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COMBINATORIAL LEMMAS FOR POLYHEDRONS I

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Abstract

We formulate general boundary conditions for a labelling of vertices of a triangulation of a polyhedron by vectors to assure the existence of a balanced simplex. The condition is not for each vertex separately, but for a set of vertices of each boundary simplex. This allows us to formulate a theorem, which is more general than the Sperner lemma and theorems of Shapley; Idzik and Junosza-Szaniawski; van der Laan, Talman and Yang. A generalization of the Poincaré-Miranda theorem is also derived.

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1. Preliminaries

For $n \in \mathbb{N}$, let $N = \{1, \dots, n\}$ and $N_0 = \{0, \dots, n\}$. By a *polyhedron* we understand the convex hull of a finite set of $\mathbb{R}^n$. Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension n. A *face* of the polyhedron P is the intersection of P with some of its supporting hyperplanes. Denote the set of all k-dimensional faces of the polyhedron P by $\mathbf{F}_k(P)$ $(k < n)$, the set of all faces of the polyhedron P by $\mathbf{F}(P)$ (hence $\mathbf{F}(P) = \bigcup_{k=0}^{n-1} \mathbf{F}_k(P)$) and the set of all vertices of the polyhedron P by $V(P)$ $(V(P) = \mathbf{F}_0(P))$. The maximal dimension proper faces of the polyhedron P are called *facets*. For a finite set $A = \{a_0, \dots, a_m\} \subset \mathbb{R}^n$ a set $\operatorname{co} A = \{\alpha_0 a_0 + \dots + \alpha_m a_m : \ a_i \in A, \ \sum_{i=0}^m \alpha_i = 1, \ \alpha_i \geq 0 \text{ for } i \in \{0, \dots, m\}\}$ is the *convex hull* of A, $\operatorname{aff} A = \{\alpha_0 a_0 + \dots + \alpha_m a_m : \ \sum_{i=0}^m \ \alpha_i = 1, \ \ a_i \in A, \ \alpha_i \in \mathbb{R} \text{ for } i \in \{0, \dots, m\}\}$ is the *affine hull* of A. And if for a finite set $A = \{a_0, \dots, a_m\} \subset \mathbb{R}^n$ $(m \in \{0, \dots, n\})$ the dimension of $\operatorname{aff} A$ is equal to m, then $\operatorname{co} A$ is called a *simplex* (precisely an *m-simplex*). Let Tr_n be a finite family of n-simplexes such that $P = \bigcup_{\delta \in Tr_n} \delta$ and for any $\delta_1, \delta_2 \in Tr_n$, $\delta_1 \cap \delta_2$ is the empty set or their common face. A *triangulation* of the polyhedron P (we denote it by Tr) is a family consisting of simplexes of Tr_n and all their faces. Let Tr_m $(m \in N_0)$ denote the family of m-simplexes belonging to a triangulation Tr. Hence $Tr = \bigcup_{i=0}^n Tr_i$. Let $V = Tr_0$ be the set of vertices of all simplexes of Tr. Notice, that $V = \bigcup_{\delta \in Tr_n} V(\delta)$. An $(n-1)$-simplex of Tr_{n-1} is a *boundary* $(n-1)$*-simplex* if it is a facet of exactly one n-simplex of Tr_n. For a triangulation Tr^P of the polyhedron P and a triangulation Tr^Q of a polyhedron Q a function $f : V(Tr^P) \to V(Tr^Q)$ is a *simplicial function* if for every $\sigma \in Tr^P$ there exists $\delta \in Tr^Q$ such that $f(V(\sigma)) = V(\delta)$.

2. Main Result

We start with the following

Definition 2.1. Let $\sigma \subset \mathbb{R}^n$ be a simplex, $l : V(\sigma) \to \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$. A simplex σ is *b-balanced* if the point b belongs to $\operatorname{co}(l(V(\sigma)))$ and *b-subbalanced with respect to Z*, if the point b belongs to $\operatorname{co}(l(V(\sigma)) \cup Z)$. If $Z = \{x\}$, then we write b-subbalanced with respect to x instead of with respect to $\{x\}$. For $b = 0$ we say balanced and subbalanced instead of b-balanced and b-subbalanced, respectively.

Notice that in the case Z is a polyhedron, a simplex σ is b-subbalanced with respect to Z if and only if σ is b-subbalanced with respect to $V(Z)$.

Lemma 2.2. *Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension n, Tr be a triangulation of the polyhedron P, $l : Tr_0 \to \mathbb{R}^n$, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. If the triangulation Tr contains neither a b-balanced simplex of dimension less than n nor a simplex of dimension less than $n-1$ which is b-subbalanced with respect to x, then the number of b-balanced simplexes in Tr is congruent modulo 2 to the number of b-subbalanced with respect to x boundary simplexes in Tr.*

Proof. For this proof by a b-subbalanced simplex we understand a b-subbalanced simplex with respect to x. Consider a graph $G = (W, E)$ where W is the set of b-balanced n-simplexes and b-subbalanced $(n-1)$-simplexes in Tr and there is an edge between two different simplexes $\sigma_1, \sigma_2 \in W$ if and only if there exists a simplex $\sigma \in Tr$ containing σ_1 and σ_2 (in particular $\sigma = \sigma_1$). We will show that

$$\deg_G(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is a } b\text{-balanced or a boundary } b\text{-subbalanced simplex,} \\ 2 & \text{if } \sigma \text{ is a } b\text{-subbalanced simplex not in the boundary.} \end{cases}$$

Let σ be a b-balanced simplex of Tr. By our assumption σ is an n-dimensional simplex. Let $V(\sigma) = \{v_0, \ldots, v_n\}$, $u_i = l(v_i)$ for $i \in N_0$ and let $A_i = \operatorname{co}\{u_0, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_n\}$ for $i \in N_0$. There is at least one $j \in N_0$ such that $b \in A_j$ since $b \in \operatorname{co}\{u_0, \ldots, u_n\} \subseteq \bigcup_{i=0}^n A_i$. If there exists $j, k \in N_0$, $j < k$ such that $b \in A_j$ and $b \in A_k$, then it is easy to show that $b \in \operatorname{co}\{x, u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n\}$, so that the simplex $\operatorname{co}\{v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$ is b-subbalanced and of dimension less than $n-1$. This contradicts our assumption.

Now let σ be a b-subbalanced simplex in Tr of dimension $n-1$ and let σ_1 be an n-simplex containing σ, $V(\sigma) = \{v_1, \ldots, v_n\}$, $V(\sigma_1) \setminus V(\sigma) = \{v_0\}$, $u_i = l(v_i)$ for $i \in N_0$, $B_0 = \operatorname{co}\{u_0, u_1, \ldots, u_n\}$, $B_i = \operatorname{co}\{x, u_1, \ldots, u_{i-1}, u_0, u_{i+1}, \ldots, u_n\}$ for $i \in N_0$. Since $b \in \operatorname{co}\{x, u_1, \ldots, u_n\} \subseteq \bigcup_{i=0}^n B_i$, then there exists $i \in N_0$ such that $b \in B_i$. If $b \in B_0$, then σ_1 is b-balanced and σ and σ_1 form an edge in G. If $b \in B_i$ for some $i \in N$, then $\sigma_2 = \operatorname{co}\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$ is b-subbalanced and σ and σ_2 form an edge in G. If $b \in B_0 \cap B_j$ for some $j \in N$, then the simplex $\operatorname{co}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$ is b-subbalanced of dimension less that $n-1$, but this is impossible. If $b \in B_j \cap B_k$ for some $j, k \in N$, $j < k$, then the simplex

$\mathrm{co}\,\{v_0,\ldots,v_{j-1},v_{j+1},\ldots,v_{k-1},v_{k+1},\ldots,v_n\}$ is b-subbalanced of dimension less that $n-1$, but this is also impossible. In all cases, σ_1 defines an adjacent edge to σ in G. Hence, if σ is a boundary simplex (it is a face of exactly one n-simplex), then $\deg_G(\sigma)=1$ and if σ is not a boundary simplex (it is a face of exactly two n-simplexes), then $\deg_G(\sigma)=2$.

Graph G has vertices of degree one and two only. Thus the number of vertices of degree one is even and hence the number of b-balanced simplexes in Tr is congruent modulo two to the number of b-subbalanced with respect to x boundary simplexes in Tr. ■

Remark 2.3. Let $S\subset\mathbb{R}^n$ be a polyhedron, $\widetilde{Tr}$ be a triangulation of $\mathrm{bd}\,S$ and $p\in\mathrm{ri}\,S$, then $Tr=\{co(\{p\}\cup\sigma):\sigma\in\widetilde{Tr}\}\cup\widetilde{Tr}\cup\{p\}$ is a triangulation of the polyhedron S.

Definition 2.4. Two n-dimensional polyhedrons P and Q are *dual to each other through* ψ if $\psi:\mathbf{F}(P)\to\mathbf{F}(Q)$ is a one-to-one inclusion-reversing mapping, i.e., $F_1\subset F_2$ if and only if $\psi(F_1)\supset\psi(F_2)$ for any $F_1,F_2\in\mathbf{F}(P)$. Polyhedrons P and Q are dual to each other if there exists $\psi:\mathbf{F}(P)\to\mathbf{F}(Q)$ such that P and Q are dual to each other through ψ.

A simplex of any dimension is dual to itself and a 3-dimensional cube and octahedron are dual to each other. For more examples and properties of dual polyhedrons see Grunbaum [4], pp. 46–48. Notice that $\dim F+\dim\psi(F)=n-1$ for any $F\in\mathbf{F}(P)$.

Duality of polyhedrons may be defined in many ways (see e.g. Alexandrov [1], pp. 49):

Definition 2.5. Two n-dimensional polyhedrons P and Q are *dual to each other through* ϕ, if $\phi:\mathbf{F}_0(P)\to\mathbf{F}_{n-1}(Q)$ fulfils the following condition: for $v_1,v_2\in\mathbf{F}_0(P)$, $\mathrm{co}\,\{v_1,v_2\}$ is a face of P if and only if $\phi(v_1)$ and $\phi(v_2)$ have a common $(n-2)$-dimensional face.

Observe that both definitions are equivalent.

Theorem 2.6. *Let $P,Q\subset\mathbb{R}^n$ be n-dimensional polyhedrons, dual to each other through a mapping ψ, Tr be a triangulation of the polyhedron P, $V=Tr_0$, $b\in\mathrm{ri}\,Q$ and $l:V\to\mathbb{R}^n$ be a labelling. If for every $G\in\mathbf{F}(P)$ and every simplex $\sigma\in Tr$ and $\sigma\subseteq G$, σ is not b-subbalanced with respect to the set $\psi(G)$, then there exists a b-balanced simplex in Tr.*

Proof. For $n = 1$ the boundary condition implies that the labels of two vertices of P lie on opposite sides of the point b. Thus there is a vertex $v \in Tr_0$ such that $l(v) = b$ or the number of b-balanced simplexes in Tr is odd.

Consider the case $n > 1$. Assume that there is no b-balanced simplex in Tr of dimension less than n. We show that there exists a b-balanced simplex of dimension n in Tr.

We define a triangulation Tr^Q of $\operatorname{bd} Q$. For every face $H \in \mathbf{F}_1(Q)$ we choose a point $u_H \in \operatorname{ri} H$ and apply Remark 2.3 to get a triangulation of the face H. Then inductively for $k = 2, \ldots, n-1$: for every face $H \in \mathbf{F}_k(Q)$ we choose a point $u_H \in \operatorname{ri} H$ and apply Remark 2.3 to get a triangulation of the face H. Finally we obtain a triangulation of $\operatorname{bd} Q$.

Let $V(P) = \{a_0, \ldots, a_k\}$ $(k \geq n)$. For $i \in \{0, \ldots, k\}$ and $c \in \operatorname{ri} P$, let $a'_i = 2a_i - c$ and $P' = co\,\{a'_0, \ldots, a'_k\}$. Notice that $P \subset P'$.

Now we define a triangulation of P', which is an extension of the triangulation Tr of the polyhedron P. We define a triangulation of the set $P' \setminus \operatorname{ri} P$.

For every face $F = co\,\{a_{i(0)}, \ldots, a_{i(l)}\}$ (defined by some: $\{a_{i(0)}, \ldots, a_{i(l)}\} \subset V(P)$) of the polyhedron P we denote $F' = co\,\{a'_{i(0)}, \ldots, a'_{i(l)}\}$. Every face F of P has one-to-one correspondence to the face F' of P'.

Let us denote $FF' = co\,\{F \cup F'\}$. Thus $P' \setminus \operatorname{ri} P = \bigcup_{F \in \mathbf{F}_{n-1}(P)} FF'$.

For every face $F_1 \in \mathbf{F}_1(P)$ we choose a point $v_{F'_1} \in \operatorname{ri} F'_1$. By Remark 2.3 we receive a triangulation of F'_1. Then for every face $F_1 \in \mathbf{F}_1(P)$ we choose a point $v_{F_1F'_1} \in \operatorname{ri} F_1F'_1$. By Remark 2.3 we receive a triangulation of $F_1F'_1$.

Now we apply the induction for $k \in \{2, \ldots, n-1\}$: for any face $F_k \in \mathbf{F}_k(P)$ we choose a point $v_{F'_k} \in \operatorname{ri} F'_k$ and by Remark 2.3 we get a triangulation of the face F'_k. Analogously we choose a point $v_{F_kF'_k} \in \operatorname{ri} F_kF'_k$ and get a triangulation of $F_kF'_k$.

Finally we obtain a triangulation of $P' \setminus \operatorname{ri} P$ and denote it by Tr''. Hence $Tr' = Tr \cup Tr''$ is a triangulation of P', which is an extension of the triangulation Tr on P.

Let $V' = Tr'_0$. If $v \in V' \setminus V$, then $v \in V(P') \cup \{v_{GG'}, v_{G'} : G \in \mathbf{F}_k(P), k \in \{1, \ldots, n-1\}\}$. For $G' \in \mathbf{F}_0(P')$ we also denote $v_{G'} := G'$.

Now, we define a labelling $l' : V' \to \mathbb{R}^n$:

$$l'(v) = \begin{cases} l(v) & \text{for } v \in V, \\ u_{\psi(G)} & \text{for } v = v_{GG'} \text{ or } v = v_{G'}\,. \end{cases}$$

We prove that there is no b-balanced simplex in Tr''. Consider an n-simplex $\sigma \in Tr''$. If $\sigma \cap P = \emptyset$, then there is exactly one vertex v of σ, which is also a vertex of P'. Let $v = a'_j \in V(P')$ $(j \in \mathbb{N})$ and thus $l'(V(\sigma)) \subset \psi(a_j)$, where $\psi(a_j)$ is a facet of Q so σ is not b-balanced. Now consider the case $\sigma \cap P \neq \emptyset$: let $\tau = \sigma \cap P$, G_τ be the smallest face of P (in the sense of inclusion) containing τ. Let $v \in V(\sigma) \setminus V(\tau)$, thus $v = v_{G'}$ or $v = v_{GG'}$ for some $G \in \mathbf{F}(P)$. Notice that $G_\tau \subseteq G$ and thus $\psi(G_\tau) \supseteq \psi(G)$. From definition of Tr^Q we have $u_H \in H$ for any $H \in \mathbf{F}(Q)$ and from definition of labelling l' we have $l'(v) = u_{\psi(G)} \in \psi(G) \subseteq \psi(G_\tau)$. Thus $l'(V(\sigma) \setminus V(\tau)) \subseteq \psi(G_\tau)$ and $l'(V(\sigma)) = l'((V(\sigma) \setminus V(\tau)) \cup V(\tau)) = l'((V(\sigma) \setminus V(\tau))) \cup l'(V(\tau)) \subseteq \psi(G_\tau) \cup l'(V(\tau)) = \psi(G_\tau) \cup l(V(\tau))$.

From the assumption $b \notin \mathrm{co}\,(l(V(\tau)) \cup \psi(G_\tau))$. Therefore $b \notin \mathrm{co}\, l'(V(\sigma))$ and σ is not a b-balanced simplex.

Let σ be an $(n-1)$-simplex, $V(\sigma) = \{v_1, \ldots, v_n\}$, $v'_i = 2b - v_i$ $(i \in N)$, $C(\sigma) = \mathrm{cone}(\{v'_1, \ldots, v'_n\}, b)$. An $(n-1)$-simplex σ is b-subbalanced with respect to x if and only if $x \in C(\sigma)$. The set $C(\sigma)$ is an $(n-1)$-dimensional set and the union $\bigcup_{\sigma \in Tr'_{n-1}, \sigma \subset \mathrm{bd}\, P'} C(\sigma)$ is also an $(n-1)$-dimensional set. Hence, we can choose $x \in \mathbb{R}^n$, $x \neq b$ in such a way that Tr' does not contain a b-subbalanced simplex with respect to x of dimension smaller that $n-1$. Consider a line going through x and b. This line meets $\mathrm{bd}\, Q$ in two points. By x' we denote the common point of this line and $\mathrm{bd}\, Q$ such that $b \in \mathrm{co}\,\{x, x'\}$ and by $\sigma_Q \in Tr^Q$ we denote the $(n-1)$-dimensional boundary simplex containing x'. The function l' restricted to the set $\mathrm{bd}\, P' \cap V'$ is a one-to-one simplicial function. The simplex $\sigma_P := \mathrm{co}\, l'^{-1}(V(\sigma_Q))$ is b-subbalanced with respect to x and it is the only such simplex on $\mathrm{bd}\, P$.

Now, from Lemma 2.2 it follows that the number of b-balanced simplexes in Tr' is odd. Since $Tr' = Tr \cup Tr''$ and there is no b-balanced simplex in Tr'', there exists b-balanced simplex in Tr. ∎

3. Corollaries and Applications

In this section we present corollaries to Theorem 2.6 in order to show the strength of this theorem. First we apply Theorem 2.6 to the simplex:

Corollary 3.1. *Let $P = \mathrm{co}\,\{d_0, \ldots, d_n\} \subset \mathbb{R}^n$ be an n-dimensional simplex, $m_{F_i} = \sum_{j \neq i} \frac{d_j}{n}$ be the gravity center of a facet $F_i = \mathrm{co}\,\{d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n\}$, $m_P = \sum_{j=0}^{n} \frac{d_j}{n+1}$ be the gravity center of P, Tr be a triangulation* *of the simplex P, $V = Tr_0$ and $l : V \to \mathbb{R}^n$ be a labelling. If for every face $F = \mathrm{co}\{d_i : i \in M\}$ and every simplex $\sigma \subset F$, $\sigma \in Tr$, σ is not m_P-subbalanced with respect to the set $\{m_{F_i} : i \in M\}$, then there exists an m_P-balanced simplex in Tr.*

Corollary 3.1 is more general than the Sperner lemma [12] and the Shapley lemma (Lemma 7.2 in [11]).

Corollary 3.2. *Let $P = \mathrm{co}\{d_0, \ldots, d_n\} \subset \mathbb{R}^n$ be an n-dimensional simplex, $m_P = \sum_{j=0}^{n} \frac{d_j}{n+1}$ be the gravity center of P, Tr be a triangulation of the simplex P, $V = Tr_0$ and $l : V \to \mathbb{R}^n$ be a labelling. If for every face $F = \mathrm{co}\{d_i : i \in M\}$ $(M \subset N_0)$ and every simplex $\sigma \subset F$, $\sigma \in Tr$, σ is not m_P-subbalanced with respect to the set $\{d_i : i \notin M\}$, then there exists an m_P-balanced simplex in Tr.*

Corollary 3.2 is more general than the Scarf lemma ([10]; see also Theorem 3.4 in [9]) and the Garcia lemma ([3], see also Theorem 3.6 in [9]).

The next result is on an n-dimensional cube. Let $I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : -1 \le x_i \le 1,\ i \in N\}$ be an n-dimensional cube and for $k \in N$, $i_1, \ldots, i_k \in N$, $i_1 < i_2 < \cdots < i_k$, $s_{i_1}, \ldots, s_{i_k} \in \{-1, 1\}$ let $I(s_{i_1} i_1, \ldots, s_{i_k} i_k)$ $= \{(x_1, \ldots, x_n) \in I^n : x_{i_j} = s_{i_j}, j \in \{1, \ldots, k\}\}$ be an $(n-k)$-dimensional face of I^n.

Corollary 3.3. *Let Tr be a triangulation of the cube I^n, $V = Tr_0$ and $l : V \to \mathbb{R}^n$ be a labelling. If for all $k \in N$, $i_1, \ldots, i_k \in N$, $i_1 < i_2 < \cdots < i_k$, $s_{i_1}, \ldots, s_{i_k} \in \{-1, 1\}$ and every simplex $\sigma \in Tr$ and $\sigma \subseteq I(s_{i_1} i_1, \ldots, s_{i_k} i_k)$, σ is not subbalanced with respect to the set $\{s_{i_j} e_{i_j} : j \in \{1, \ldots k\}\}$, then there exists a balanced simplex in Tr.*

Proof. It follows directly from Theorem 2.6 for $P = I^n$, $Q = \mathrm{co}\{e_i, -e_i : i \in N\}$ and $\psi(\{s_{i_j} e_{i_j} : j \in \{1, \ldots, k\}\}) = \mathrm{co}\{s_{i_j} e_{i_j} : j \in \{1, \ldots, k\}\}$ for all $k \in N$, $i_1, \ldots, i_k \in N$, $i_1 < i_2 < \cdots < i_k$, $s_{i_1}, \ldots, s_{i_k} \in \{-1, 1\}$. ■

Corollary 3.3 is more general than the Freund lemma (Lemma 1 in [2], see also Lemma 3.7 in [9]). Our next result is a generalization of the Poincaré-Miranda theorem [8]:

Theorem 3.4. *Let $f : I^n \to \mathbb{R}^n$ be a continuous function, such that for all $k \in N$, $i_1, \ldots, i_k \in N$, $i_1 < i_2 < \cdots < i_k$, $s_{i_1}, \ldots, s_{i_k} \in \{-1, 1\}$*

$$f(I(s_{i_1}i_1,\dots,s_{i_k}i_k))\cap \operatorname{cone}\{-s_{i_j}e_{i_j} : j\in\{1,\dots,k\}\}=\emptyset,$$

then there exists $x\in I^n$ *such that* $f(x)=0$.

Proof. Consider a sequence of triangulations Tr^m of I^n ($m\in\mathbb{N}$) with mesh tending to zero, when m tends to infinity. Let $V^m=V(Tr^m)$ and $l^m=f|_{V^m}$. We show that the labelling l^m fulfils the condition of Corollary 3.3. Let $I(s_{i_1}i_1,\dots,s_{i_k}i_k)$ be a face of I^n for $k\in N$, $i_1,\dots,i_k\in N$, $i_1<i_2<\dots<i_k$, $s_{i_1},\dots,s_{i_k}\in\{-1,1\}$. Take $\sigma^m\subseteq I(s_{i_1}i_1,\dots,s_{i_k}i_k)$. Because f is continuous and for sufficiently large m the mesh of Tr^m is small enough, the condition $f(I(s_{i_1}i_1,\dots,s_{i_k}i_k))\cap\operatorname{cone}\{-s_{i_j}e_{i_j} : j\in\{1,\dots,k\}\}=\emptyset$ implies $l^m(V(\sigma^m))\cap\operatorname{cone}\{-s_{i_j}e_{i_j} : j\in\{1,\dots,k\}\}=\emptyset$. This is equivalent to the condition that σ^m is not subbalanced with respect to the set $\{s_{i_j}e_{i_j} : j\in\{1,\dots,k\}\}$. Corollary 3.3 implies that there exists a balanced simplex in Tr^m. Now, if the mesh of Tr^m tends to zero, the sequence of simplexes σ^m tends to a point z. But each σ^m is a balanced simplex so we have $f(z)=0$. ■

Theorem 2.6 is more general than our previous result:

Corollary 3.5 (Theorem 3.4 in [6])**.** *Let* $P\subset\mathbb{R}^n$ *be a polyhedron of dimension* n*,* Tr *be a triangulation of the polyhedron* P*,* $V=Tr_0$*,* $b\in\operatorname{ri}P$ *and* $l:V\to\mathbb{R}^n$ *be a labelling. If for every facet* F *of the polyhedron* P *there exists an* $(n-1)$*-dimensional hyperplane* h_b^F *containing the point* b *and disjoint with* F *such that* $l(V\cap F)\subset H_b^F$*, where* H_b^F *is an open halfspace containing* F *such that* h_b^F *is in its boundary, then there exists a* b*-balanced* n*-simplex in the triangulation* Tr*.*

Proof. For every polyhedron P and any point $b\in P$ there exists a dual polyhedron Q such that every face of Q is perpendicular to the ray issuing from b through the vertices of P and whose vertices lie on the rays issuing from b and perpendicular to the faces of P (for the proof see [1], pp. 45). Hence P,Q and l fulfil the condition of Theorem 2.6 and we get our corollary. ■

For every n-dimensional polyhedron $P\subset\mathbb{R}^n$ and any point $x_0\in\operatorname{ri}P$ there exist vectors a_i ($i\in I\subset\mathbb{N}$), such that $P=\{x\in\mathbb{R}^n : a_ix\le 1+a_ix_0, i\in I\}$. Let $\operatorname{car}B=\{i\in I : a_ix=1+a_ix_0 \text{ for all } x\in B\}$ for $B\subset P$.

Two theorems below, proved by van der Laan, Talman and Yang, follow also from our Theorem 2.6:

Corollary 3.6 (Theorem 4.1 in [9]). *Let* $P = \{x \in \mathbb{R}^n : a_i x \leq 1 + a_i x_0, i \in I\} \subset \mathbb{R}^n$ *be an* n*-dimensional polyhedron,* $Q = \operatorname{co}\{a_j \in \mathbb{R}^n : i \in I\}$, $b \in \operatorname{ri} Q$, Tr *be a triangulation of* P, $V = Tr_0$ *and* $l : V \to \mathbb{R}^n$ *be a labelling. There exists a simplex* $\sigma \in Tr$ *such that* $b \in \operatorname{co}(l(V(\sigma)) \cup \{a_i : i \in \operatorname{car}\sigma\})$.

Proof. For a polyhedron $P = \{x \in \mathbb{R}^n : a_i x \leq 1 + a_i x_0, i \in I\} \subset \mathbb{R}^n$ the polyhedron $Q = \operatorname{co}\{a_j \in \mathbb{R}^n : i \in I\}$ is dual to P through a mapping $\psi : \mathbf{F}(P) \to \mathbf{F}(Q)$ defined by $\psi(F) = \operatorname{co}\{a_i \in \mathbb{R}^n : i \in \operatorname{car} F\}$, (for the proof see Grunbaum [4] pp. 46–49).

Notice that for any boundary simplex $\sigma \in Tr$ the condition $b \in \operatorname{co}(l(V(\sigma)) \cup \{a_i : i \in \operatorname{car}\sigma\})$ says that σ is b-subbalanced with respect $\operatorname{co}\{a_i \in \mathbb{R}^n : i \in \operatorname{car}\sigma\}$. Hence, if there is no boundary simplex σ such that $b \in \operatorname{co}(l(V(\sigma)) \cup \{a_i : i \in \operatorname{car}\sigma\})$, then the assumptions of Theorem 2.6 are satisfied and we get the thesis. ■

For any k-dimensional polyhedron $P \subset \mathbb{R}^n$ there exists m vectors $a_i \in \mathbb{R}^n$, m real numbers $\alpha_i \in \mathbb{R}$ ($m > k$, $i \in I \subset \mathbb{N}$), and $n-k$ vectors $d_h \in \mathbb{R}^n$, $n-k$ real numbers $\delta_h \in \mathbb{R}$ ($h \in N_k \subset \mathbb{N}$) such that $P = \{x \in \mathbb{R}^n : a_i x \leq \alpha_i \text{ for } i \in I,\ d_h x = \delta_h \text{ for } h \in N_k\}$.

Corollary 3.7 (Theorem 3.1 in [9]). *Let* $P = \{x \in \mathbb{R}^n : a_i x \leq \alpha_i \text{ for } i \in I,\ d_h x = \delta_h \text{ for } h \in N_k\}$ *be a* k*-dimensional polyhedron,* $W = \operatorname{aff}\{d_h : h \in N_k\}$, $W^* = \{x \in \mathbb{R}^n : xy = 0 \text{ for all } y \in W\}$, Tr *a triangulation of* P, $l : V \to \mathbb{R}^n$ *be a labelling such that* $(\operatorname{co} l(V)) \cap W = \{0\}$. *If for every* $F \in \mathbf{F}(P)$ *and every simplex* $\sigma \in Tr$, $\sigma \subseteq F$ *the intersection* $(\operatorname{co} l(V(\sigma))) \cap (\operatorname{cone}(0, \{a_i : i \in \operatorname{car}(F)\}) + W)$ *is empty or contains the point* $0 \in \mathbb{R}^n$, *then there exists a balanced simplex in* Tr.

Proof. Without loss of generality we may assume that the vectors a_i for $i \in I$ are parallel to the hyperplane W^*. We can consider projection of the polyhedron P, labels $l(V)$ and vectors a_i for $i \in I$ on the hyperplane W^* parallel to the hyperplane W. Hence, we reduce this theorem to the full-dimensional case. Analogously vectors a_i ($i \in I$) can be scaled in such a way that $Q = \operatorname{co}\{a_i : i \in I\}$ is a polyhedron dual to P through a mapping $\psi : \mathbf{F}(P) \to \mathbf{F}(Q)$ defined by $\psi(F) = \operatorname{co}\{a_i \in \mathbb{R}^n : i \in \operatorname{car} F\}$. If there exists a simplex σ such that $\operatorname{co} l(V(\sigma))$ contains $0 \in \mathbb{R}^n$, then σ is a balanced simplex. If for any $F \in \mathbf{F}(P)$ and any simplex $\sigma \in Tr$, $\sigma \subseteq F$ the intersection $\operatorname{co} l(V(\sigma)) \cap \operatorname{cone}(0, \{a_i : i \in \operatorname{car}(F)\})$ is empty, then σ is not a subbalanced simplex with respect to the set $\psi(F)$ and by Theorem 2.6 we get our theorem. ■

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