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A LOWER BOUND ON THE INDEPENDENCE NUMBER OF A GRAPH IN TERMS OF DEGREES

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Abstract

For a connected and non-complete graph, a new lower bound on its independence number is proved. It is shown that this bound is realizable by the well known efficient algorithm MIN.

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1. Introduction and Theorem

Let G be a finite, undirected, simple, non-complete, and connected graph on its vertex set $V(G) = \{1, 2, \ldots, n\}$. For a subgraph H of G and for a vertex $i \in V(H)$ let $d_H(i)$ be the degree of i in H, i.e., the cardinality of the neighbourhood $N_H(i) \subset V(H)$ of i in H, and let $\delta(H)$ be the minimum degree of H. A subset I of $V(G)$ is called *independent* if the subgraph of G spanned by I is edgeless. The *independence number* $\alpha(G)$ is the largest cardinality $|I|$ among all independent sets I of G. The following algorithm MIN (cf. [8]) is a well known procedure to construct an independent set of a graph G.

Algorithm MIN:
1. $G_1 := G, j := 1$
2. while $V(G_j) \neq \emptyset$ do
begin
choose $i_j \in V(G_j)$ with $d_{G_j}(i_j) = \delta(G_j)$, delete $\{i_j\} \cup N_{G_j}(i_j)$ to obtain G_{j+1} and set $j := j + 1$;
end;
3. $k := j - 1$
STOP

Obviously, the set $\{i_1, i_2, \ldots, i_k\} \subset V(G)$ is an independent set of G and therefore $\alpha(G) \geq k$ for every output k of algorithm MIN. Let k_{MIN} be the smallest k Algorithm MIN provides for a fixed graph G. In the following Theorem a new lower bound on k_{MIN} is established.

Theorem. *Let G be a finite, simple, connected, and non-complete graph on n vertices with maximum degree Δ, n_j be the number of vertices of degree j in G, and*

$$x(j) = \frac{j(j+1)}{j(j+1)-1}\Bigg[\left(\frac{1}{j+1} - (\Delta - j)\right)n_\Delta + \left(\frac{1}{j+1} - (\Delta - j - 1)\right)n_{\Delta-1} + \ldots + \left(\frac{1}{j+1} - 1\right)n_{j+1} + \frac{n_j}{j+1} + \frac{n_{j-1}}{j} + \ldots + \frac{n_1}{2} - 1\Bigg]$$

for $j \in \{\Delta, \Delta - 1, \ldots, 1\}$.

(i) *Then there is a unique $j_0 \in \{\Delta, \Delta - 1, \ldots, 1\}$ such that $0 \leq x(j_0) < n_\Delta + \ldots + n_{j_0}$ and*

(ii)
$$k_{MIN} \geq \left(\sum_{j=1}^{\Delta} \frac{n_j}{j+1}\right) + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta-1)\Delta} + \ldots + \frac{n_\Delta + \ldots + n_{j_0+1}}{(j_0+2)(j_0+1)} + \frac{x(j_0)}{(j_0+1)j_0}$$
$$= 1 + x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \ldots + (\Delta - j_0)n_\Delta.$$

2. Proof

Let $d_i = d_G(i), i = 1, \ldots, n$ and for $1 \leq k \leq d_1 + \ldots + d_n + 1$ let $f(k) = \min \sum_{i=1}^{n} \frac{1}{d_i+1-x_i}$, where the minimum is taken over integers x_i with $0 \leq x_i \leq d_i$ and $\sum_{i=1}^{n} x_i = k - 1$. Lemma 1 and Lemma 2 are proved in [7].

Lemma 1. $k_{MIN} \geq f(k_{MIN})$.

Lemma 2. *The following algorithm A calculates $f(k)$:*
Input: $F = \{d_1, d_2, \ldots, d_n\}$, $k \in \{1, 2, \ldots, d_1 + \ldots + d_n + 1\}$, $j := 0$;
while $j < k - 1$ do begin $F := (F \setminus \{\max(F)\}) \cup \{\max(F) - 1\}$; $j := j + 1$ end. Output: $f(k) = \sum_{f \in F} \frac{1}{f+1}$.

Note that F is a family, i.e., a member of F may occur more than once. Given $k \in \{1, 2, \ldots, d_1 + \ldots + d_n + 1\}$, in each of the $k-1$ steps of algorithm A a maximum member f of the current family F is replaced by $f - 1$.

If $k = d_1 + \ldots + d_n + 1$ then $f(k) = n$. If $1 \leq k \leq d_1 + \ldots + d_n = n_1 + 2n_2 + \ldots + \Delta n_\Delta$ then there are unique integers j and x with $j \in \{\Delta, \Delta - 1, \ldots, 1\}$ and $0 \leq x < n_\Delta + \ldots + n_j$ such that $k - 1 = x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_\Delta = n_\Delta + (n_\Delta + n_{\Delta-1}) + \ldots + (n_\Delta + n_{\Delta-1} + \ldots + n_{j+1}) + x$. With this expression for $k - 1$ the part cut away by algorithm A is illustrated in Figure 1.

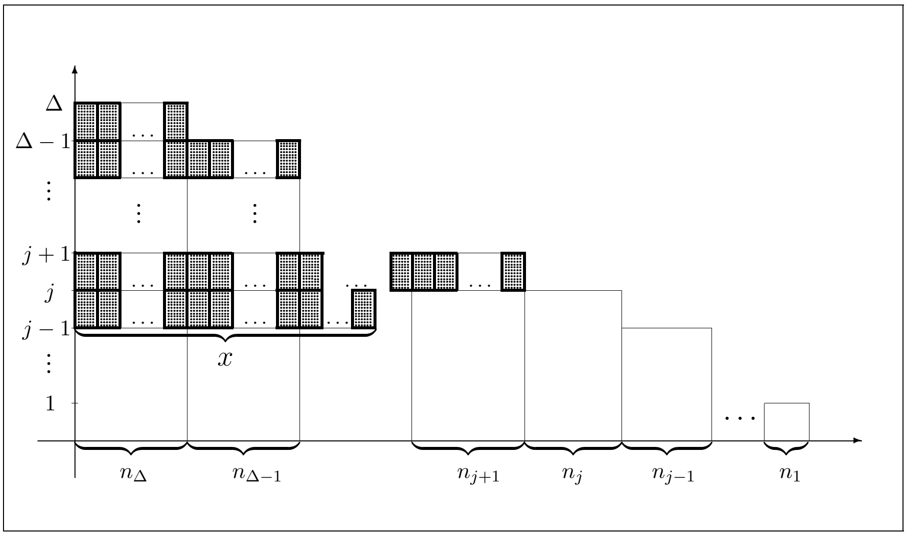

Figure 1

Hence, after applying algorithm A, the family F contains the member $j-1$ exactly $x+n_{j-1}$ times, the member j exactly $n_\Delta+\ldots+n_j-x$ times, and all other members of F being smaller than $j-1$ at the beginning remain unchanched. Thus, the following Lemma 3 is proved.

Lemma 3.

(i) Given $k\in\{1,\ldots,d_1+\ldots+d_n\}$, there are unique integers j and x with $j\in\{\Delta,\Delta-1,\ldots,1\}$ and $x\in\{0,\ldots,n_\Delta+\ldots+n_j-1\}$ such that

$$k-1=n_\Delta+(n_\Delta+n_{\Delta-1})+\ldots+(n_\Delta+n_{\Delta-1}+\ldots+n_{j+1})+x$$

$$=x+n_{j+1}+2n_{j+2}+\ldots+(\Delta-j)n_\Delta$$

and

(ii) $$f(k)=(n_\Delta+\ldots+n_j-x)\frac{1}{j+1}+\frac{x}{j}+\frac{n_{j-1}}{j}+\ldots+\frac{n_1}{2}$$

$$=(n_\Delta+\ldots+n_j)\frac{1}{j+1}+\frac{x}{j(j+1)}+\frac{n_{j-1}}{j}+\ldots+\frac{n_1}{2}\text{ for that }k.$$

Lemma 4. *If* $k=1+x+n_{j+1}+2n_{j+2}+\ldots+(\Delta-j)n_\Delta$ *with* $j\in\{\Delta,\Delta-1,\ldots,1\}$ *and* $x\in\{0,\ldots,n_\Delta+\ldots+n_j-1\}$*, then* $f(k+1)-f(k)=\frac{1}{j(j+1)}$.

Proof of Lemma 4. If $x\le n_\Delta+\ldots+n_j-2$ then $k+1=1+(x+1)+n_{j+1}+2n_{j+2}+\ldots+(\Delta-j)n_\Delta$ and if $x=n_\Delta+\ldots+n_j-1$ then $k+1=1+n_j+2n_{j+1}+\ldots+(\Delta-j+1)n_\Delta$. In both cases Lemma 3 implies Lemma 4. ■

Using Lemma 3, the calculation of $f(k)$ is possible now without taking a minimum and without using algorithm A. In the sequel, we will define the function f for real $k\in[1,d_1+\ldots+d_n+1)$ and show that the function $g(k)=k-f(k)$ is continuous and strictly increasing on $[1,d_1+\ldots+d_n+1)$. Finally, using $g(1)<0$ and $g(k_{MIN})\ge 0$, the lower bound k_0 on k_{MIN} is the unique solution of the equation $k=f(k)$.

Thus, for given integer $j\in\{\Delta,\Delta-1,\ldots,1\}$ and real number x with $0\le x<n_\Delta+\ldots+n_j$ let the real numbers k and $f(k)$ (implicitely) be defined as $k=1+x+n_{j+1}+2n_{j+2}+\ldots+(\Delta-j)n_\Delta$ and $f(k)=(n_\Delta+\ldots+n_j)\frac{1}{j+1}+\frac{x}{j(j+1)}+\frac{n_{j-1}}{j}+\ldots+\frac{n_1}{2}$.

Lemma 5. *The function g with $g(k) = k - f(k)$ is continuous and strictly increasing on $[1, d_1 + \ldots + d_n + 1)$.*

***Proof of Lemma* 5.** First, let $j \in \{\Delta, \Delta - 1, \ldots, 1\}$ be fixed. Then $k = 1 + x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_\Delta$ with $0 \leq x < n_\Delta + \ldots + n_j$ belongs to the interval $I(j) = [1 + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_\Delta, 1 + n_j + 2n_{j+1} + \ldots + (\Delta - j + 1)n_\Delta)$. Obviously g is continuous on $I(j)$ and, because $g(k + \epsilon) - g(k) = \epsilon - \frac{\epsilon}{j(j+1)}$ and $j(j+1) \geq 2$, g is strictly increasing on $I(j)$.

Now consider g on $[1, \ldots, d_1 + \ldots + d_n + 1)$ and note that $I(\Delta) \cup \ldots \cup I(1) = [1, \ldots, d_1 + \ldots + d_n + 1)$ and $I(j) \cap I(j') = \emptyset$ if $j \neq j'$. It is easy to see that g is also continuous in $k = 1 + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_\Delta$ for $j \in \{\Delta - 1, \Delta - 2, \ldots, 2\}$ and we are done. ■

In [2, 12] the well known Caro-Wei-bound $CW = \sum_{j=1}^{\Delta} \frac{n_j}{j+1}$ is proved to be a lower bound on $\alpha(G)$ and being tight if and only if G is complete. With our assumption that G is non-complete, $g(1) = 1 - \sum_{j=1}^{\Delta} \frac{n_j}{j+1} < 0$ and $g(k_{MIN}) \geq 0$ by Lemma 1. As a consequence of Lemma 5 there is a unique zero $k_0 = 1 + x(j_0) + n_{j_0+1} + 2n_{j_0+2} + \ldots + (\Delta - j_0)n_\Delta$ of g with $1 < k_0 \leq k_{MIN}$ and $0 \leq x(j_0) < n_\Delta + \ldots + n_{j_0}$. Considering the equation $f(k) = k$ we obtain

Lemma 6. *If $j \in \{\Delta, \Delta - 1, \ldots, 1\}$ and $k = 1 + x + n_{j+1} + 2n_{j+2} + \ldots + (\Delta - j)n_\Delta$ with $0 \leq x < n_\Delta + \ldots + n_j$, then $f(k) = k$ if and only if*

$$x = \frac{j(j+1)}{j(j+1) - 1}\left[\left(\frac{1}{j+1} - (\Delta - j)\right)n_\Delta + \ldots + \left(\frac{1}{j+1} - 1\right)n_{j+1} + \frac{n_j}{j+1} + \ldots + \frac{n_1}{2} - 1\right].$$

Now we complete the proof of the Theorem. Assume there is $j_1 \in \{\Delta, \Delta - 1, \ldots, 1\}$ with $j_1 \neq j_0$, $x = \frac{j_1(j_1+1)}{j_1(j_1+1)-1}[(\frac{1}{j_1+1} - (\Delta - j_1))n_\Delta + \ldots + (\frac{1}{j_1+1} - 1)$ $n_{j_1+1} + \frac{n_{j_1}}{j_1+1} + \ldots + \frac{n_1}{2} - 1]$, and $0 \leq x < n_\Delta + \ldots + n_{j_1}$. Then $k_1 = 1 + x(j_1) + n_{j_1+1} + 2n_{j_1+2} + \ldots + (\Delta - j_1)n_\Delta$ is a solution of the equation $f(k) = k$ by Lemma 6 and $k_0 \neq k_1$ by Lemma 3 (i) contradicting the uniqueness of k_0.

With $k_0 = f(k_0) = f(1) + (f(2) - f(1)) + \ldots + (f(\lfloor k_0 \rfloor) - f(\lfloor k_0 \rfloor - 1)) + (f(k_0) - f(\lfloor k_0 \rfloor))$ and Lemma 4 we have $f(k_0) = (\sum_{j=1}^{\Delta} \frac{n_j}{j+1}) + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta-1)\Delta} + \ldots + \frac{n_\Delta + \ldots + n_{j_0+1}}{(j_0+2)(j_0+1)} + \frac{x(j_0)}{(j_0+1)j_0}$ and the Theorem is proved. ■

Many lower bounds on $\alpha(G)$ are known (cf. [1, 2, 3, 4, 5, 6, 8, 9, 10, 11]). If we compare them with k_0, let us remark here that, by the Theorem,

$$k_0 = CW + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{(\Delta-1)\Delta} + \ldots + \frac{n_\Delta + \ldots + n_{j_0+1}}{(j_0+2)(j_0+1)} + \frac{x(j_0)}{(j_0+1)j_0}$$

$$\geq CW + \frac{n_\Delta}{\Delta(\Delta+1)} + \frac{n_\Delta + n_{\Delta-1}}{\Delta(\Delta+1)} + \ldots + \frac{n_\Delta + \ldots + n_{j_0+1}}{\Delta(\Delta+1)}$$

$$+ \frac{x(j_0)}{\Delta(\Delta+1)} = CW + \frac{k_0 - 1}{\Delta(\Delta+1)}.$$

This implies $k_0 \geq CW + \frac{CW-1}{\Delta(\Delta+1)-1}$ improving the well known lower bound $CW + \frac{CW-1}{\Delta(\Delta+1)}$ on $\alpha(G)$ by O. Murphy ([8]).

In [6] it was established $\alpha \geq \frac{CW^2}{CW - \sum_{ij\in E(G)} (d_i-d_j)^2 q_i^2 q_j^2}$, and S.M. Selkow ([9]) proved $\alpha \geq \sum_{i=1}^n q_i(1 + \max\{0, d_i q_i - \sum_{ij\in E(G)} q_j\})$, where $q_i = \frac{1}{d_i+1}$ and $E(G)$ is the edge set of G. Both bounds equal CW if the graph is regular, however, Murphy's bound and therefore also k_0 are considerably larger in that case. For a star $K_{1,p}$ on $p+1$ vertices we have the converse situation, i.e., k_0 is not comparable with these bounds in [6, 9].

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