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A NOTE ON PACKING OF TWO COPIES OF A HYPERGRAPH

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Abstract

A 2-*packing* of a hypergraph $\mathcal{H}$ is a permutation σ on $V(\mathcal{H})$ such that if an edge e belongs to $\mathcal{E}(\mathcal{H})$, then $\sigma(e)$ does not belong to $\mathcal{E}(\mathcal{H})$.

We prove that a hypergraph which does not contain neither empty edge $\emptyset$ nor complete edge $V(\mathcal{H})$ and has at most $\frac{1}{2}n$ edges is 2-packable.

A 1-uniform hypergraph of order n with more than $\frac{1}{2}n$ edges shows that this result cannot be improved by increasing the size of $\mathcal{H}$.

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1. Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where V is the *vertex set* and $\mathcal{E} \subset 2^V$ is the *edge set*. We shall assume that V and $\mathcal{E}$ are non-empty but allow in general empty edges for technical reasons. So, a complete simple hypergraph of order n has 2^n edges. We consider only finite hypergraphs.
An edge $e \in \mathcal{E}$ is called a *singleton* if $|e| = 1$. A vertex is *isolated* if no edge contains it. The number $d(v)$ of edges containing a vertex v is called the *degree of* $v \in V$. A hypergraph is *t-uniform* if $|e| = t$ for all $e \in \mathcal{E}$.

Let $\mathcal{H}$ be a hypergraph of order n. A *packing* of two copies of $\mathcal{H}$ (2-*packing* of $\mathcal{H}$) is a permutation σ on $V(\mathcal{H})$ such that if an edge $e = \{x_1, \ldots, x_k\}$ belongs to $\mathcal{E}(\mathcal{H})$, then the edge $\sigma(e) = \{\sigma(x_1), \ldots, \sigma(x_k)\}$ does not belong to $\mathcal{E}(\mathcal{H})$. Such a permutation (a *packing permutation*) is called also an *embedding* of $\mathcal{H}$ into its complement.

Let us consider a hypergraph $\mathcal{H}$ and a permutation σ on V. We have $\sigma(V) = V$ and $\sigma(\emptyset) = \emptyset$. So, if $V \in \mathcal{E}$ or $\emptyset \in \mathcal{E}$ then $\mathcal{H}$ cannot be packable. The hypergraph $\mathcal{H}$ such that neither $\emptyset \in \mathcal{E}(\mathcal{H})$ nor $V \in \mathcal{E}(\mathcal{H})$ is called *admissible*. We consider only admissible hypergraphs.

Let $\mathcal{H}$ be an admissible hypergraph. Let us consider a hypergraph $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ with the same vertex set V and the edge set, $\tilde{\mathcal{E}}$ obtained from $\mathcal{E}$ in the following way: if $e \in \mathcal{E}$ has at most $\frac{n}{2}$ vertices then e belongs to $\tilde{\mathcal{E}}$ and if e has more than $\frac{n}{2}$ vertices, then e is replaced by $V \setminus e$ with convention that each double edge eventually created in this way is replaced by a single one.

Remark. Let $\mathcal{H}$ be an admissible hypergraph of order n and size at most $\frac{1}{2}n$. Let us observe that a hypergraph $\tilde{\mathcal{H}}$ is 2-packable iff the hypergraph $\mathcal{H}$ is 2-packable.

Therefore, we will consider 2-packing of $\tilde{\mathcal{H}}$ in the proof of Theorem 2, although we shall write $\mathcal{H}$.

A 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H.P. Yap ([2, 7]) or survey papers by H.P. Yap or M. Woźniak ([8, 5, 6] and [4])). One of the first results in this area was the following theorem (see [3]).

Theorem 1. *A graph G of order n and size at most $n - 2$ is 2-packable.*

2. Main Result

The aim of this note is to prove the following theorem.

Theorem 2. *An admissible hypergraph of order n and size at most $\frac{1}{2}n$ is 2-packable.*

First, let us observe that this bound is best possible. Namely, if $\mathcal{H}$ is a hypergraph of order n and has more than $\frac{1}{2}n$ edges and each edge is a singleton, then evidently $\mathcal{H}$ is not packable.

***Proof of Theorem* 2.** It is easy to see that the theorem is true for $n = 2$ and $n = 3$. So, let $n \geq 4$.

By Remark in the previous section, we may consider only hypergraphs which have only edges of cardinality at most $\frac{n}{2}$. Let $\mathcal{H}$ be an admissible hypergraph. Denote by m_k the number of edges of cardinality k and let m be the size of $\mathcal{H}$. Thus

$$\frac{n}{2} \geq m = m_1 + m_2 + \ldots + m_{\lfloor \frac{n}{2} \rfloor}.$$

The proof will be divided into two parts.

Case 1. $m_1 = 0$

First, by using a 'probabilistic' argument we shall show that the packing permutation exists if $\mathcal{H}$ has no singleton.

Let e and f be two edges of $\mathcal{H}$ of the same cardinality and let σ be a random permutation on V. We say that an edge e *covers* an edge f (with respect to σ), if $\sigma(e) = f$. We write: $(e \curvearrowright f)$.

Let e and f be two edges of cardinality k. The probability of the event A that e covers f (denoted by $A(e \curvearrowright f))$) is equal to

$$Pr(A(e \curvearrowright f)) = \frac{k!(n-k)!}{n!} = \binom{n}{k}^{-1}.$$

Let us observe, that the number of events that an edge of cardinality k cover some edge in $\mathcal{H}$ of cardinality k is equal to m_k^2.

So, we have

$$Pr\left(\bigcup_{e,f\in\mathcal{H}} A(e \curvearrowright f)\right) \leq \sum_{e,f\in\mathcal{H}} Pr\left(A(e \curvearrowright f)\right)$$

$$= m_2^2\binom{n}{2}^{-1} + m_3^2\binom{n}{3}^{-1} + \ldots + m_{\lfloor\frac{n}{2}\rfloor}^2\binom{n}{\lfloor\frac{n}{2}\rfloor}^{-1}.$$

Since $k \leq \frac{n}{2}$, the sequence $\left(\binom{n}{2}^{-1}, \binom{n}{3}^{-1}, \ldots\right)$ is decreasing and we have

$$m_2^2\binom{n}{2}^{-1} + m_3^2\binom{n}{3}^{-1} + \ldots + m_{\lfloor\frac{n}{2}\rfloor}^2\binom{n}{\lfloor\frac{n}{2}\rfloor}^{-1}$$

$$\leq \binom{n}{2}^{-1}\left(m_2^2 + m_3^2 + \ldots + m_{\lfloor\frac{n}{2}\rfloor}^2\right)$$

$$\leq \binom{n}{2}^{-1}\left(\frac{n}{2}\right)^2 = \frac{n}{2(n-1)}.$$

It is easy to see that $\frac{n}{2(n-1)} < 1$ for $n > 2$. In consequence, there exists a packing of an admissible hypergraph $\mathcal{H}$ of order n and size at most $\frac{1}{2}n$ into its complement, if $\mathcal{H}$ does not have any singletons.

Case 2. $m_1 \geq 1$
In this case we use the induction with respect to n. Let $\mathcal{H}$ be an admissible hypergraph of order n and suppose that the theorem holds for $n' < n$.

Let $\{x\}$ be a singleton and let y be a vertex of $\mathcal{H}$ such that $\{y\} \notin \mathcal{E}$ and $\{x, y\} \notin \mathcal{E}$. Such a y exists. For, otherwise each vertex other than x would be either a singleton or the end of an edge joining it with x, and we would get a contradiction with the size of $\mathcal{H}$.

Now, we construct a hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ such that $V' = V - \{x, y\}$ and the set of edges is obtained from $\mathcal{E}$ as follows: we delete the edge $\{x\}$ and we replace all edges containig x or y (or x and y) by new edges without these vertices. So $\mathcal{H}'$ has one edge and two vertices less than $\mathcal{H}$. If $m_1 \neq 0$ in $\mathcal{H}'$ then a packing permutation σ' exists by the induction hypothesis. If $m_1 = 0$ in $\mathcal{H}'$ then a packing permutation σ' exists by Case 1.

By the choice of x and y and the property of σ', it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$ where (xy) denotes a transposition, is a packing permutation of $\mathcal{H}$ ∎

3. Some Open Problems

One of the objectives of this paper is to draw attention of the reader to some open problems.

I. It would be interesting to consider an analogous problem for uniform hypergraphs. For $t = 1$ the above result is still best possible. For $t = 2$ the answer is given by Theorem 1. In the case $t = 3$ we suppose (together with E. Győri) that the *right* bound of the size of a packable hypergraph is $k = \frac{1}{6}(n-2)(n+3)$. It is easy to see that this bound cannot be improved.

II. Does Theorem 2 remain true if instead of packing of two copies of the same hypergraph of order n we pack two distinct hypergraphs of order n, both of size at most $\frac{1}{2}n$?

III. A. Benhocine and A.P. Wojda in [1] proved that a graph of order n and size at most $n-1$ is 2-packable if and only if G is embeddable into a self complementary graph of the same order (for $n \equiv 0, 1 \pmod 4$). It would be interesting to get an analogous result for hypergraphs.

References

[1] A. Benhocine and A.P. Wojda, *On self-complementation*, J. Graph Theory **8** (1985) 335–341.

[2] B. Bollobás, Extremal Graph Theory (Academic Press, London, 1978).

[3] D. Burns and S. Schuster, *Every* $(n,\ n-2)$ *graph is contained in its complement*, J. Graph Theory **1** (1977) 277–279.

[4] M. Woźniak, *Embedding graphs of small size*, Discrete Applied Math. **51** (1994) 233–241.

[5] M. Woźniak, *Packing of graphs*, Dissertationes Math. **362** (1997) 1–78.

[6] M. Woźniak, *Packing of graphs and permutations — a survey*, Discrete Math. **276** (2004) 379–391.

[7] H.P. Yap, Some Topics in Graph Theory, London Math. Society, Lecture Notes Series, Vol. 108 (Cambridge University Press, Cambridge, 1986).

[8] H.P. Yap, *Packing of graphs — a survey*, Discrete Math. **72** (1988) 395–404.

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