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TOTAL EDGE IRREGULARITY STRENGTH OF TREES

Jaroslav Ivančo and Stanislav Jendrol'

Institute of Mathematics
P.J. Šafárik University
Jesenná 5, SK–041 54 Košice, Slovak Republic

e-mail: ivanco@science.upjs.sk
e-mail: jendrol@kosice.upjs.sk

Abstract

A total edge-irregular k-labelling $\xi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ of a graph G is a labelling of vertices and edges of G in such a way that for any different edges e and f their weights $\mathrm{wt}(e)$ and $\mathrm{wt}(f)$ are distinct. The weight $\mathrm{wt}(e)$ of an edge $e = xy$ is the sum of the labels of vertices x and y and the label of the edge e. The minimum k for which a graph G has a total edge-irregular k-labelling is called the total edge irregularity strength of G, $\mathrm{tes}(G)$. In this paper we prove that for every tree T of maximum degree Δ on p vertices

$$\mathrm{tes}(T) = \max\{\lceil (p+1)/3 \rceil, \lceil (\Delta+1)/2 \rceil\}.$$

Keywords: graph labelling, tree, irregularity strength, total labellings, total edge irregularity strength.

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1. Introduction

In [7], Chartrand *et al.* proposed the following problem:

Assign positive integer labels to the edges of a simple connected graph of order at least 3 in such a way that the graph becomes irregular, i.e., the weights (label sums) at each vertex are distinct. What is the minimum value of the label over all such irregular assignments?

This parameter of a graph is well known as the *irregularity strength* of the graph G, $s(G)$. Finding the irregularity strength of a graph seems to be hard even for simple graphs, see e.g., [1, 2, 5, 6, 7, 8, 11] and a survey article by Lehel [10]. For example, Amar and Togni proved the following result.

Theorem 1 [2]**.** *Let T be a tree having t leaves and no vertex of degree 2. Then*

$$s(T) = t.$$

Motivated by total labellings mentioned in a survey paper of Gallian [9] and a book of Wallis [12], Bača *et al.* [4] started to investigate total edge-irregular labellings of graphs.

For a simple graph G, a labelling $\xi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ is called a *total k-labelling.* The *weight* of an edge xy under a total k-labelling ξ is defined as

$$\mathrm{wt}(xy) = \xi(x) + \xi(xy) + \xi(y).$$

A total k-labelling is defined to be a *total edge-irregular k-labelling* of a graph G if, for different edges e and f of G,

$$\mathrm{wt}(e) \neq \mathrm{wt}(f).$$

The minimum k for which a graph G has a total edge-irregular k-labelling is called the *total edge irregularity strength* of G, $\mathrm{tes}(G)$.

It is not difficult to prove (see [4]) that for every graph G with q edges

$$\left\lceil \tfrac{1}{3}(q+2) \right\rceil \leq \mathrm{tes}(T) \leq q.$$

The authors of [4] present also a few families of graphs G for which they found the exact value of $\mathrm{tes}(G)$. Among other results they proved

Theorem 2 [4]**.** *Let P_p and S_p be a path and a star $K_{1,p-1}$ on p vertices, $p \geq 3$. Then*

$$\mathrm{tes}(P_p) = \left\lceil \frac{p+1}{3} \right\rceil,$$
$$\mathrm{tes}(S_p) = \left\lceil \frac{p}{2} \right\rceil.$$

Motivated by results on irregularity strength of trees by Aigner and Triesch [1], Amar and Togni [2], Bohman and Kravitz [5] and Cammack *et al.* [6], Bača *et al.* in [4] posed the problem to determine the total edge-irregularity strength of trees. In a recent paper [3] there is proved

Theorem 3 [3]**.** *Let T be a tree on p vertices, $p \geq 3$. Then*

$$\left\lceil \frac{p+1}{3} \right\rceil \leq \operatorname{tes}(T) \leq \left\lceil \frac{p}{2} \right\rceil .$$

Moreover, both bounds are tight.

Let us recall that, in the sequel, $V(G)$, $E(G)$, and $\Delta(G)$ will denote the vertex set, the edge set, and the maximum degree of a graph G, respectively. The main result of this paper is the following

Theorem 4. *Let T be a tree. Then*

$$\operatorname{tes}(T) = \max\left\{ \left\lceil \frac{|E(T)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(T)+1}{2} \right\rceil \right\}.$$

2. FOUR LEMMAS

Lemma 1. *Suppose that a graph G has a total edge-irregular k-labelling. Then*

$$3k-2 \geq |E(G)| \quad \textit{and} \quad 2k-1 \geq \Delta(G).$$

Proof. Let ξ be a total edge-irregular k-labelling of G. The weight of any edge $xy \in E(G)$ satisfies: $3 \leq \operatorname{wt}(xy) = \xi(x)+\xi(xy)+\xi(y) \leq 3k$. As weights of different edges are distinct, we get $3k-2 \geq |E(G)|$.

Let $u \in V(G)$ be a vertex of G with degree $\Delta(G)$. For two different vertices x, y adjacent to u it holds: $\xi(u)+\xi(ux)+\xi(x) = \operatorname{wt}(ux) \neq \operatorname{wt}(uy) = \xi(u)+\xi(uy)+\xi(y)$. Then $\xi(ux)+\xi(x) \neq \xi(uy)+\xi(y)$. As $2 \leq \xi(uz)+\xi(z) \leq 2k$, for every vertex z adjacent to u, we get $2k-1 \geq \Delta(G)$. ∎

Given a mapping φ from the vertex set of a graph G to $\{0,1\}$. Put $E_i(\varphi) := \{xy \in E(G) : \varphi(x)+\varphi(y) = i\}$, for $i \in \{0,1,2\}$.

Lemma 2. *Suppose that u is a vertex of a tree T with q edges. Then, for every integer k, $0 \leq k \leq q$, and $i \in \{0,1\}$ there is a mapping $\varphi : V(T) \to \{0,1\}$ such that $\varphi(u) = i$, $|E_{2i}(\varphi)| = k$ and $|E_1(\varphi)| = q-k$.*

Proof. As every non-trivial tree contains at least two leaves (i.e., the vertices of degree 1), there is an ordering v_0, v_1,…,v_q of $V(T)$ such that $v_0 = u$ and the set $\{v_0, v_1, \ldots, v_j\}$ induces a subtree of T with the leaf v_j, $j = 1, \ldots, q$.

Let $\varphi : V(T) \to \{0,1\}$ be a mapping defined by

$$\varphi(v_t) = \begin{cases} i & \text{for } 0 \le t \le k, \\ 1 - \varphi(v_t^*) & \text{for } k < t \le q, \end{cases}$$

where v_t^* is a vertex of a subtree induced by $\{v_0, v_1, \ldots, v_t\}$ which is adjacent to v_t. Evidently, φ is the desired mapping. ■

For a graph G, a mapping $\varphi : V(G) \to \{0,1\}$ is called *k-irregularisable* if

$$|E(G)| \le 3k-2, |E_0(\varphi)| \le k, |E_1(\varphi)| \le k, |E_2(\varphi)| \le k,$$
$$|E_0(\varphi)| + |E_1(\varphi)| \le 2k-1 \text{ and } |E_1(\varphi)| + |E_2(\varphi)| \le 2k-1.$$

Lemma 3. *Suppose that a graph G admits a k-irregularisable mapping. Then G has a total edge-irregular k-labelling.*

Proof. Let $\varphi : V(G) \to \{0,1\}$ be a k-irregularisable mapping. Let e_1^i, $e_2^i, \ldots, e_{r_i}^i$, $i \in \{0,1,2\}$, $r_i = |E_i(\varphi)|$, be any ordering of edges belonging to $E_i(\varphi)$. Consider a mapping α from $E(G)$ to positive integers defined by

$$\alpha(e_j^0) = j,$$
$$\alpha(e_j^1) = \begin{cases} 1+j & \text{if } r_0 = k, \\ j & \text{if } r_0 < k, \end{cases}$$
$$\alpha(e_j^2) = \begin{cases} 1+j & \text{if } r_1 = k \text{ or } r_0 + r_1 = 2k-1, \\ j & \text{if } r_1 < k \text{ and } r_0 + r_1 < 2k-1. \end{cases}$$

Clearly, $1 \le \alpha(e_j^i) \le k$.

Now define a labelling ξ from $V(G) \cup E(G)$ into positive integers by

$$\xi(x) = \begin{cases} k^{\varphi(x)} & \text{if } x \in V(G), \\ \alpha(x) & \text{if } x \in E(G). \end{cases}$$

One can easily check that ξ is a total edge-irregular k-labelling of G. ■

For a vertex u of a graph G, let $N(u)$ denote the set of vertices adjacent to u. The set of vertices of $N(u)$ with degree at least 2 is denoted by $N^*(u)$.

Lemma 4. *Let u be a maximum degree vertex of a graph G. Let k be a positive integer such that $\Delta = \deg(u) \in \{2k-2, 2k-1\}$, $|E(G)| \le 3k-2$, $|N^*(u)| \le k$, and $|E(G)| - |N(u)| + |N^*(u)| \le 2k-1$. Then $\operatorname{tes}(G) = k$.*

Proof. Let E^* denote the set of all edges not incident with u. Clearly, every vertex of $N^*(u)$ is incident with some edge of E^* and $|N(u)|+|E^*| = |E(G)|$, i.e., $|E^*| \le k$. Moreover, there is a set U, $N^*(u) \subseteq U \subseteq N(u)$, such that $|U| = k-1$ if $|E^*| = k$ and $|U| = k$ if $|E^*| < k$. Put $W := \{u\} \cup (N(u) - U)$ and define a mapping $\varphi : V(G) \to \{0, 1\}$ by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in W, \\ 1 & \text{if } x \in V(G) - W. \end{cases}$$

As $E_1(\varphi) = \{uy : y \in U\}$ and $E_2(\varphi) = E^*$, φ is a k-irregularisable mapping of G. Combining Lemma 1 and Lemma 3 we obtain the desired assertion. ■

3. PROOF OF THEOREM 4

Put $k = \max\{\lceil \frac{1}{3}(|E(T)|+2)\rceil, \lceil \frac{1}{2}(\Delta(T)+1)\rceil\}$. According to Lemma 1 and Lemma 3 it is enough to find some k-irregularisable mapping of T. Consider the following cases.

A. $k = \lceil \frac{1}{2}(\Delta(T)+1)\rceil$. Then $\Delta(T) \in \{2k-2, 2k-1\}$ and $|E(T)| \le 3k-2$. Let u be a maximum degree vertex of T and let E^* be the set of all edges not incident with u. So, $|E^*| = |E(T)| - |N(u)| \le k$. As T is a tree, at most one end vertex of an edge of E^* belongs to $N(u)$. Thus, $|N^*(u)| \le |E^*| \le k$. If $|N^*(u)| < k$, then the assertion follows from Lemma 4. If $|N^*(u)| = k$, then $\Delta(T) = 2k-2$, $|E(T)| = 3k-2$ and every edge of E^* is incident with exactly one vertex of $N(u)$. Therefore, we can denote vertices of T by u, $v_1, \ldots, v_{2k-2}$, $w_1, \ldots, w_k$ in such a way that $E(T) = \{uv_1, \ldots, uv_{2k-2}, v_1w_1, \ldots, v_kw_k\}$. In this case, a mapping $\varphi : V(T) \to \{0, 1\}$ defined by

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in \{u, v_k, \ldots, v_{2k-2}\}, \\ 1 & \text{for } x \in \{v_1, \ldots, v_{k-1}, w_1, \ldots, w_k\}, \end{cases}$$

is k-irregularisable.

B. $k > \lceil \frac{1}{2}(\Delta(T)+1) \rceil$. Then $\Delta(T) < 2k-2$, $|E(T)| \in \{3k-2, 3k-3, 3k-4\}$ and so, without loss of generality $|E(T)| = 3k-2$.

For an edge xy, $T(xy,x)$ denotes the maximal subtree of T, which contains x and does not contain xy. The number of edges in $T(xy,x)$ is denoted by $t(xy,x)$. Let $\mu := \min\{|t(xy,x) - t(xy,y)| : xy \in E(T)\}$ and let vw be an edge of T such that $t(vw,w) - t(vw,v) = \mu$.

B1. Suppose that $\mu \le k$. Then $t(vw,w) - \mu = t(vw,v) \ge k-1$. By Lemma 2, there are mappings $\varphi_v : V(T(vw,v)) \to \{0,1\}$ and $\varphi_w : V(T(vw,w)) \to \{0,1\}$ such that $\varphi_v(v) = 0$, $|E_0(\varphi_v)| = k-1$, $|E_1(\varphi_v)| = t(vw,v)-k+1$, $\varphi_w(w) = 1$, $|E_2(\varphi_w)| = k-1$ and $|E_1(\varphi_w)| = t(vw,w)-k+1$. Evidently, a mapping $\varphi : V(T) \to \{0,1\}$, given by

$$\varphi(x) = \begin{cases} \varphi_v(x) & \text{if } x \in V(T(vw,v)), \\ \varphi_w(x) & \text{if } x \in V(T(vw,w)), \end{cases}$$

is k-irregularisable.

B2. Suppose that $\mu > k$. Then $t(vw,v) \le k-2$. Denote vertices of $N(w)$ by $v_1, v_2, \ldots, v_d$ in such a way that $t(v_1w,v_1) \ge t(v_2w,v_2) \ge \cdots \ge t(v_dw,v_d)$. As $t(v_1w,v_1) + t(v_2w,v_2) + \cdots + t(v_dw,v_d) = |E(T)| - \deg(w) > k$, there is an integer $\varrho := \min\{j : \sum_{i=1}^{j} t(v_iw,v_i) \ge k-1\}$. Put $\kappa := \sum_{i=1}^{\varrho} t(v_iw,v_i) - (k-1)$. Clearly, $0 \le \kappa < t(v_\varrho w, v_\varrho)$.

Let T^* be the maximal subtree of T, which contains w and does not contain $v_1, \ldots, v_\varrho$. Since $\sum_{i=1}^{\varrho-1} t(v_iw,v_i) \le k-2$ and $k-2 \ge t(v_iw,v_i) \ge 1$ for every $i \in \{1,\ldots,\varrho\}$ we have

$$\begin{aligned} \sum_{i=1}^{\varrho}(1+t(v_iw,v_i)) &\le \sum_{i=1}^{\varrho-1} 2t(v_iw,v_i) - (t(v_1w,v_1)-1) + t(v_\varrho w,v_\varrho) + 1 \\ &\le 2(k-2) + t(v_\varrho w,v_\varrho) - t(v_1w,v_1) + 2 \\ &= 2k-2 + (t(v_\varrho w,v_\varrho) - t(v_1w,v_1)) \le 2k-2. \end{aligned}$$

Then $|E(T^*)| = |E(T)| - \sum_{i=1}^{\varrho}(1+t(v_iw,v_i)) \ge k$. By Lemma 2, there are mappings $\varphi^* : V(T^*) \to \{0,1\}$ and $\varphi^\circ : V(T(v_\varrho w,v_\varrho)) \to \{0,1\}$ such that $\varphi^*(w) = 0$, $|E_0(\varphi^*)| = k-1$, $|E_1(\varphi^*)| = |E(T^*)| - k + 1$, $\varphi^\circ(v_\varrho) = 1$, $|E_1(\varphi^\circ)| = \kappa$ and $|E_2(\varphi^\circ)| = t(v_\varrho w,v_\varrho) - \kappa$. Evidently, a mapping $\varphi : V(T) \to \{0,1\}$, given by

$$\varphi(x) = \begin{cases} \varphi^*(x) & \text{if } \ x \in V(T^*), \\ \varphi^\circ(x) & \text{if } \ x \in V(T(v_\varrho w, v_\varrho)), \\ 1 & \text{if } \ x \in V(T(v_i w, v_i)) \ \text{ for } \ i \in \{1, \ldots, \varrho - 1\}, \end{cases}$$

is k-irregularisable.

4. Appendix

Using Lemma 1 and Lemma 3 it is easy to determine the total edge irregularity strength of some special graphs.

The generalized Petersen graph $P(n, k)$ is a graph with the vertex set $V = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and the edge set $E = \{u_i u_{i+1}, v_i v_{i+k}, u_i v_i : i = 1, \ldots, n\}$ (indices are taken modulo n). The mapping $\varphi : V \to \{0, 1\}$ given by

$$\varphi(u_i) = 0, \quad \varphi(v_i) = 1 \quad \text{for} \ \ i = 1, \ldots, n$$

is clearly $(n+1)$-irregularisable. As $|E| = 3n < 3(n+1) - 2$, Lemmas 1 and 3 immediately imply

Theorem 5. $\operatorname{tes}(P(n, k)) = n + 1$.

Using the same idea for the Cartesian product $G \times K_2$ of a graph G and a complete graph K_2 (details are left to the reader) we get

Theorem 6. *Let G be a graph with p vertices and q edges. If $p - 1 \le q \le p$, then*

$$\operatorname{tes}(G \times K_2) = q + 1.$$

Similarly, for a graph $3G$ consisting of three disjoint copies of a bipartite graph G we have

Theorem 7. *Let G be a bipartite graph with q edges. Then*

$$\operatorname{tes}(3G) = q + 1.$$

We believe that the following conjecture is true.

Conjecture. Let G be an arbitrary graph different from K_5. Then

$$\operatorname{tes}(G) = \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}.$$

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