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GRUNDY NUMBER OF GRAPHS

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Abstract

The Grundy number of a graph G is the maximum number k of colors used to color the vertices of G such that the coloring is proper and every vertex x colored with color i, $1 \leq i \leq k$, is adjacent to $(i-1)$ vertices colored with each color j, $1 \leq j \leq i-1$. In this paper we give bounds for the Grundy number of some graphs and cartesian products of graphs. In particular, we determine an exact value of this parameter for n-dimensional meshes and some n-dimensional toroidal meshes. Finally, we present an algorithm to generate all graphs for a given Grundy number.

Keywords: Grundy coloring, vertex coloring, cartesian product, graph product.

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1. Introduction

We consider graphs without loops or multiple edges. Let G be a graph on vertices $x_0, x_1, \ldots, x_{n_g-1}$, with vertex set $V(G)$ and edge set $E(G)$. Let $d(x)$ be the degree of the vertex x of G and let $\Delta(G)$ be the maximum degree of G.

The cartesian product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, denoted $G \,\square\, H$, has the vertex set $V_1 \times V_2$, and the neighborhood of each vertex (x_1, x_2) is $N_{G \square H}((x_1, x_2)) = (\{x_1\} \times N_H(x_2)) \cup (N_G(x_1) \times \{x_2\})$. Thus, in the graph $G \square H$ we find several copies of graphs G and H denoted by G^i and H^j, where i represents the rows of $G \square H$, with $0 \leq i \leq n_H - 1$, and j represents the columns of $G \square H$, with $0 \leq j \leq n_G - 1$. The set $\{x_0^j, x_1^j, \ldots, x_{n_G-1}^j\}$ denotes the vertices of the j^{th} copy of G in $G \square H$ (i.e., G^j), with $0 \leq j \leq n_H - 1$.

Next, we define a *k-coloring* of G as a function c defined on $V(G)$ into a set of colors $C = \{1, 2, \ldots, k\}$ such that for each vertex x_i, with $0 \leq i \leq n_G - 1$, $c_{x_i} \in C$. A *proper k-coloring* is a k-coloring verifying the condition $c_x \neq c_y$ for any pair of adjacent vertices $x, y \in V(G)$. A *Grundy coloring* is a proper k-coloring satisfying the following property P: every vertex v, colored with color i ($1 \leq i \leq k$), is adjacent to vertices colored by each color j such that $1 \leq j \leq i - 1$. The *Grundy number* $\Gamma(G)$ of a graph G is then defined as the maximum number of colors among all Grundy colorings of G. If we color only a set of vertices of a graph G, we will say that the coloring of G is *partial*.

This parameter was introduced by Christen and Selkow [2] in 1979. They proved that determining the Grundy number is NP-complete for general graphs (also studied by McRae in 1994 [11]). In [8], Hedetniemi et *al.* gave a linear algorithm for the Grundy number of a tree and established a relation between the chromatic number, the Grundy number and the achromatic number: $\chi(G) \leq \Gamma(G) \leq \psi(G)$, where the achromatic number $\psi(G)$ is the maximum number of colors used for a proper coloring of G such that each pair of colors appears on at least one edge of G. In 1997, Telle and Proskurowski [14] gave an algorithm for the Grundy number of partial k-trees in $O(n^{3k^2})$ and bounded this parameter for these graphs by the value $1 + k \log_2 n$, where n is the graph order. In 2000, Dunbar et *al.* used the Grundy number to bound new parameters that they introduced in [4], the chromatic and the achromatic numbers of a fall coloring. Recently, Germain and Kheddouci studied in [5, 6], the Grundy coloring of power graphs. They gave bounds for the Grundy number of the power graphs of a path, a cycle, a caterpillar and a complete binary tree. Such colorings are also explored for other graphs like chessboard graphs [12].

The cartesian product of graphs is widely studied in literature since it generates interesting classes of graphs like grids (obtained by the cartesian product of paths and cycles) or hypercubes, which are used to model problems on interconnection networks or multiprocessor networks [1, 7, 9, 13]. In particular, several coloring parameters were evaluated for graphs resulting from the cartesian product of graphs. Čižek and Klavžar [3] computed chromatic numbers of the cartesian sum of two odd cycles. In [10], Kouider and Mahéo were interested in the b-chromatic number of the cartesian product of two graphs. In [15] and [16], Zhu presented several bounds for respectively the star and the fractional chromatic number of graph products.

In our case, we are interested in the Grundy number of the cartesian product of graphs. This parameter has a lot of applications in fields like scheduling or multiprocessor architectures. For instance, suppose a set of processes such that a process P_i could be computed if processes $P_1, P_2, \ldots, P_{i-1}$ are already computed. Such a law on the processes can be modeled by a Grundy coloring. Thus, if we consider a given architecture G, the study of the Grundy number brings a solution for two questions. First, how many processes can we put into this architecture ? This is directly given by the Grundy number $\Gamma(G)$ of the architecture G. Second, how many times must we load processes in the architecture to compute P_n ? The relation $\frac{n-\Gamma(G)}{\Gamma(G)-1} + 1$ gives a solution to this second question.

Thus in this paper we will decompose our study in several parts. First in Section 2, we will present some properties of the Grundy number, by comparing it to other graph parameters. Then we will discuss in Section 3 the Grundy number of several cartesian products of two graphs (paths, cycles, complete and bipartite graphs,...). In Section 4, this parameter will be studied for the cartesian product of several graphs. Thus, we will determine an exact value of the Grundy number for n-dimensional meshes and some n-dimensional toroidal meshes. In these sections, in addition to determining exact values and bounds for the Grundy number, we will also propose constructions of these colorings. Finally in Section 5, we will present an algorithm to generate graphs for a given Grundy number.

2. Grundy Number of a Graph

First, the following obvious fact enables us to find a proper coloring for a graph G from a proper coloring of G', where G' is a subgraph of G.

Fact 1. *Let G' be an induced subgraph of G given by a set of vertices $V' \subseteq V$. Any proper coloring of G' can be extended to a proper coloring of G.*

Proof. We extend the coloring of G' to G as follows. Let x be a vertex of G such that $x \notin V'$. Let C be the set of colors of $N_{G'}(x)$. Let c be the smallest color such that $c \notin C$. We put $c_x := c$ and $V' := V' \cup \{x\}$. Then, we repeat this process until $V' = V$. ■

Then, we present some results for the Grundy number of simple graphs.

Proposition 2. *Let S_n, K_n, P_n, C_n and $K_{n,p}$ be respectively the stable graph, the complete graph, the path, the cycle on order n and the complete bipartite graph on $n+p$ vertices. Let G be a non connected graph with connected components $G_1, G_2, \dots, G_p$. Then, we have:*

1. $\Gamma(S_n) = 1$ and $\Gamma(K_n) = n$,
2. $\Gamma(P_n) = \begin{cases} 2 \text{ if } 2 \leq n \leq 3, \\ 3 \text{ if } n \geq 4, \end{cases}$
2. $\Gamma(C_n) = \begin{cases} 2 \text{ if } n = 4, \\ 3 \text{ if } n \neq 4, \end{cases}$
4. $\Gamma(G) \geq \max\{\Gamma(G_i) : 1 \leq i \leq p\}$,
5. $\Gamma(K_{n,p}) = 2$.

Proof. For cases 1 to 4, the proofs are obvious.

For the complete bipartite graph, we prove the result by contradiction. Let $V(K_{n,p}) = A \cup B$, with $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_p\}$. Let a_i be a vertex colored by a color c, with $1 \leq i \leq n$ and $c \geq 3$. There exists two vertices b_j, b_k colored respectively by colors 1 and 2, with $1 \leq j \neq k \leq p$. Observe that b_k must be adjacent to color 1 on a vertex $a_{i'}$, with $1 \leq i' \neq i \leq n$. So $a_{i'}$ and b_j admit the same color 1, which is a contradiction. ■

Next, we show two results where the Grundy number of a graph G is bounded by other parameters of G. Firstly, we give a relation between the Grundy number and the stability number of G.

Theorem 3. *Let G be a connected graph on order n and stability number α. Then $\Gamma(G) \leq n + 1 - \alpha$.*

Proof. Let A be an independent set in G of size α. Suppose that there is a Grundy coloring of G with $k \geq n - \alpha + 2$ colors. Then at least two color classes should be subsets of A. This contradicts the definition of a Grundy coloring. ■

Secondly, we present the inequality of "Nordhaus-Gaddum"-type for Grundy number of some graph classes.

Proposition 4. *Let G be a graph on order n and $\overline{G}$ its complement. Let Γ and $\overline{\Gamma}$ be the Grundy number of G and $\overline{G}$, respectively. Let x and y be two vertices of G (noted $\overline{x}$ and $\overline{y}$ in $\overline{G}$) such that $c_x = \Gamma$ and $c_{\overline{y}} = \overline{\Gamma}$. Then, $\Gamma + \overline{\Gamma} \leq n+1$ if one of these assertions is verified:*

1. *G is a k-regular graph, with $k \geq 1$,*
2. *$d(x) \leq d(y)$, or*
3. *$c_x = \Gamma$ and $c_{\overline{x}} = \overline{\Gamma}$.*

Proof. 1. The graph G is k-regular. By definition we have $\Gamma(G) \leq \Delta(G)+1$ and $\Gamma(\overline{G}) \leq \Delta(\overline{G})+1$. Since the graph is k-regular, we deduce that $\Delta(G) = k$ and $\Delta(\overline{G}) = n-1-k$. Thus, $\Gamma(G)+\Gamma(\overline{G}) \leq \Delta(G)+\Delta(\overline{G})+2 \leq n+1$.

2. By definition we have

$$d(x) \geq \Gamma - 1, \tag{1}$$

and

$$d(\overline{y}) \geq \overline{\Gamma} - 1. \tag{2}$$

Then, inequality (2) implies

$$d(y) = n-1-d(\overline{y}) \leq n - \overline{\Gamma}. \tag{3}$$

As $d(x) \leq d(y)$, inequalities (1) and (3) give

$$\Gamma - 1 \leq d(x) \leq d(y) \leq n - \overline{\Gamma},$$

$$\Gamma + \overline{\Gamma} \leq n+1.$$

3. From inequalities (1) and (2) we can deduce

$$\Gamma + \overline{\Gamma} \leq d(x) + d(\overline{x}) + 2 = d(x) + (n - d(x) - 1) + 2 = n+1.$$

■

Remark. Observe that this inequality is not verified for any graph. Indeed, consider a graph G on order n composed by a complete graph K_3 (vertices denoted x_1, x_2 and x_3), where every node x_i is the center of a star K_{1,n_i}, with $1 \leq i \leq 3$ and $n_i \geq 1$. By coloring G such that $c(x_1) = 2$, $c(x_2) = 3$, $c(x_3) = 4$ and each endvertex of the stars is colored by 1 (see Figure 1.a), we find $\Gamma(G) \geq 4$. Then in the complement graph $\overline{G}$, the endvertices of the stars form a clique (denoted $K_{n_1+n_2+n_3}$) and vertices x_1, x_2 and x_3 are the centers of the stars respectively K_{1,n_2+n_3}, K_{1,n_1+n_3} and K_{1,n_1+n_2}. Therefore by coloring $c(x_1) = c(x_2) = c(x_3) = 1$ and every vertex of the clique with a different color c, with $2 \leq c \leq n-2$ (see Figure 1.b), we have $\Gamma(\overline{G}) \geq n-2$. Thus, $\Gamma(G) + \Gamma(\overline{G}) \geq n+2$. This remark presents a counterexample[1] to the inequality of "Nordhaus-Gaddum"-type for any graph.

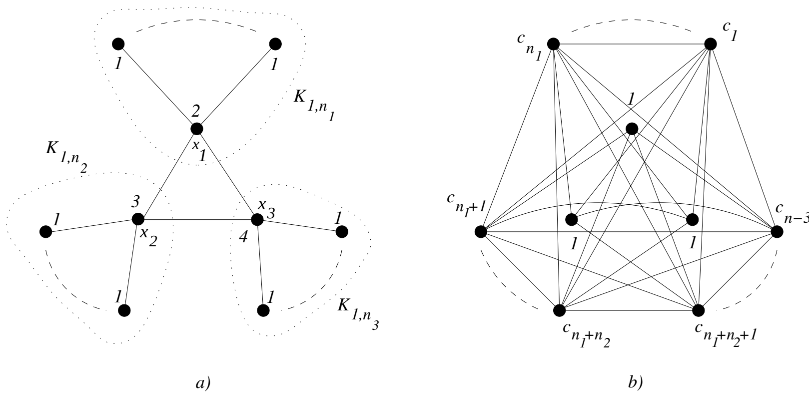

Figure 1. Grundy colorings of a) G and b) $\overline{G}$ where $2 \leq c_1 \neq c_2 \neq \ldots \neq c_{n-3} \leq n-2$.

3. Grundy Number of the Cartesian Product of Two Graphs

In this section, we will discuss the Grundy number of the cartesian product of two graphs. In particular we will study the cartesian product of two paths, two cycles, a path by a cycle, a bipartite graph by other graphs and a complete graph by any graph G.

Proposition 5. *Let G be a bipartite graph. Let P_n be a path on order $n \geq 3$ and C_m be a cycle on order $m \geq 4$. Then,*

$$\Gamma(G \,\square\, P_n) \geq \Gamma(G) + 2,$$

$$\Gamma(G \,\square\, C_m) \geq \Gamma(G) + 2.$$

[1]Authors thank referees for this counterexample.

Proof. On the copy G^1, we put the coloring of G incremented by 2. Thus, on G^1 we find the colors 3 to $\Gamma(G)+2$. Let X_i and Y_i be the two independent sets of the copy G^i. We put the color 1 on every vertex of X_0 and Y_2, and the color 2 on every vertex of X_2 and Y_0. If some vertices are not colored, Fact 1 allows us to color them with a proper coloring. Thus we can deduce $\Gamma(G \,\square\, P_n) \geq \Gamma(G) + 2$ and $\Gamma(G \,\square\, C_m) \geq \Gamma(G) + 2$. ■

As we studied the cartesian product of G by a path or a cycle, where G is a bipartite graph, we discuss the same products if G is not bipartite.

Proposition 6. *Let G be a non bipartite graph on order n_G. Let P_n be a path on order $n \geq 4$ and C_m be a cycle on order $m \geq 4$. Then,*

$$\Gamma(G \,\square\, P_n) \geq \Gamma(G) + 1,$$
$$\Gamma(G \,\square\, C_m) \geq \Gamma(G) + 1.$$

Proof. The proof is given by construction. We color G^0, G^1 and G^2 with the coloring of G. Let x_p^0 be a vertex of G^0 colored by $\Gamma(G)$. Then we put $c_{x_p^0} = \Gamma(G) + 1$, $c_{x_p^1} = \Gamma(G)$ and $c_{x_p^2} = \Gamma(G) - 1$. Next we can remove the colors of vertices x_j^1 on G^1 for which $c_{x_j^1} = 1$ (with $0 \leq j \neq p \leq n_G - 1$) and the colors of vertices x_i^2 on G^2 for which $c_{x_i^2} = \Gamma(G) - 1$ (with $0 \leq i \neq p \leq n_G - 1$). Then for every remaining colored vertex on G^1, we compute $c_{x_j^1} = c_{x_j^1} - 1$. Finally, by Fact 1, we color the remaining non colored vertices with a proper coloring. Thus we can deduce $\Gamma(G \,\square\, P_n) \geq \Gamma(G) + 1$ and $\Gamma(G \,\square\, C_m) \geq \Gamma(G) + 1$. ■

Corollary 7. *Let P_n and P_m be the paths on order n and m respectively. And let C_n, C_m and C_k be the cycles on respectively n, m and k vertices. Then,*

a)
$$\Gamma(P_n \,\square\, P_m) = \begin{cases} (\leq)4 & \text{if } n = 2 \text{ or } n = m = 3, \\ 5 & \text{otherwise,} \end{cases}$$

b)
$$\Gamma(P_n \,\square\, C_k) = \begin{cases} 4 & \text{if } n = 2 \text{ or } n = k = 3, \\ 5 & \text{otherwise,} \end{cases}$$

c)
$$\Gamma(C_n \,\square\, C_m) = \begin{cases} (\leq)4 & \text{if } n = 3 \text{ and } m = 4 \text{ or } n = m = 3, \\ 5 & \text{otherwise.} \end{cases}$$

Proof. In each case, the maximum degree is 4. So the Grundy number is less or equal to 5. The lower bounds are obvious, they are deduced from constructions (see Figure 2). ■

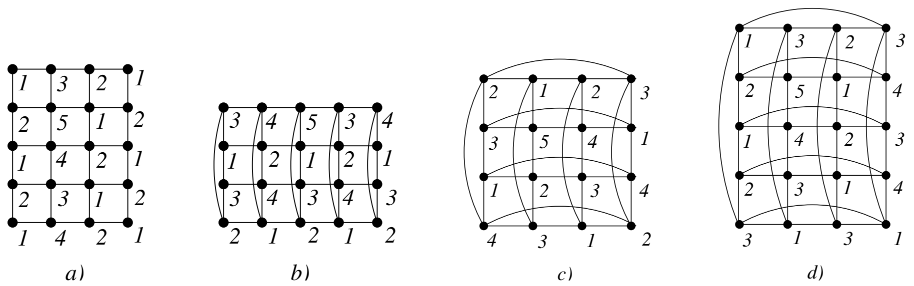

Figure 2. Grundy colorings of a) $P_4 \,\square\, P_5$, b) $P_5 \,\square\, C_4$, c) $C_4 \,\square\, C_4$ and d) $C_4 \,\square\, C_5$.

In the following theorems we study the Grundy number for the cartesian product of a complete graph by another graph. Firstly, we present the cartesian product of a bipartite graph by a complete graph.

Theorem 8. *Let G be a bipartite graph on order n_G. Let K_p be a complete graph on order $p \geq 3$. Then,*

$$\Gamma(G) + p - 1 \leq \Gamma(K_p \,\square\, G) \leq p + \Delta(G).$$

Proof. We prove by construction that $\Gamma(K_p \,\square\, G) \geq \Gamma(G) + p - 1$. Let X_i and Y_i be the two independent sets of the copy G^i. We color G^0 by $c_{x_0^i} := c_{x_i} + p - 1$ where c_{x_i} is the color of the vertex x_i of G, with $0 \leq i \leq n_G - 1$. Then, for each $1 \leq i \leq p-1$, we color each vertex of X_i with the color i, and each vertex of Y_i with the color $i+1$ (mod$(p-1)$). For example, Figure 3.b gives a Grundy coloring of $K_4 \,\square\, G$. As $\Delta(K_p \,\square\, G) = \Delta(G) + p - 1$, we can deduce that $\Gamma(K_p \,\square\, G) \leq \Delta(G) + p$. ■

Remark. For all the classes of bipartite graphs such that $\Gamma(G) = \Delta(G) + 1$, the equality $\Gamma(K_p \,\square\, G) = \Gamma(G) + p - 1$ holds.

Secondly we bound the Grundy number of the cartesian product of a complete graph by any graph G.

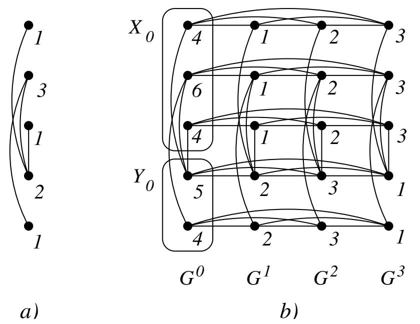

Figure 3. Grundy colorings of a) G and b) $K_4 \,\square\, G$.

Theorem 9. *Let G be a graph on order n_G. Let K_n be a complete graph on order n. Then,*

- $\Gamma(K_n \,\square\, G) \geq \begin{cases} n + \Gamma(G) - 1 & if \;\; \Gamma(G) \leq n-1, \\ 2n-2 & if \;\; n \leq \Gamma(G) \leq 2n-3, \\ \Gamma(G) & if \;\; \Gamma(G) \geq 2n-2, \end{cases}$
- $\Gamma(K_n \,\square\, G) \leq n + \Delta(G)$.

Proof. As $\Delta(K_n \square G) = \Delta(G) + n - 1$, we have $\Gamma(K_n \square G) \leq \Delta(K_n \square G) + 1 \leq n + \Delta(G)$. The proof of the lower bound is given by construction.

- $\Gamma(G) \leq n-1$. Let $k = n + \Gamma(G) - 1$. For each vertex x_0^i, with $0 \leq i \leq n_G - 1$, we shift the color of the vertex x_i on G by $(n-1)$ (i.e., we put $c_{x_0^i} := c_{x_i} + n - 1$). Then, for each vertex x_1^i, with $0 \leq i \leq n_G - 1$, we put $c_{x_1^i} := c_{x_i}$. Next, for each remaining vertex, we put $c_{x_i^j} := c_{x_{i-1}^j} + 1$ $(\mathrm{mod}(n-1))$, with $2 \leq i \leq n-1$ and $0 \leq j \leq n_G - 1$. Figure 4.b shows a Grundy coloring of $K_5 \,\square\, G$.
- $n \leq \Gamma(G) \leq 2n-3$. Let $k = 2n-2$. Let x_i, with $0 \leq i \leq n_G - 1$, be a vertex of G colored by $\Gamma(G)$. As $\Gamma(G) \geq n$, then x_i (resp. x_0^i) has at least $n-1$ neighbors in G (resp. G^0). Let $\{v_1, v_2, \ldots, v_{n-1}\}$ be a set of $n-1$ vertices of $N_{G^0}(x_0^i)$. To color $K_n \,\square\, G$ we put the colors $(k, k-1, \ldots, n)$ respectively on vertices $x_0^i, x_1^i, \ldots, x_{k-n}^i$. Then, on each copy K_n^j containing a vertex v_m, with $0 \leq j \leq n_G - 1$ and $1 \leq m \leq n-1$, we put the colors $(m, m+1, \ldots, m+n-2)$, where each color is taken modulo $(n-1)$, on vertices $x_0^j, x_1^j, \ldots, x_{n-2}^j$ respectively. Then, Fact 1 gives a proper coloring for the non colored vertices. Figure 4.c presents a Grundy coloring of $K_4 \,\square\, G$.

- $\Gamma(G) \geq 2n-2$. We put the coloring of G on G^0 and Fact 1 completes the coloring of $K_n \,\square\, G$ to have a proper coloring. ■

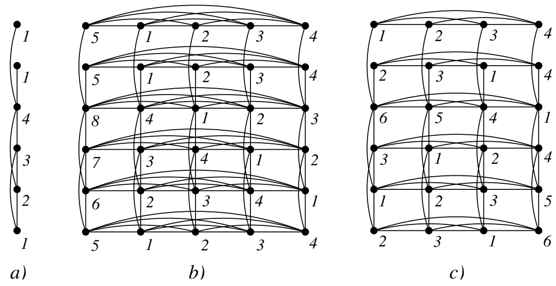

Figure 4. Grundy colorings of a) G, b) $K_5 \,\square\, G$ and c) $K_4 \,\square\, G$.

4. Grundy Number of the Cartesian Product of Several Graphs

The results presented in the previous sections enable us to deduce some results for the cartesian product of several graphs. Firstly, we determine exact values for the Grundy number of the cartesian product of several paths and the cartesian product of several even cycles.

Proposition 10. *Let $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$ be the paths of sizes respectively $n_1, n_2, \ldots, n_k$ such that $k \geq 2$, $n_i \geq 3$ for each $1 \leq i \leq k$ and $\max\{n_i : 1 \leq i \leq k\} > 3$. Let $C_{m_1}, C_{m_2}, \ldots, C_{m_k}$ be the cycles of sizes respectively $m_1, m_2, \ldots, m_k$, where $k \geq 2$, $m_i \geq 4$ and m_i is even, for every $1 \leq i \leq k$. Then,*

$$\Gamma(P_{n_1} \,\square\, P_{n_2} \,\square\, \ldots \,\square\, P_{n_k}) = 2k+1,$$

$$\Gamma(C_{m_1} \,\square\, C_{m_2} \,\square\, \ldots \,\square\, C_{m_k}) = 2k+1.$$

Proof. The proof is given by induction. Suppose $k=2$. Since $n_i \geq 3$ and either $n_1 > 3$ or $n_2 > 3$, and as $m_i \geq 4$, Corollary 7 shows that $\Gamma(P_{n_1} \square P_{n_2}) = \Gamma(C_{m_1} \square C_{m_2}) = 5 = 2k+1$. Suppose that $\Gamma(P_{n_1} \square P_{n_2} \square \ldots \square P_{n_{k-1}}) = 2k-1$ and $\Gamma(C_{m_1} \,\square\, C_{m_2} \,\square\, \ldots \,\square\, C_{m_{k-1}}) = 2k-1$. From its structure, the graph $P_{n_1} \,\square\, P_{n_2} \,\square\, \ldots \,\square\, P_{n_{k-1}}$ is a bipartite graph. Moreover, as m_i is even, with $1 \leq i \leq k$, the graph $C_{m_1} \,\square\, C_{m_2} \,\square\, \ldots \,\square\, C_{m_{k-1}}$ is also a bipartite graph. Thus, Proposition 5 gives,

$$\Gamma(P_{n_1} \,\square\, P_{n_2} \,\square\, \ldots \,\square\, P_{n_k}) \geq \Gamma(P_{n_1} \,\square\, P_{n_2} \,\square\, \ldots \,\square\, P_{n_{k-1}}) + 2 \geq 2k+1,$$

and

$$\Gamma(C_{m_1} \square C_{m_2} \square \ldots \square C_{m_k}) \geq \Gamma(C_{m_1} \square C_{m_2} \square \ldots \square C_{m_{k-1}}) + 2 \geq 2k + 1.$$

Finally, as $\Delta(P_{n_1} \square P_{n_2} \square \ldots \square P_{n_k}) = \Delta(C_{m_1} \square C_{m_2} \square \ldots \square C_{m_k}) = 2k$, we deduce $\Gamma(P_{n_1} \square P_{n_2} \square \ldots \square P_{n_k}) \leq 2k+1$ and $\Gamma(C_{m_1} \square C_{m_2} \square \ldots \square C_{m_k}) \leq 2k+1$. ■

Remark. The graph $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_k}$ is a n_k-dimensional mesh. Therefore, Proposition 10 gives the Grundy number of an n-dimensional mesh.

Secondly, we give a bound to the parameter for the cartesian product of odd cycles.

Corollary 11. *Let $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ be the cycles of size respectively n_1, $n_2, \ldots, n_k$, where $k \geq 2$, n_i is odd, for each $1 \leq i \leq k$, and* $\max\{n_i : 1 \leq i \leq k\} > 3$. *Then*

$$\Gamma(C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}) \geq k + 3.$$

Proof. The proof is given by induction. For $k = 2$, as either $n_1 > 3$ or $n_2 > 3$, Corollary 7 shows that $\Gamma(C_{n_1} \square C_{n_2}) = 5 = k + 3$. Suppose that $\Gamma(C_{n_1} \square C_{n_2} \square \ldots \square C_{n_{k-1}}) \geq (k-1) + 3 = k + 2$. As n_i is odd, for each $1 \leq i \leq k$, the graph $C_{n_1} \square C_{n_2} \square \ldots \square C_{n_{k-1}}$ is not a bipartite graph. Then Proposition 6 gives

$$\Gamma(C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}) \geq \Gamma(C_{n_1} \square C_{n_2} \square \ldots \square C_{n_{k-1}}) + 1 \geq k + 3. \quad ■$$

However, an exact value of the parameter can be determined for the cartesian product of the k first odd cycles.

Proposition 12. *Let $C_3, C_5, \ldots, C_{2k+1}$ be the cycles of size $3, 5, \ldots, 2k+1$ respectively. The Grundy number of $C_3 \square C_5 \square \ldots \square C_{2k+1}$ is given by*

$$\Gamma(C_3 \square C_5 \square \ldots \square C_{2k+1}) = 2k + 1.$$

Proof. As $\Delta(C_3 \square C_5 \square \ldots \square C_{2k+1}) = 2k$, we have $\Gamma(C_3 \square C_5 \square \ldots \square C_{2k+1}) \leq 2k+1$. Next, the lower bound is deduced by construction. We define a *basic element* $E = C_3 \square C_5$ and we use the partial coloring of E given in Figure 5.a. Let $G = C_3 \square C_5 \square \ldots \square C_{2k+1}$ and $H = C_3 \square C_5 \square \ldots \square C_{2k-1}$. Thus in G we find $(2k+1)$ copies of H, denoted by $H^0, H^1, \ldots, H^{2k}$. A partial coloring of G is obtained from that of H. The principle of the coloring of G is the following. We put a partial coloring of H on H^{k-2}, H^{k-1}, H^k and H^{k+1}. Then, to keep the coloring proper, we permute some colors in the basic elements of these copies, and we put the colors $2k$ and $2k+1$ on respectively H^{k-1} and H^k, as shown in Figure 5.b (a partial coloring of $C_3 \square C_5 \square C_7$). Thus, a partial coloring of G is deduced. Finally, by Fact 1 we color the remaining vertices. ∎

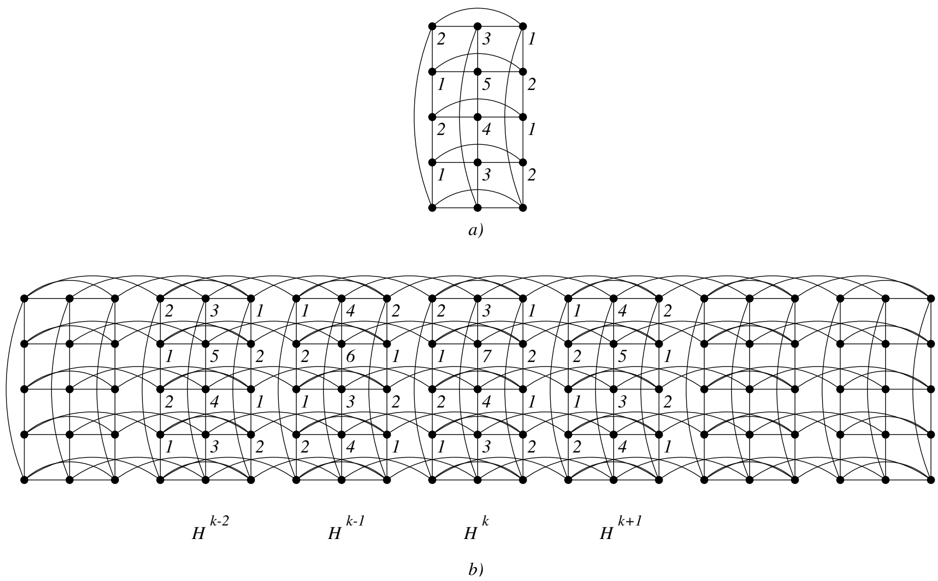

Figure 5. Partial colorings of a) $C_3 \square C_5$ and b) $C_3 \square C_5 \square C_7$.

Remark. The graph $C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}$ is a n_k-dimensional toroidal mesh. So Proposition 10 and Proposition 12 give the Grundy number of a n-dimensional toroidal mesh in two particular cases.

5. Generating Algorithm

We present a simple recursive algorithm to generate all graphs G with the minimum number of edges such that $\Gamma(G) = k$. The main idea is the following : we start from a tree with 2^{k-1} vertices and we join together some vertices having the same color. By computing all the possible groupings, we find a set of graphs with a Grundy number equals to k. Recursively we start again from each graph of this set while the computation can be done. Figure 6 shows the generation of graphs for which $\Gamma(G) = 4$.

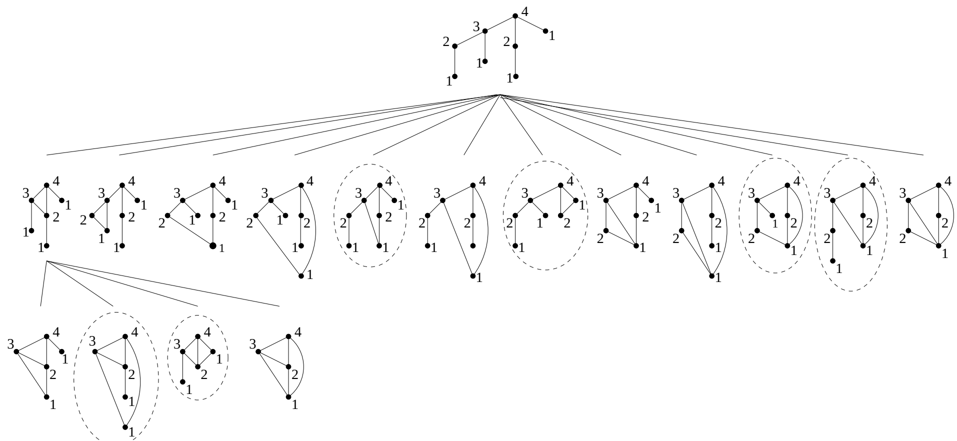

Figure 6. Generation of graphs verifying $\Gamma(G) = 4$ (surrounded graphs are duplications).

6. Conclusion

In this article, we first positioned the Grundy number of a graph G compared to other graph parameters (stability number, complement graph of G). Then we presented several bounds and values for the Grundy number of the cartesian product of two graphs. In particular we studied the cartesian product of a bipartite graph by a path or a cycle, a bipartite graph by a complete graph and a complete graph by any graph G. Then, we deduced exact values for the Grundy number of a n-dimensional mesh and particular cases of a n-dimensional toroidal mesh.

Thus, for every graph H containing an induced subgraph G presented in this paper, Fact 1 shows that $\Gamma(H) \geq \Gamma(G)$ and gives a proper coloring to H.

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