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GLOBAL ALLIANCES AND INDEPENDENCE IN TREES

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Abstract

A global defensive (respectively, offensive) alliance in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ with the properties that every vertex in $V - S$ has at least one neighbor in S, and for each vertex v in S (respectively, in $V - S$) at least half the vertices from the closed neighborhood of v are in S. These alliances are called strong if a strict majority of vertices from the closed neighborhood of v must be in S. For each kind of alliance, the associated parameter is the minimum cardinality of such an alliance. We determine relationships among these four parameters and the vertex independence number for trees.

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1. Introduction

We begin with some terminology. For a vertex v of a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is $|N(v)|$. A vertex of degree one is called a *leaf*, and its neighbor is a *support vertex*. The *boundary* of S is the set $\partial S = N[S] - S$. A set S is a dominating set if $S \cup \partial S = V$.

In [7] Hedetniemi, Hedetniemi, and Kristiansen introduced several types of alliances in graphs, including defensive and offensive alliances, defined as follows. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* (respectively, a *strong defensive alliance*) if for every $v \in S$, $|N[v] \cap S| \geqslant |N[v] - S|$ (respectively, $|N(v) \cap S| \geqslant |N(v) - S|$). Since each vertex in a defensive alliance S has at least as many vertices from its closed neighborhood in S as it has in $V - S$, by strength of numbers, we say that every vertex in S can be *defended* from possible attack by neighboring vertices in $V - S$. A non-empty set of vertices $S \subseteq V$ is called an *offensive alliance* if for every $v \in \partial S$, $|N[v] \cap S| \geqslant |N[v] - S|$. The set S is a *strong offensive alliance* if the inequality is strict. An alliance S is called *global* [4] if it effects every vertex in $V - S$, that is, every vertex in $V - S$ is adjacent to at least one member of the alliance S. In other words, S is both an alliance and a dominating set. The *global defensive alliance number* $\gamma_a(G)$ (respectively, *global strong defensive alliance number* $\gamma_{\hat{a}}(G)$) is the minimum cardinality of a global defensive alliance (respectively, global strong defensive alliance) of G. The entire vertex set is a global (strong) defensive alliance for any graph G, so every graph G has a global (strong) defensive alliance number. Similarly, the *global offensive alliance number* $\gamma_o(G)$ (respectively, *global strong offensive alliance number* $\gamma_{\hat{o}}(G)$) is the minimum cardinality of a global offensive alliance (respectively, global strong offensive alliance) of G, and they exist for every graph G. We abbreviate global defensive alliance as GDA and global strong defensive alliance as GSDA. We will use similar notation for offensive alliances.

Let $\beta_0(G)$ denote the vertex independence number, $i(G)$ denote the independent domination number, and $\gamma(G)$ the domination number of G. Clearly, for any graph G, $\gamma(G) \leqslant \gamma_a(G) \leqslant \gamma_{\hat{a}}(G)$ and $\gamma(G) \leqslant \gamma_o(G) \leqslant \gamma_{\hat{o}}(G)$. For terminology not defined here and a thorough treatment of domination and its variations, see the books [5, 6]. For other graph theory terminology and notation, we generally follow [2].

The following well-known inequality chain [5] relates some basic domination invariants:

$$ir(G) \leqslant \gamma(G) \leqslant i(G) \leqslant \beta_0(G) \leqslant \Gamma(G) \leqslant IR(G). \tag{1}$$

Much research has been focused on when equality is achieved between pairs of parameters in the chain and also on where other parameters “fit” in the chain. In this paper, we consider if and where the four global alliance parameters fit in the inequality chain for trees. We note that for trees T, the upper parameters of the chain are equal, that is, $\beta_0(T) = \Gamma(T) = IR(T)$ [3]. Hence it suffices to consider the relationships between the alliance parameters and independence numbers. We show that both $\gamma_a(T)$ and $\gamma_o(T)$ are bounded above by $\beta_0(T)$. However, we will see that this bound does not hold for the strong versions of the alliance numbers. In fact, we will show in Section 4 that $\gamma_{\hat{o}}(T)$ is bounded below by $\beta_0(T)$. We will show in Section 3 that although $\gamma_{\hat{a}}(T)$ and $\beta_0(T)$ are incomparable, $\gamma_{\hat{a}}(T) \leqslant 3/2(\beta_0(T) - 1)$ and $\gamma_{\hat{a}}(T) \leqslant \beta_0(T) + s - 1$ for every tree of order at least three with s support vertices.

Before presenting our results, we introduce some more terminology. For a generic parameter $\mu(G)$, we call a set satisfying the property for the parameter and having cardinality $\mu(G)$, a $\mu(G)$-set. In particular, a GDA with minimum cardinality $\gamma_a(G)$ is called a $\gamma_a(G)$-set. For a support vertex w, let L_w denote the set of leaves adjacent to w. A *double star* is a tree of order n with exactly two support vertices and $n-2$ leaves. The *corona* of a graph G is the graph formed from a copy of G by attaching for each $v \in V$, a new vertex v' and edge vv'. In general, the *k-corona* of a graph G is the graph of order $k|V(G)|$ obtained from G by adding a path of length k to each vertex of G so that the resulting paths are vertex disjoint.

We will use the following observation.

Observation 1. If T is a tree obtained from a tree T' by adding a star $K_{1,p}$ $(p \geqslant 1)$ of center vertex w and an edge wv for some v of T', then $\beta_0(T) = \beta_0(T') + p$.

2. Global Defensive Alliances

For general graphs, the global defensive alliance number can be much larger than the independence number. For example, the complete graph K_n has

$\beta_0(K_n) = 1 \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor = \gamma_a(K_n)$. However, our first theorem shows that for trees T, $\gamma_a(T)$ is bounded above by the independence number.

Theorem 2. *For any tree T, $\gamma_a(T) \leqslant \beta_0(T)$, and this bound is sharp.*

Proof. We proceed by induction on the order of T. Clearly the result holds for $n = 1, 2$. Let T be a tree of order $n \geqslant 3$, and assume that for every tree T' of order $n' < n$, we have $\gamma_a(T') \leqslant \beta_0(T')$. If T is a star, then $\gamma_a(T) = \lceil n/2 \rceil \leqslant \beta_0(T) = n - 1$, and hence the result is valid. So assume that T is not a star, and let v be a support vertex of T with exactly one nonleaf neighbor, say w. Let $T' = T - (L_v \cup \{v\})$. Since T is not a star, T' has order at least two. We consider two cases.

Case 1. $\deg_T(v) \geqslant 3$, that is, v is adjacent to at least two leaves. Then every $\gamma_a(T')$-set S' can be extended to a GDA of T by adding v and either $\lfloor (|L_v| - 1)/2 \rfloor$ or $\lceil |L_v|/2 \rceil$ leaves depending on whether w is contained in S' or not, respectively. Thus, $\gamma_a(T) \leqslant \gamma_a(T') + \lceil |L_v|/2 \rceil + 1$. Also by Observation 1, $\beta_0(T) = \beta_0(T') + |L_v|$. Now, applying the inductive hypothesis to T', we obtain $\gamma_a(T) \leqslant \gamma_a(T') + \lceil |L_v|/2 \rceil + 1 \leqslant \beta_0(T') + \lceil |L_v|/2 \rceil + 1 \leqslant \beta_0(T) - |L_v| + \lceil |L_v|/2 \rceil + 1$, and therefore $\gamma_a(T) \leqslant \beta_0(T)$.

Case 2. $\deg_T(v) = 2$, that is, v is adjacent to exactly one leaf, say v'. Then every $\gamma_a(T')$-set S' can be extended to a GDA of T by adding v or v' depending on whether $w \in S'$. Thus, $\gamma_a(T) \leqslant \gamma_a(T') + 1$. By applying the inductive hypothesis to T' and using Observation 1, we obtain the inequality.

That this bound is sharp may be seen by considering the tree H_k, formed from a path $P_{2k+1} (k \geqslant 0)$ labelled $1, 2, \ldots, 2k+1$, where for each odd labelled vertex v of the path, a new P_5 is added by identifying its center vertex with v. Then $\gamma_a(H_k) = \beta_0(H_k) = 3(k+1)$. For example, see H_3 in Figure 1. ■

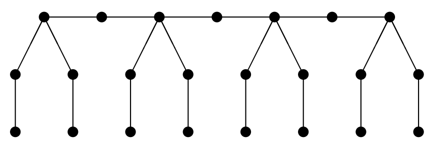

Figure 1. The tree H_3.

The following upper bound on the global alliance number of a tree is given in [4].

Theorem 3 [4]**.** *If T is a tree of order $n \geqslant 4$, then $\gamma_a(T) \leqslant \frac{3n}{5}$.*

Since $\beta_0(T) \leqslant (n+\ell-1)/2$ for every nontrivial tree T with ℓ leaves [1], our next corollary is an improvement on the bound of Theorem 3 for $\ell \leqslant n/5$.

Corollary 4. *For every nontrivial tree T with ℓ leaves, $\gamma_a(T) \leqslant (n+\ell-1)/2$.*

Before concluding this section, we note that $\gamma_a(T)$ and $i(T)$ are incomparable. For a star T of order $n \geq 3$, $1 = i(T) < \lceil n/2 \rceil = \gamma_a(T)$. On the other hand, for the caterpillar T_{6k} with $6k$ support vertices each adjacent to exactly two leaves has $\gamma_a(T_k) = 6k$, while $i(T_k) = 9k$.

3. Global Strong Defensive Alliances

The examples in the previous section also show that $\gamma_{\hat{a}}(T)$ is incomparable to $i(T)$. Next we show that the global strong defensive alliance number and the vertex independence number are also incomparable in trees. In fact, the differences can be arbitrarily large. For example, $\beta_0(T) = p > \lceil p/2 \rceil + 1 = \gamma_{\hat{a}}(T)$ if T is a star $K_{1,p}$ ($p \geqslant 4$) and $\gamma_{\hat{a}}(T) = 4k > 3k = \beta_0(T)$ if T is the 2-corona of a path P_{2k}. However, we establish the following upper bounds on $\gamma_{\hat{a}}(T)$ in terms of $\beta_0(T)$. We will use rooted trees, and let T_v denote the subtree of the rooted tree T induced by v and its descendants.

Theorem 5. *If T is a tree of order $n \geqslant 3$ with s support vertices, then*

(a) $\gamma_{\hat{a}}(T) \leqslant \frac{3\beta_0(T)-1}{2}$,

(b) $\gamma_{\hat{a}}(T) \leqslant \beta_0(T) + s - 1$,

and these bounds are sharp.

Proof. First note that any GSDA of a graph G contains the support vertices of G. We proceed by induction on the order n of T. If $\operatorname{diam}(T) = 2$, then T is a star $K_{1,p}$ ($p \geqslant 2$) where $\gamma_{\hat{a}}(K_{1,p}) = \lceil p/2 \rceil + 1$, $\beta_0(T) = p$, and $s = 1$, so the result is valid. If $\operatorname{diam}(T) = 3$, then T is a double star $S_{p,q}$ where $\gamma_{\hat{a}}(S_{p,q}) = \lfloor p/2 \rfloor + \lfloor q/2 \rfloor + 2$, $\beta_0(T) = p + q$, and $s = 2$.

Assume that for every tree T' of order n' with $n > n' \geqslant 3$ and s' support vertices, we have $2\gamma_{\hat{a}}(T') \leqslant 3\beta_0(T') - 1$ and $\gamma_{\hat{a}}(T') \leqslant \beta_0(T') + s' - 1$. Let T be a tree of order n. If any support vertex, say x, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf say x' adjacent to x. Then every $\gamma_{\hat{a}}(T')$-set can be extended to a GSDA of T by adding the vertex x' and so $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 1$. Also it can be seen that $\beta_0(T) = \beta_0(T') + 1$ and $s' = s$. Applying the inductive hypothesis to T', we obtain $2\gamma_{\hat{a}}(T) \leqslant 2(\gamma_{\hat{a}}(T') + 1) \leqslant 3\beta_0(T') + 1 < 3\beta_0(T) - 1$, and $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 1 \leqslant (\beta_0(T') + s' - 1) + 1 = \beta_0(T) + s - 1$. Thus we assume that every support vertex is adjacent to exactly one leaf.

Recall that $\mathrm{diam}(T) \geqslant 4$, and root T at a vertex r of maximum eccentricity. Let v be a support vertex of maximum distance from r and u the parent of v in the rooted tree. Then v has degree two. Let y be the child of v and consider the following two cases.

Case 1. $\deg_T(u) \geqslant 3$. Then every child of u is either a leaf or a support vertex of degree two. Let $T' = T - T_v$. Since $\mathrm{diam}(T) \geqslant 4$, T' has order at least three. Let S' be a $\gamma_{\hat{a}}(T')$-set. If u is a support vertex, then $u \in S'$. If u is not a support vertex, then either S' contains u and all its children, or S' contains all the descendants of u and not u. The minimality of S' implies that the second case occurs if and only if u has degree two in T'. Then removing the leaf descendant of u from S' and adding u yields a $\gamma_{\hat{a}}(T')$-set containing u. Thus we can assume there is a $\gamma_{\hat{a}}(T')$-set that contains u. Such a set can be extended to a GSDA of T by adding the vertex v, and hence, $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 1$. Clearly we also have $s' = s - 1$. Applying the inductive hypothesis to T' and using Observation 1, we obtain $2\gamma_{\hat{a}}(T) \leqslant 2(\gamma_{\hat{a}}(T') + 1) \leqslant 3\beta_0(T') + 1 = 3\beta_0(T) - 3 + 1 < 3\beta_0(T) - 1$, and $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 1 \leqslant (\beta_0(T') + s' - 1) + 1 < \beta_0(T) + s - 1$.

Case 2. $\deg_T(u) = 2$. Let w be the parent of u in the rooted tree. Based on the previous case, we may assume that every descendent of w has degree at most two.

Assume first that w is a support vertex or there is a path $P_3 = abc$ besides yvu attached to w by a. Let $T' = T - T_u$. We may assume that T' has order at least three else the result holds. Then $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 2$ since every $\gamma_{\hat{a}}(T')$-set can be extended to a GSDA of T by adding to it v and u. On the other hand, it is a routine matter to check that there is a $\beta_0(T')$-set S' that does not contain w, implying that $S' \cup \{u, y\}$ is an independent set of T. Thus, $\beta_0(T) \geqslant \beta_0(T') + 2$. Also $s' \leqslant s$. Applying the inductive hypothesis to T', we have $2\gamma_{\hat{a}}(T) \leqslant 2(\gamma_{\hat{a}}(T') + 2) \leqslant 3\beta_0(T') + 3 < 3\beta_0(T) - 1$, and $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 2 \leqslant (\beta_0(T') + s' - 1) + 2 \leqslant \beta_0(T) + s - 1$.

Assume now that $\deg_T(w) \geqslant 3$ and every path attached to w except the $P_3 = uvy$ is a path P_2, that is, T_w is obtained from a star $K_{1,p}$ with

$p \geqslant 2$ where exactly one edge is subdivided twice and the remaining edges once. Let $T' = T - T_w$. Since w is not a support vertex, T' has order at least two and if T' has order two, then $\gamma_{\hat{a}}(T) = p + 3, \beta_0(T) = p + 2$ and $s = p + 1$, and the result is valid. We assume that T' has order at least three. Then $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + p + 2$, $\beta_0(T) \geqslant \beta_0(T') + p + 1$, and $s - p \leqslant s' \leqslant s - p + 1$. Applying the inductive hypothesis to T', we have $2(\gamma_{\hat{a}}(T) - p - 2) \leqslant 2\gamma_{\hat{a}}(T') \leqslant 3\beta_0(T') - 1 \leqslant 3(\beta_0(T) - p - 1) - 1$. Therefore $2\gamma_{\hat{a}}(T) < 3\beta_0(T) - 1$ since $p \geqslant 2$. Also, $\gamma_{\hat{a}}(T) - p - 2 \leqslant \gamma_{\hat{a}}(T') \leqslant \beta_0(T') + s' - 1 \leqslant (\beta_0(T) - p - 1) + (s - p + 1) - 1$, and hence $\gamma_{\hat{a}}(T) \leqslant \beta_0(T) + s - 1$ since $p \geqslant 2$.

Finally assume that $\deg_T(w) = 2$. Let $T' = T - T_w$. We assume that T' has order at least three for otherwise $T \in \{P_5, P_6\}$ and the result holds. Then every $\gamma_{\hat{a}}(T')$-set can be extended to a GSDA of T by adding the vertices v, u, w, and so $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 3$. We also have $\beta_0(T) \geqslant \beta_0(T') + 2$ and $s - 1 \leqslant s' \leqslant s$. Applying the inductive hypothesis to T', we have $2(\gamma_{\hat{a}}(T) - 3) \leqslant 2\gamma_{\hat{a}}(T') \leqslant 3\beta_0(T') - 1 \leqslant 3(\beta_0(T) - 2) - 1$. Therefore, $2\gamma_{\hat{a}}(T) \leqslant 3\beta_0(T) - 1$.

Moreover, if $s' = s - 1$, then $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 3 \leqslant (\beta_0(T') + s' - 1) + 3 \leqslant \beta_0(T) + s - 1$. Now if $s' = s$, then the parent of w, say w', in the rooted tree has degree two in T and is a leaf in T'. Let S' be any $\gamma_{\hat{a}}(T')$-set. If $w' \notin S'$, then $S' \cup \{v, u\}$ is a GSDA of T. If $w' \in S'$, then since S' also contains the support vertex adjacent to w', $S' \cup \{v, u\}$ is a GSDA of T. In both cases, $\gamma_{\hat{a}}(T) \leqslant \gamma_{\hat{a}}(T') + 2$. Applying the inductive hypothesis to T', we obtain the desired inequality. This achieves the proof.

That both bounds are sharp may be seen by the caterpillar T, where T has $k \geqslant 2$ support vertices each adjacent to exactly one leaf and the distance between every pair of consecutive support vertices is three. Then $\beta_0(T) = 2k - 1$, $\gamma_{\hat{a}}(T) = 3k - 2$, and $s = k$. ■

4. Offensive Alliances

We begin with the following observations.

Observation 6. If G is a graph of order n with no isolated vertices, then $\beta_0(G) + \gamma_o(G) \leqslant n$.

Proof. Let S be a $\beta_0(G)$-set. Then $V - S$ is a global offensive alliance of G and so $\gamma_o(G) \leqslant |V - S|$. ■

The next corollary follows from Theorem 2 and Observation 6.

Corollary 7. *If T is a nontrivial tree, then $\gamma_a(T) + \gamma_o(T) \leqslant n$.*

We note that for any graph G, the leaves of G are contained in every $\gamma_{\hat{o}}(G)$-set.

Theorem 8. *For any tree T, $\gamma_o(T) \leqslant \beta_0(T) \leqslant \gamma_{\hat{o}}(T)$, and these bounds are sharp.*

Proof. The first inequality follows directly from Observation 6 and the fact that $\beta_0(T) \geq n/2$ for trees. To prove the upper bound on $\beta_0(T)$, we proceed by induction on the order n of T. Clearly, the result holds for $n = 1$. For $n = 2$, $\beta_0(T) < 2 = \gamma_{\hat{o}}(T)$. Let $n \geqslant 3$, and assume that for every tree T' of order $n' < n$, we have $\beta_0(T') \leqslant \gamma_{\hat{o}}(T')$. If T is a star of order $n \geqslant 3$, then $\beta_0(T) = n - 1 = \gamma_{\hat{o}}(T)$. Therefore assume that T is not a star, and let v be a support vertex of T with exactly one nonleaf neighbor w. Root T at vertex w, and let $T' = T - T_v$. Since T is not a star, T' has order at least two.

From our observation, L_v is contained in every $\gamma_{\hat{o}}(T)$-set. Hence, without loss of generality, we can choose S to be a $\gamma_{\hat{o}}(T)$-set that does not contain v. Then if S' is the subset of S restricted to T', S' is a GSOA of T'. Hence, $\gamma_{\hat{o}}(T') \leqslant |S| - |L_v| = \gamma_{\hat{o}}(T) - |L_v|$. Applying the inductive hypothesis to T', we have $\beta_0(T') \leqslant \gamma_{\hat{o}}(T')$ and so $\beta_0(T) = \beta_0(T') + |L_v| \leqslant \gamma_{\hat{o}}(T') + |L_v| \leqslant \gamma_{\hat{o}}(T)$.

Stars of order $n \geqslant 3$ achieve the upper bound. Moreover, the trees T_p formed from a star $K_{1,p+1}$ by subdividing p of its edges exactly three times each have $\gamma_o(T_p) = 2p + 1 = \beta_0(T_p)$. ∎

Note that $i(T)$ and $\gamma_o(T)$ are incomparable, and the differences can be arbitrarily large. To see this, let T be the 2-corona of a path P_{3k}. Then $i(T) = 3k < 4k = \gamma_o(T)$. On the other hand, for the double star $S_{p,q}$, $3 \leqslant p \leqslant q$, $\gamma_o(S_{p,q}) = 2 < p + 1 = i(S_{p,q})$.

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