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NEW SUFFICIENT CONDITIONS FOR HAMILTONIAN AND PANCYCLIC GRAPHS*

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Abstract

For a graph G of order n we consider the unique partition of its vertex set $V(G) = A \cup B$ with $A = \{v \in V(G) : d(v) \geq n/2\}$ and $B = \{v \in V(G) : d(v) < n/2\}$. Imposing conditions on the vertices of the set B we obtain new sufficient conditions for hamiltonian and pancyclic graphs.

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1. Introduction

We use [4] for terminology and notation not defined here and consider finite and simple graphs only.

A graph of order n is called *hamiltonian* if it contains a cycle of length n and is called *pancyclic* if it contains cycles of all lengths from 3 to n.

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Let $\omega(G)$ denote the number of components of a graph G. A graph G is called 1-tough if, for every nonempty proper subset S of $V(G)$, we have $\omega(G - S) \leq |S|$.

Various sufficient conditions for a graph to be hamiltonian have been given in terms of vertex degrees. Recall some of them.

Theorem 1 (Dirac [5])**.** *Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is hamiltonian.*

Theorem 2 (Ore [9])**.** *Let G be a graph of order $n \geq 3$. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then G is hamiltonian.*

In [7] Theorem 2 was extended as follows.

Theorem 3 (Flandrin, Li, Marczyk, Woźniak [7])**.** *Let $G = (V, E)$ be a 2-connected graph on n vertices with minimum degree δ. If $uv \in E(G)$ for every pair of vertices $u, v \in V(G)$ with $d(u) = \delta$ and $d(v) < n/2$, then G is hamiltonian.*

With respect to its vertex degrees, the vertex set of every graph G has a unique partition $V(G) = A \cup B$ with $A = \{v \in V(G) : d(v) \geq n/2\}$ and $B = \{v \in V(G) : d(v) < n/2\}$. In terms of A and B we make the following observations:

- If a graph G satisfies Dirac's condition then $B = \emptyset$.
- If a graph G satisfies Ore's condition, then $G[B]$ is complete and $|B| \leq \delta + 1$.
- If a graph G satisfies the condition of Theorem 3, then $G[B]$ is connected, $G[u \in B : d(u) = \delta]$ is complete and $|B| \leq \delta + 1$.

2. Results

We define three classes of graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ as follows.

Let $\mathcal{G}_1$ be the class of all 2-connected graphs G such that $uv \in E(G)$ for every pair of vertices $u, v \in B$ with $d(u) = \delta(G)$.

Let $\mathcal{G}_2$ be the class of all 2-connected graphs G such that there exists a vertex $u \in B$ with $d(u) = \delta(G)$ and $uv \in E(G)$ for all vertices $v \in B - \{u\}$.

Let $\mathcal{G}_3$ be the class of all 2-connected graphs G such that $|B| \leq \delta(G) + 1$ and $\Delta(G[B]) \geq \min\{2, |B| - 1\}$.

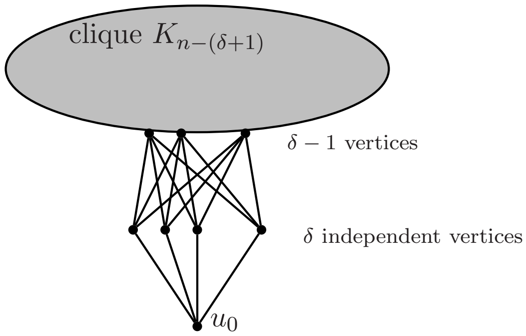

Figure 1. Graph $\mathcal{F}_{n,\delta}$

For all n, δ with $2 \le \delta \le \frac{n-1}{2}$ define $\mathcal{F}_{n,\delta}$ as a graph of order n, minimum degree δ and vertex set $V(\mathcal{F}_{n,\delta}) = \{u_0, u_1, \ldots, u_\delta, w_1, \ldots, w_{n-\delta-1}\}$ such that $d(u_0) = \delta$, $N(u_0) = \{u_1, \ldots, u_\delta\}$, vertices $u_1, \ldots, u_\delta$ are independent, vertices $w_1, \ldots, w_{n-\delta-1}$ induce a clique and $u_i w_j \in E(G)$ for all $1 \le i \le \delta$ and $1 \le j \le \delta - 1$. Now, for $S = \{u_0, w_1, \ldots, w_{\delta-1}\}$ we have

$$\omega(\mathcal{F}_{n,\delta} - S) = \delta + 1 > \delta = |S|.$$

Hence, $\mathcal{F}_{n,\delta}$ is not 1-tough and therefore not hamiltonian.

For all n, δ with $2 \le \delta \le \frac{n-1}{2}$ define $\mathcal{H}_{n,\delta}$ as a supergraph of $\mathcal{F}_{n,\delta}$ such that $V(\mathcal{H}_{n,\delta}) = V(\mathcal{F}_{n,\delta})$ and $E(\mathcal{H}_{n,\delta}) = E(\mathcal{F}_{n,\delta}) \cup \{u_0 w_i : 1 \le i \le n-\delta-1\}$.

Hence, $\mathcal{H}_{n,\delta}$ is not 1-tough and therefore not hamiltonian, too.

Theorem 3 can be now restated as follows.

Theorem 3 (restated)**.**
If $G \in \mathcal{G}_1$, then G is hamiltonian. ■

Using closure operations we obtain the following extension of Theorem 3.

Theorem 4. *If $G \in \mathcal{G}_2$, then G is hamiltonian or $G \subset \mathcal{F}_{n,\delta}$.*

The proof of the above theorem is given in Section 3. It provides a further extension which can be formulated as follows.

Theorem 5. *If $G \in \mathcal{G}_3$, then G is hamiltonian or $G \subset \mathcal{H}_{n,\delta}$.*

Since both $\mathcal{F}_{n,\delta}$ and $\mathcal{H}_{n,\delta}$ are not 1-tough, we obtain the following corollary.

Corollary 6. *If* $G \in \mathcal{G}_3$ *is* 1*-tough, then* G *is hamiltonian.*

Bondy suggested the interesting "meta-conjecture" in [2] that almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs). He proved the following result concerning Ore's condition.

Theorem 7 ([2])**.** *Let* G *be a graph of order* $n \geq 3$. *If* $d(u) + d(v) \geq n$ *for every pair of nonadjacent vertices* $u, v \in V(G)$, *then* G *is pancyclic or isomorphic to the complete bipartite graph* $K_{\frac{n}{2},\frac{n}{2}}$.

In [7] it was shown that Theorem 7 can be extended as follows.

Theorem 8. *If* $G \in \mathcal{G}_1$, *then* G *is pancyclic or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$. ■

Theorem 3 extends the following result of Jin, Liu and Wang [8].

Corollary 9 ([8])**.** *Let* G *be a* 2*-connected graph of order* $n \geq 3$. *If* $d(u) + d(v) \geq n + \delta$ *for every pair of nonadjacent vertices* $u, v \in V(G)$, *then* G *is pancyclic or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$. ■

Concerning pancyclicity we will prove the following theorems.

Theorem 10. *If* $G \in \mathcal{G}_2$, *then* G *is pancyclic or* $G \cong K_{\frac{n}{2},\frac{n}{2}}$ *or* $G \subset \mathcal{F}_{n,\delta}$.

Theorem 11. *If* $G \in \mathcal{G}_3$, *then* G *is pancyclic or bipartite or* $G \subset \mathcal{H}_{n,\delta}$.

3. Proofs

3.1. Hamiltonicity

The closure concept of Bondy and Chvátal [3] is based on the following result of Ore [9].

Theorem 12 (Ore [9])**.** *Let* G *be a graph on* n *vertices such that the edge* $e = uv$ *does not belong to* $E(G)$ *and* $d(u) + d(v) \geq n$. *Then, the graph* G *is hamiltonian if and only if the graph* $G + e$ *is hamiltonian.*

By successively joining pairs of nonadjacent vertices having degree sum at least n as long as this is possible (in the new graph(s)), the unique so called n-closure $cl_n(G)$ is obtained. Using Theorem 12 it is easy to prove the following result.

Theorem 13 (Bondy and Chvátal [3]). *Let G be a graph of order $n \geq 3$. Then G is hamiltonian if and only if $cl_n(G)$ is hamiltonian.*

Corollary 14 (Bondy and Chvátal [3]). *Let G be a graph of order $n \geq 3$. If $cl_n(G)$ is complete ($cl_n(G) = K_n$), then G is hamiltonian.*

Ainouche and Christofides [1] established the following generalization of Theorem 12.

Theorem 15 (Ainouche and Christofides [1]). *Let G be a 2-connected graph on n vertices such that the edge $e = uv$ does not belong to $E(G)$. Let $T = T(u, v) = \{w \in V(G) \setminus (N[u] \cup N[v])\}$ and let $t = |T|$. Suppose that*

$$(*) \qquad d(w) \geq t + 2 \quad \text{for all vertices of } T.$$

Then, the graph G is hamiltonian if and only if the graph $G + e$ is hamiltonian.

In [1] the corresponding (unique) closure of G is called the 0-dual closure $cl^*(G)$. Since Theorem 15 is more general than Theorem 12 (cf. [1]), $G \subseteq cl_n(G) \subseteq cl^*(G)$. The counterpart of Corollary 14 is

Corollary 16 (Ainouche and Christofides [1]). *Let G be a 2-connected graph of order n. If $cl^*(G)$ is complete ($cl^*(G) = K_n$), then G is hamiltonian.*

***Proof of Theorem* 4.** Observe first that if $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian by Dirac's theorem. So, assume that $B \neq \emptyset$.

Step 0. Applying the Bondy-Chvátal closure to the set A we get the graph G_0 with the set A complete.

Step 1. By using cl^* we are able to add to G_0 all edges connecting the vertex u with the set A. Indeed, it suffices to verify the hypothesis of Theorem 15. Suppose there exists a vertex $x \in A$ such that $ux \notin E$. Since u is adjacent to all vertices of B and x is adjacent (in G_0) to all vertices of A, we have $T_{G_0}(u, x) = \emptyset$. Denote the graph obtained in this step by G_1.

Step 2. Let $x \in B, x \neq u$. We put $a(x) = a_G(x) = |N_G(x) \cap A|$. Denote by B' the vertices of B with $a(x) < \delta - 1$. Consider now a vertex $x \in B'$ and a vertex $y \in A$ such that $xy \notin E$. Then x has at least one neighbour in B different from u. This implies $|T(x, y)| \leq \delta - 2$. Hence, the condition $(*)$ of Theorem 15 is satisfied. This means that we can add all edges between B' and A. We denote the graph obtained in this step by G_2.

Step 3. Denote by B_1 the vertices of B different from u that are joined to all vertices of A in G_2. Note that $B' \subset B_1$. We put $B_2 = B - (B_1 \cup \{u\})$. Let $\xi = |B_1|$ and $\eta = |B_2|$. We have $1 + \xi + \eta = |B| \leq \delta + 1$. Consider now a vertex $x \in B_2$ and $y \in A$ such that $xy \notin E$. By Step 2, $a_{G_2}(x) \geq \delta - 1$. Since the vertices of A as well as the vertices of B_1 and the vertex u are the neighbours of y we get $|T(x, y)| \leq \eta - 1$. So, $t + 2 \leq \eta + 1$. If the condition $(*)$ of Theorem 15 is not fulfilled then $\delta \leq \eta$. This implies in particular that $\xi = 0$ and $\delta = \eta$. Moreover, x has no neighbour in B other than u, for otherwise $|T(x, y)| \leq \eta - 2$ and $(*)$ would be satisfied. Observe that either

(a) the above statements concern all vertices of B_2 (see Step 5), or

(b) we can add all edges between B_2 and A.

In the later case we can continue the closure operation (see Step 4 below).

Step 4. Denote by G_3 the graph obtained in Step 3b. Let x, y be two vertices of B such that $xy \notin E$. Then at most $\delta - 2$ vertices of B belong to $T(x, y)$ and we can finish the closure operation with the conclusion that $\mathrm{cl}^*(G) = K_n$.

Step 5. Suppose now that no edge can be added in Step 3. Then B consists of the vertex u and its δ neighbours, say $u_1, u_2, \ldots, u_\delta$, forming an independent set. This implies that each of the vertices $u_1, u_2, \ldots, u_\delta$ sends at least $\delta - 1$ edges to A. Suppose, that there exists a vertex $x \in A$ such that $u_i x \notin E$ and $u_j x \in E$ for some $j \neq i$.. Then $|T(u_i, x)| \leq \delta - 2$ and the edge $u_i x$ can be added to G_2. Denote by G_5 the graph obtained from G_2 by adding all edges as above. We may conclude that in G_5 all vertices $u_1, u_2, \ldots, u_\delta$ have the same neighbourhood. It is now easy to see that only in the case where this neighbourhood contains exactly $\delta - 1$ vertices of A the graph G_5 is not hamiltonian. Observe that in this case $G_5 \subset \mathcal{F}_{n,\delta}$. ■

***Proof of Theorem* 5.** As in the previous proof observe first that if $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian by Dirac's theorem. So, assume that $B \neq \emptyset$. Applying the Bondy-Chvátal closure to the set A we get the graph G_0 with the set A complete. It is easy to verify the hamiltonicity of the graph G_0 if $|B| \leq 2$. So, suppose $|B| \geq 3$. Then $\Delta(G[B]) \geq 2$. Let u be a vertex of B having the maximum number of neighbours in B. Consider a vertex $x \in A$ with $ux \notin E(G_0)$. Then $|T(u,x)| \leq \delta - 2$. This implies that the operation cl* can be applied. That means we can add all edges between u and A. Denote the graph obtained in this way by G_1.

Suppose now that there exists a vertex $x \in B$ such that $ux \notin E(G_0)$. It is easy to see that $|T(u,x)| \leq \delta - 3$. Therefore, we can add all edges between u and other vertices of B. It suffices now to observe that the graph obtained in this way has the same properties as the graph G_1 in the proof of the previous theorem. Now we can follow that proof. ■

In terms of the 0-dual closure Theorem 5 can be restated as follows.

Theorem 5 (restated).
If $G \in \mathcal{G}_3$, then G is hamiltonian or cl$^*(G) = (\overline{K}_\delta \cup K_{n-2\delta}) + K_\delta$.

3.2. Pancyclicity

For the proof of Theorem 11 we will apply the following three theorems.

Theorem 17 (Faudree, Häggkvist, Schelp [6]). *Every hamiltonian graph of order n and size $e(G) > \frac{(n-1)^2}{4} + 1$ is pancyclic or bipartite.* ■

Lemma 18 (Bondy [2]). *Let G be a hamiltonian graph of order n with a Hamilton cycle $v_1 v_2 \dots v_n v_1$ such that $d(v_1) + d(v_n) \geq n + 1$. Then G is pancyclic.*

Theorem 19 (Schmeichel-Hakimi [10]). *If G is a hamiltonian graph of order $n \geq 3$ with a Hamilton cycle $v_1 v_2 ... v_n v_1$ such that $d(v_1) + d(v_n) \geq n$, then G is either*

- *pancyclic,*
- *bipartite, or*
- *missing only an $(n-1)$-cycle.*

Moreover, in the last case we have $d(v_{n-2}), d(v_{n-1}), d(v_2), d(v_3) < n/2$.

Remark. Actually, the Schmeichel-Hakimi result gives some more information about the possible adjacency structure near the vertices v_1 and v_n, but the above version is sufficient for our proof.

***Proof of Theorem* 11.** If $\delta \geq n/2$, then G is pancyclic or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ by Theorem 7. Hence we may assume that $2 \leq \delta \leq \frac{n-1}{2}$. If $G \in \mathcal{H}_{n,\delta}$, then G is not hamiltonian and thus not pancyclic. Hence we may further assume that G is hamiltonian.

If $\delta = \frac{n-1}{2}$, then

$$e(G) \geq \frac{1}{2}\left(\frac{n+1}{2} \cdot \frac{n-1}{2} + \frac{n-1}{2} \cdot \frac{n+1}{2}\right) = \frac{n^2-1}{4} > \frac{(n-1)^2}{4} + 1$$

for all $n \geq 5$. Thus G is pancyclic or bipartite by Theorem 17 (since n is odd, G cannot be bipartite).

If $\delta = \frac{n-2}{2}$, then

$$e(G) \geq \frac{1}{2}\left(\frac{n}{2} \cdot \frac{n-2}{2} + \frac{n}{2} \cdot \frac{n}{2}\right) = \frac{n^2-n}{4} > \frac{(n-1)^2}{4} + 1$$

for all $n \geq 6$. Thus G is pancyclic or bipartite by Theorem 17. In the later case we conclude that $G \cong K_{\frac{n}{2},\frac{n}{2}} - \frac{n}{4}K_2$.

If $2 \leq \delta \leq \frac{n-3}{2}$, then $|A| \geq \frac{n+1}{2} > |B|$, since $|B| \leq \delta + 1 \leq \frac{n-1}{2}$. In this case the third alternative of Theorem 19 cannot occur since a simple counting argument gives $|A| \leq |B|$, a contradiction. Hence G is pancyclic or bipartite by Theorem 19. ■

***Proof of Theorem* 10.** Since $\mathcal{G}_2 \subset \mathcal{G}_3$ we can apply Theorem 11. If $G \subset \mathcal{H}_{n,\delta}$ then we conclude that $G \subset \mathcal{F}_{n,\delta}$, since there is a vertex $u \in B$ with $d(u) = \delta$. Suppose $G \not\subset \mathcal{F}_{n,\delta}$. Then G is hamiltonian by Theorem 4. Thus, if G is bipartite, then G is balanced bipartite with partite sets V_1 and V_2. Suppose $u \in V_1$ for a vertex u with $d(u) = \delta$ and $uv \in E(G)$ for all vertices $v \in B - \{u\}$. Since $\delta < n/2$, there exists a vertex $w \in V_2$ with $w \notin N(u)$. But then $d(w) \leq \frac{n}{2} - 1 < \frac{n}{2}$, a contradiction. Thus G cannot be bipartite. Therefore, by Theorem 11, G is pancyclic. ■

4. Concluding Remarks

Our results presented in Section 2 all imply that $|B| \leq \delta + 1$ for the considered graphs. Thus it is a natural question to study hamiltonicity (and pancyclicity) of graphs with $|B| \leq \delta + k$ for some positive integer $k \geq 2$.

For all n, δ and k with $2 \leq \delta \leq \frac{n-1}{2}$ and $1 \leq k \leq \delta - 1$ define $\mathcal{I}_{n,\delta,k}$ as a graph of order n, minimum degree δ and vertex set

$$V(\mathcal{I}_{n,\delta,k}) = \{u_1, \ldots, u_k, v_1, \ldots, v_\delta, w_1, \ldots, w_{n-\delta-k}\}$$

such that $d(u_i) = n - 1$ for $1 \leq i \leq k$, the vertices $\{v_1, \ldots, v_\delta\}$ are independent, $G[\{w_1, \ldots, w_{n-\delta-k}\}]$ is complete and $v_i w_j \in E(G)$ for all $1 \leq i \leq \delta$ and $1 \leq j \leq \delta - k$.

Now, for $S = \{u_1, \ldots, u_k, w_1, \ldots, w_{\delta-k}\}$ we have

$$\omega(\mathcal{I}_{n,\delta,k} - S) = \delta + 1 > \delta = |S|.$$

Hence, $\mathcal{I}_{n,\delta,k}$ is not 1-tough and thus not hamiltonian. Note that $\mathcal{I}_{n,\delta,1} = \mathcal{H}_{n,\delta}$.

Following the proof of Theorem 5 we have obtained the following theorem.

Theorem 20. *Let G be a 2-connected graph of order n. If for some k with $1 \leq k \leq \delta - 1$*

(i) *$G[B]$ is complete for $|B| \leq k + 1$ or*

(ii) *there are at least k vertices of degree at least $k + 1$ in B for $k + 2 \leq |B| \leq \delta + k$,*

then G is hamiltonian or $\subset \mathcal{I}_{n,\delta,k}$. ■

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