 that $\gamma(G) = \lfloor n(G)/2 \rfloor$.

Subcase 1.1. Assume that G_0 is isomorphic to $K_p \circ K_1$. Since G and G_0 are of the same parity, it follows from Theorem 2.5 that $G = H \circ K_1$, where H is a connected block graph. If H has more than one cutvertex, then we observe that $\gamma_2(G) \geq |L(G)| + 2$, a contradiction to the hypothesis $\gamma_2(G) = \gamma(G) + 1 = |L(G)| + 1$. Thus G is of the structure described in (a).

Subcase 1.2. Assume that G_0 is isomorphic to $(K_p \circ K_1) - w$, where w is a leaf of $K_p \circ K_1$. Then G is of odd order, and one of the cases (1)–(3) of Theorem 2.5 has to be satisfied.

Case (1) in Theorem 2.5 is only possible when $G = K_{1,2} = (K_2 \circ K_1) - w$.

Case (2) in Theorem 2.5 shows that G is of the form $(H \circ K_1) - w$ for a connected block graph H with, as in the proof of Subcase 1.1, at most one cutvertex. If H is a block, then we are done. It remains the case that H has a cutvertex s. If there is a vertex $v \neq s$ in H with $N_G(v) \cap L(G) = \emptyset$, then we arrive at the contradiction $\gamma_2(G) > \gamma(G) + 1$. This shows that G has structure described in (b).

In Case (3) of Theorem 2.5 let $G - N_G[L(G)]$ be the star with vertex set a_1, a_2, v and edge set va_1 and va_2. Since a_1 and a_2 are not adjacent, we deduce that $G - v$ consists of exactly two connected block graphs G_1' and G_2' such that $G[V(G_1) \cup \{v\}] = H_1 \circ K_1$ and $G[V(G_2) \cup \{v\}] = H_1 \circ K_1$, where H_1 and H_2 are connected block graphs. As above, it is a simple matter to verify that H_1 as well as H_2 has at most one cutvertex, and hence G has the form described in (c).

Case 2. Assume that $G_0 = K_p$. Assume that there are three different vertices u, v, w in $V(G_0)$ with the property that they also belong to other blocks B_1, B_2 and B_3 of G. Then we can reduce G, as in Corollary 3.4, to a graph G'' that consists of G_0, the blocks B_1, B_2, B_3 together with the individual leaves to every vertex in $V(B_1 \cup B_2 \cup B_3) - \{u, v, w\}$. It is evident that $\gamma_2(G'') = |L(G'')| + 3$ and $\gamma(G'') = |L(G'')| + 1$, a contradiction to $\gamma_2(G'') = \gamma(G'') + 1$.

This implies that there are at most two different vertices in G_0 which belong to another block of G. Since the cases $p = 1, 2$ are contained in the cases discussed above, we assume in the following that $p \geq 3$. Assume next that there exists a vertex u in $V(G_0)$ which belongs to another block B_1 of $G - L(G)$ and that there exists a vertex $v \neq u$ in B_1 which belongs to a further block B_2 of $G - L(G)$.

Subcase 2.1. Assume that $n(B_1) \geq 3$ or $n(B_2) \geq 3$. Then we can reduce G, as in Corollary 3.4, to a graph G'' that consists of G_0, the blocks B_1, B_2 together with the individual leaves to every vertex in $V(B_1 \cup B_2) - \{u\}$. It is evident that $\gamma_2(G'') = |L(G'')| + 3$ and $\gamma(G'') = |L(G'')| + 1$, a contradiction to $\gamma_2(G'') = \gamma(G'') + 1$.

Subcase 2.2. Assume that $V(B_1) = \{u, v\}$ and $V(B_2) = \{v, w\}$. Since B_2 is no endblock in G, there exists a block B_3 in G such that $w \in V(B_3)$. If $n(B_3) = 2$, then let $V(B_3) = \{w, x\}$. In this case we can reduce G to a graph G'' that consists of G_0, the blocks B_1, B_2, B_3 and with either a leaf to the vertex v or to the vertex x. Next assume that $n(B_3) \geq 3$ and that there is no other block B' with $w \in V(B')$ and $n(B') = 2$. Then we can reduce G to a graph G'' that consists of G_0 and the blocks B_1, B_2, B_3 together with the individual leaves to every vertex in $V(B_3 - w)$. Both cases lead to the contradiction $\gamma_2(G'') = |L(G'')| + 3$ and $\gamma(G'') = |L(G'')| + 1$.

In the remaining cases, the block graph G is of the structure described in (d), and the proof is complete. ■

References

[1] M. Blidia, M. Chellali and L. Volkmann, *Bounds of the* 2*-domination number of graphs*, Utilitas Math. **71** (2006) 209–216.

[2] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, in: Graph Theory with Applications to Algorithms and Computer Science (John Wiley and Sons, New York, 1985), 282–300.

[3] J.F. Fink and M.S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, in: Graph Theory with Applications to Algorithms and Computer Science (John Wiley and Sons, New York, 1985), 301–311.

[4] J.F. Fink, M.S. Jacobson, L.F. Kinch and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985) 287–293.

[5] T. Gallai, *Über extreme Punkt-und Kantenmengen*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **2** (1959) 133–138.

[6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[7] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater (eds.), Domination in Graphs: Advanced Topics (Marcel Dekker, New York, 1998).

[8] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982) 23–32.

[9] B. Randerath and L. Volkmann, *Characterization of graphs with equal domination and covering number*, Discrete Math. **191** (1998) 159–169.

[10] J. Topp and L. Volkmann, *On domination and independence numbers of graphs*, Results Math. **17** (1990) 333–341.

[11] L. Volkmann, Foundations of Graph Theory (Springer, Wien, New York, 1996) (in German).

[12] L. Volkmann, *Some remarks on lower bounds on the p-domination number in trees*, J. Combin. Math. Combin. Comput., to appear.

[13] B. Xu, E.J. Cockayne, T.W. Haynes, S.T. Hedetniemi and S. Zhou, *Extremal graphs for inequalities involving domination parameters*, Discrete Math. **216** (2000) 1–10.

Received 7 December 2005
Revised 18 October 2006